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# Magnetoelastic buckling of a rectangular block in plane strain

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**Abstract**

Of interest here is the stability of a rectangular block subjected to a uniform magnetic field perpendicular to its longitudinal axis. The two ends of the block are frictionless and kept parallel to each other. This boundary value problem is motivated by the classical problem of magnetoelastic buckling in which a cantilever beam subjected to a transverse magnetic field buckles when the applied field reaches a critical value.

This work presents a finite strain continuum mechanics formulation of the stability problem of a homogeneous, compressible, magnetoelastic rectangular block in plane strain subjected to a uniform transverse magnetic field. The applied variational approach employs an unconstrained energy minimization recently proposed by the authors.

The analytical solution for the critical buckling fields for both the antisymmetric and symmetric modes are obtained for three different constitutive laws. The corresponding result for thin beams is extracted asymptotically for a special material and the solution is compared to previously published results. The critical magnetic field is shown to increase monotonically with the block's aspect ratio for each material and mode type. Antisymmetric modes are always the critical buckling modes for stress saturated and neo-Hookean materials, except for a narrow range of moderate aspect ratios (about 0.25) where symmetric modes become critical. For strain-saturated solids no buckling is possible above a maximum aspect ratio.


*Keywords:* Electromechanical processes; Finite strain; Particulate reinforced material; Energy methods; Stability

## 1. Introduction and motivation

Magnetoelastic solids exhibit coupling between their mechanical and magnetic responses. Their study in the context of continuum mechanics goes back a few decades to Truesdell and Toupin (1960), Tiersten (1964), Brown (1966), and Maugin and Eringen (1972). Due to novel technological applications, such as magnetoelastic elastomers, there has been a renewed interest in these materials (e.g. DeSimone and James, 2002; Dorfmann and Ogden, 2003; Kankanala and Triantafyllidis, 2004; and Ericksen, 2006).

The solution of basic nontrivial boundary value problems is the obvious next step in further examining the nature of the underlying coupling between magnetic and elastic effects. As such, attention is here focused on the classical magnetoelastic buckling problem in which a bar in a transverse magnetic field buckles when the

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magnitude of the field reaches a critical value. The first authors to systematically address this problem are Moon and Pao (1968) who present a mathematical model and experimental observations for the magnetoelastic buckling of an elastic beam–plate. Above the critical field the bar rotates from its straight equilibrium position while below the critical field, the bar is stable in the straight configuration.

These authors employ the classical thin-plate theory, assume a linear ferromagnetic material and neglect magnetostrictive effects to find the critical field at which bifurcation in the equilibrium configuration of the beam–plate exists. Their experimental observations indicate a critical value that is about a half of the theoretical value. Pao and Yeh (1973) revisit this problem using a general theory of magnetoelasticity which upon linearization yields the buckling equations for the structural beam problem. The antisymmetric buckling result of Pao and Yeh (1973) is identical to the one obtained earlier by Moon and Pao (1968) and so we will henceforth mainly reference Pao and Yeh (1973).

Subsequent articles (e.g. Wallerstein and Peach, 1972; Popelar, 1972; Dalrymple et al., 1974; Miya et al., 1978) were published in the 1970s to investigate the source of the discrepancy between observed and theoretical results. In all such published works, it seems only structural models were used to describe the magnetoelastic phenomenon. The present work is a continuum mechanics based approach to the equivalent problem of the buckling of a magnetoelastic block subjected to a transverse magnetic field. In contrast to previous works, a continuum formulation is valid for arbitrary block geometries and constitutive laws and is analogous to the hyperelastic problem (e.g. Hill and Hutchinson, 1975; Rivlin and Sawyers, 1974; Ogden, 1984).

The outline of the present work is as follows: in Section 2 we formulate the stability problem of a homogeneous, compressible, hyper-elastic, magnetic rectangular block in plane strain subjected to a uniform transverse magnetic field. More specifically, we start with an overview of a variational method in magnetoelasticity proposed by Kankanala and Triantafyllidis (2004), and based on an unconstrained energy minimization, that yields the governing equations and boundary conditions. A description of the principal solution is given next followed by the bifurcation analysis of the block for arbitrary constitutive laws and aspect ratios. For efficiency in presentation, the lengthy intermediate steps in the bifurcation analysis are given in Appendix A.

In Section 3 are given the applications of the general theory for three different constitutive laws: a Gent type solid that exhibits strain saturation in a simple shear test, a neo-Hookean material and finally a solid that saturates in stress under simple shear. The principal solution for the different materials is then discussed, followed by the presentation of the block's critical magnetic field and eigenmode, as a function of the block's aspect ratio. The critical magnetic field is found to increase monotonically with the block's aspect ratio for each material and mode type. For stress saturated and neo-Hookean materials antisymmetric modes are always the critical buckling modes, except for a narrow range of moderate aspect ratios (about 0.25) where symmetric modes become critical. In the case of strain-saturated solids no buckling is possible above a maximum aspect ratio. The section concludes with an asymptotic analysis in the slender limit using, without loss of generality, a special material and the solution is compared with the well-known result from structural approximations (e.g. Pao and Yeh, 1973). Concluding remarks and suggestions for future work are provided in Section 4.

## 2. Mathematical model

This section presents the formulation for the plane strain stability problem of a magnetoelastic rectangular block. The first subsection outlines the general 3D energy formulation for a magnetoelastic material. The next subsection presents the principal solution for the plane strain problem of the rectangular magnetoelastic block subjected to a transverse magnetic field. The bifurcation analysis for the block is presented subsequently and the section is concluded with an asymptotic analysis for the small aspect ratio rectangular block.

### 2.1. Variational formulation

A brief outline of the energy formulation of the isothermal, reversible finite strain magnetoelasticity is given here for completeness. Readers interested in a more detailed exposition of the energy method are referred to

Kankanala and Triantafyllidis (2004). Unless otherwise indicated, the usual continuum mechanics convention is employed henceforth, according to which all field quantities in capital letters are associated with the reference configuration while their counterparts in small letters are associated with the current configuration.

In contrast to finite elasticity, a magnetoelastic solid not only stores energy inside the volume $V$ it occupies but its presence changes the magnetic field of the free space around it. The total energy, $\mathscr{E}$, neglecting kinetic and thermal effects, is the sum of the solid's free energy plus the magnetic energy of the entire space

$$\mathscr{E} = \int_V \rho\psi \, \mathrm{d}v + \int_{\mathbf{R}^3} \frac{\mu_0}{2} (\mathbf{h} \cdot \mathbf{h}) \, \mathrm{d}v, \tag{2.1}$$

where $\rho$ is the current mass density of the solid, $\psi(\mathbf{F}, \mathbf{M})$ is the Helmholtz free energy taken to be a function of the deformation gradient $\mathbf{F} \equiv \mathbf{x}\nabla$ (where $\nabla \equiv \partial()/\partial\mathbf{X}$ is the gradient operator in the reference configuration) and the specific magnetization $\mathbf{M}(\mathbf{x})$. In addition $\mathbf{h}(\mathbf{x})$ is the magnetic field and $\mu_0$ is the magnetic permeability of vacuum ($\mu_0 = 4\pi 10^{-7} \, N \, A^{-2}$). The existence of the free energy is based on the assumption there are no hysteretic or rate effects in the magnetoelastic solid and that there is no energy dissipation in a closed loading loop in strain and magnetization space (under fixed temperature).

It is important to separate the magnetic field $\mathbf{h}$ into the externally applied field $\mathbf{h}_0$ plus the perturbation field $\widehat{\mathbf{h}}$ due to the presence of the magnetoelastic solid, namely

$$\mathbf{h} = \mathbf{h}_0 + \widehat{\mathbf{h}}. \tag{2.2}$$

To find the potential energy $\mathscr{P}$ of the magnetoelastic solid, one has to add the potential $\mathscr{W}$ of the applied loads to the total energy $\mathscr{E}$

$$\mathscr{W} = -\int_V [\mu_0(\mathbf{h}_0 \cdot \mathbf{m}) + \rho(\mathbf{f} \cdot \mathbf{u})] \, \mathrm{d}v - \int_{\partial V} \mathbf{t} \cdot \mathbf{u} \, \mathrm{d}a, \tag{2.3}$$

where the $\mathbf{h}_0 \cdot \mathbf{m}$ term is the contribution of the applied external magnetic field $\mathbf{h}_0$, $\rho(\mathbf{f} \cdot \mathbf{u})$ is the contribution due to the body force $\mathbf{f}$ ($\mathbf{u} \equiv \mathbf{x} - \mathbf{X}$ denotes the displacement field) and the $\mathbf{t} \cdot \mathbf{u}$ term is the contribution of the mechanical surface traction $\mathbf{t}$.[1] In view of Ampère's law, the magnetic energy of the entire space can be rewritten as (see Kankanala and Triantafyllidis, 2004)

$$\int_{\mathbf{R}^3} \frac{\mu_0}{2} (\mathbf{h} \cdot \mathbf{h}) \, \mathrm{d}v = \int_{\mathbf{R}^3} \frac{\mu_0}{2} (\widehat{\mathbf{h}} \cdot \widehat{\mathbf{h}}) \, \mathrm{d}v + \int_{\mathbf{R}^3} \frac{\mu_0}{2} (\mathbf{h}_0 \cdot \mathbf{h}_0) \, \mathrm{d}v. \tag{2.4}$$

Hence the potential energy $\mathscr{P}$ of the system (solid plus surrounding free space) is found to be from Eqs. (2.1), (2.3), (2.4)

$$\mathscr{P} \equiv \mathscr{E} + \mathscr{W} = \int_V (\rho\psi - \mu_0 \mathbf{h}_0 \cdot \mathbf{m} - \rho \mathbf{f} \cdot \mathbf{u}) \, \mathrm{d}v + \int_{\mathbf{R}^3} \frac{\mu_0}{2} (\widehat{\mathbf{h}} \cdot \widehat{\mathbf{h}}) \, \mathrm{d}v - \int_{\partial V} \mathbf{t} \cdot \mathbf{u} \, \mathrm{d}a + \int_{\mathbf{R}^3} \frac{\mu_0}{2} (\mathbf{h}_0 \cdot \mathbf{h}_0) \, \mathrm{d}v. \tag{2.5}$$

The last term in the potential energy expression (2.5) is fixed (it depends on the applied external magnetic field $\mathbf{h}_0$ which exists in the absence of the magnetoelastic solid) and as a constant can be omitted from the potential energy.

Using the $\mathbf{b}$ versus $\mathbf{h}$ relation, ($\mathbf{b} = \mu_0(\mathbf{h} + \mathbf{m})$), where $\mathbf{b}$ is the magnetic flux, and noting that $\mathbf{b}_0 = \mu_0 \mathbf{h}_0$, one has the following relationship for the perturbation fields $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{h}}$:

$$\widehat{\mathbf{b}} = \mu_0(\widehat{\mathbf{h}} + \mathbf{m}). \tag{2.6}$$

Since in addition the perturbation flux $\widehat{\mathbf{b}}$ has to satisfy the divergence free (or nonmonopole) condition and the corresponding boundary condition:

$$\mathbf{\nabla} \cdot \widehat{\mathbf{b}} = 0, \quad \mathbf{n} \cdot [\![ \widehat{\mathbf{b}} ]\!] = 0, \tag{2.7}$$

[1] By specifying the body force and surface traction to be mechanical in nature, no *a priori* assumptions for the magnetic terms of the body force and surface tractions are made—especially since these terms are dependent on the choice of the arguments of the free energy. The approach employed here directly yields the magnetic parts of the body force and surface traction from the divergence of the general stress measure and from Cauchy's tetrahedron relation, respectively.

one can express $\widehat{\mathbf{b}}$ in terms of a vector potential $\widehat{\mathbf{a}}$:

$$\widehat{\mathbf{b}} = \nabla \times \widehat{\mathbf{a}}. \tag{2.8}$$

By using (2.6) and (2.8), one can rewrite the potential energy (2.5), without the constant term representing the magnetic energy of the imposed external field $\mathbf{h}_0$, as

$$\mathscr{P} = \int_V (\rho\psi - \mu_0 \mathbf{h}_0 \cdot \mathbf{m} - \rho \mathbf{f} \cdot \mathbf{u})\, \mathrm{d}v + \int_{\mathbf{R}^3} \frac{1}{2\mu_0} (\nabla \times \widehat{\mathbf{a}} - \mu_0 \mathbf{m}) \cdot (\nabla \times \widehat{\mathbf{a}} - \mu_0 \mathbf{m})\, \mathrm{d}v - \int_{\partial V} \mathbf{t} \cdot \mathbf{u}\, \mathrm{d}a. \tag{2.9}$$

For addressing the stability problem of interest here, the potential energy (2.9) will be rewritten with respect to the reference configuration where all field variables are functions of $\mathbf{X}$ (the reference configuration coordinate of a material point). Thus, the potential energy which is expressed in the current configuration in Eq. (2.9), takes the following form in the reference configuration:

$$\begin{aligned} \mathscr{P} = & \int_V \rho_0 \left[ \psi(\mathbf{F}, \mathbf{M}) - \mu_0 \mathbf{M} \cdot \mathbf{h}_0 - \mathbf{f} \cdot \mathbf{u} + \frac{\mu_0}{2J} \rho_0 \mathbf{M} \cdot \mathbf{M} - \frac{1}{J} \mathbf{M} \cdot \mathbf{F} \cdot (\nabla \times \widehat{\mathbf{A}}) \right] \mathrm{d}V \\ & + \int_{\mathbf{R}^3} \frac{1}{2\mu_0 J} (\nabla \times \widehat{\mathbf{A}}) \cdot \mathbf{C} \cdot (\nabla \times \widehat{\mathbf{A}})\, \mathrm{d}V - \int_{\partial V} \mathbf{T} \cdot \mathbf{u}\, \mathrm{d}A. \end{aligned} \tag{2.10}$$

In the above expression for the potential energy of the reference configuration $\mathscr{P}(\mathbf{u}(\mathbf{X}), \mathbf{M}(\mathbf{X}), \widehat{\mathbf{A}}(\mathbf{X}))$, the following quantities have been used: $\rho_0$ the mass density of the reference configuration and $\mathbf{M}$ the specific magnetization (i.e. magnetization per unit mass) are given by

$$\rho_0 \equiv \rho J, \quad J \equiv \det \mathbf{F}, \quad \mathbf{m} = \rho \mathbf{M}, \quad \mathbf{F} = \mathbf{I} + \mathbf{u}\nabla. \tag{2.11}$$

Moreover, $\mathbf{T}$ is the reference mechanical traction on the boundary (force per unit reference area) and $\mathbf{C}$ is the right Cauchy–Green deformation tensor related to the deformation gradient, $\mathbf{F}$ by

$$\mathbf{C} \equiv \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}. \tag{2.12}$$

The current magnetic flux perturbation $\widehat{\mathbf{b}}$ in Eq. (2.8) has been replaced in Eq. (2.10) by its reference configuration counterpart $\widehat{\mathbf{B}}$, where

$$\widehat{\mathbf{B}} = J\mathbf{F}^{-1} \cdot \widehat{\mathbf{b}}, \quad \widehat{\mathbf{B}} = \nabla \times \widehat{\mathbf{A}}. \tag{2.13}$$

The first variation of $\mathscr{P}$ with respect to the independent variables $\mathbf{u}(\mathbf{X})$, $\mathbf{M}(\mathbf{X})$, and $\widehat{\mathbf{A}}(\mathbf{X})$ gives as its Euler–Lagrange equations the equilibrium and mechanical constitutive equations, the magnetization constitutive relation and Ampère's equations, plus the corresponding boundary conditions. It should also be noted here that the field of admissible flux perturbation potentials $\widehat{\mathbf{A}}(\mathbf{X})$ is any continuous vector field defined over $\mathbf{R}^3$ while $\mathbf{M}(\mathbf{X})$ is defined only on $V$ and $\mathbf{M} = 0$ for $\mathbf{X} \notin V$. The situation of the displacement field $\mathbf{u}(\mathbf{X})$ requires clarification: Although only the values of $\mathbf{u}(\mathbf{X})$ for points $\mathbf{X} \in V$ make physical sense, one can without loss of generality continuously extend the admissible displacement fields over $\mathbf{R}^3$.

The variation of the potential energy $\mathscr{P}$ with respect to $\mathbf{M}$ is considered first. By taking the extremum of $\mathscr{P}$ in Eq. (2.10) with respect to the specific magnetization $\mathbf{M}$[2]

$$\mathscr{P}_{,\mathbf{M}} \delta\mathbf{M} = \int_V \rho_0 \left\{ \frac{\partial \psi}{\partial \mathbf{M}} \cdot \delta\mathbf{M} - \mu_0 \mathbf{h}_0 \cdot \delta\mathbf{M} + \mu_0 \rho \mathbf{M} \cdot \delta\mathbf{M} - \left[ \frac{1}{J} \mathbf{F} \cdot (\nabla \times \widehat{\mathbf{A}}) \right] \cdot \delta\mathbf{M} \right\} \mathrm{d}V = 0. \tag{2.14}$$

In view of the arbitrariness of $\delta\mathbf{M}$ and by considering the relations between $\mathbf{M}$ and $\mathbf{m}$ in Eq. $(2.11)_3$ and the perturbed current and reference magnetic fluxes $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{B}}$ in Eq. (2.13), one obtains from Eq. (2.14) the following Euler–Lagrange equation:

$$\frac{\partial \psi}{\partial \mathbf{M}} = \mu_0 \left[ \mathbf{h}_0 + \left( \frac{1}{\mu_0} \widehat{\mathbf{b}} - \mathbf{m} \right) \right] = \mu_0 (\mathbf{h}_0 + \widehat{\mathbf{h}}) = \mu_0 \mathbf{h}. \tag{2.15}$$

[2]Henceforth, $\mathscr{P}_{,\mathbf{g}} \delta\mathbf{g}$, $(\mathscr{P}_{,\mathbf{gg}} \Delta\mathbf{g})\delta\mathbf{g}$ denote, respectively, the first and second Frechet derivatives of the potential, energy $\mathscr{P}$ with respect to the independent variables $\mathbf{g} \equiv (\mathbf{u}, \mathbf{M}, \mathbf{A})$.

The extremum of the potential energy $\mathscr{P}$ with respect to the potential $\widehat{\mathbf{A}}$ of the perturbed magnetic flux yields[3]

$$\mathscr{P}_{,\widehat{\mathbf{A}}}\delta\widehat{\mathbf{A}} = \int_{\mathbf{R}^3}\left\{\left[\boldsymbol{\nabla}\times\left[\frac{1}{J}\left(\frac{1}{\mu_0}(\boldsymbol{\nabla}\times\widehat{\mathbf{A}})\cdot\mathbf{C} - \rho_0\mathbf{M}\cdot\mathbf{F}\right)\right]\right]\cdot\delta\widehat{\mathbf{A}}\right\}\mathrm{d}V + \int_{\partial V}\left\{\left[\mathbf{N}\times\left[\!\left[\frac{1}{J}\left(\frac{1}{\mu_0}(\boldsymbol{\nabla}\times\widehat{\mathbf{A}})\cdot\mathbf{C} - \rho_0\mathbf{M}\cdot\mathbf{F}\right)\right]\!\right]\right]\cdot\delta\widehat{\mathbf{A}}\right\}\mathrm{d}A = 0. \tag{2.16}$$

Recalling relation (2.13) between the reference and current magnetic flux perturbations $\widehat{\mathbf{B}}$ and $\widehat{\mathbf{b}}$, definition (2.12) of the right Cauchy–Green tensor $\mathbf{C}$ and definitions $(2.11)_1$ of the reference density $\rho_0$ and $(2.11)_3$ of the specific magnetization $\mathbf{M}$, the vector field appearing in the volume and surface integrals in Eq. (2.16) is in view of Eq. (2.6)

$$\frac{1}{\mu_0}\left(\frac{1}{J}(\boldsymbol{\nabla}\times\widehat{\mathbf{A}})\cdot\mathbf{C}\right) - \rho\mathbf{M}\cdot\mathbf{F} = \left(\frac{1}{\mu_0}\widehat{\mathbf{b}} - \mathbf{m}\right)\cdot\mathbf{F} = \widehat{\mathbf{h}}\cdot\mathbf{F} \equiv \widehat{\mathbf{H}}. \tag{2.17}$$

Consequently, and in view of the arbitrariness of the vector field $\widehat{\mathbf{A}}$, one can restate (2.16) in view of Eq. (2.17) as the Euler–Lagrange differential equation

$$\boldsymbol{\nabla}\times\widehat{\mathbf{H}} = 0 \quad \text{in } \mathbf{R}^3, \tag{2.18}$$

and the accompanying boundary condition

$$\mathbf{N}\times[\![\widehat{\mathbf{H}}]\!] = 0 \quad \text{on } \partial V, \tag{2.19}$$

where $\widehat{\mathbf{H}}$ is the perturbed magnetic field in the reference configuration which is related to its current configuration counterpart $\widehat{\mathbf{h}}$ by the last expression of Eq. (2.17). It can easily be shown that (2.18) and (2.19) are the reference configuration counterparts of Ampère' s law ($\boldsymbol{\nabla}\times\mathbf{h} = \mathbf{0}$) and ($\mathbf{n}\times[\![\mathbf{h}]\!] = \mathbf{0}$), respectively.

The final step in the variational approach is the derivation of the equilibrium equations and traction boundary conditions for the magnetoelastic solid by extremizing the potential energy with respect to $\mathbf{u}(\mathbf{X})$. From Eq. (2.10), one obtains the following expression for the first variation of the potential energy with respect to the displacement:

$$\begin{aligned}\mathscr{P}_{,\mathbf{u}}\delta\mathbf{u} = &\int_V\left\{\rho_0[-\mu_0\mathbf{M}\cdot(\mathbf{h_0}\boldsymbol{\nabla})\cdot\mathbf{F}^{-1} - \mathbf{f}]\cdot\delta\mathbf{u} + \left[\rho_0\left(\frac{\partial\psi}{\partial\mathbf{F}}\right)^{\mathrm{T}} - \frac{\mu_0 J}{2}\left(\frac{1}{\mu_0 J}\mathbf{F}\cdot\widehat{\mathbf{B}} - \rho\mathbf{M}\right)\right.\right.\\ &\left.\left.\cdot\left(\frac{1}{\mu_0 J}\mathbf{F}\cdot\widehat{\mathbf{B}} - \rho\mathbf{M}\right)\mathbf{F}^{-1} + \widehat{\mathbf{B}}\left(\frac{1}{\mu_0 J}\mathbf{F}\cdot\widehat{\mathbf{B}} - \rho\mathbf{M}\right)\right]\cdot\cdot(\delta\mathbf{u}\boldsymbol{\nabla})\right\}\mathrm{d}V\\ &+\int_{\mathbf{R}^3\backslash V}\left\{\left[\frac{1}{\mu_0 J}\widehat{\mathbf{B}}(\mathbf{F}\cdot\widehat{\mathbf{B}}) - \frac{\mu_0 J}{2}\left(\frac{1}{\mu_0 J}\mathbf{F}\cdot\widehat{\mathbf{B}}\right)\cdot\left(\frac{1}{\mu_0 J}\mathbf{F}\cdot\widehat{\mathbf{B}}\right)\mathbf{F}^{-1}\right]\cdot\cdot(\delta\mathbf{u}\boldsymbol{\nabla})\right\}\mathrm{d}V\\ &+\int_{\partial V}[\mathbf{T}\cdot\delta\mathbf{u}]\,\mathrm{d}A = 0.\end{aligned} \tag{2.20}$$

It has been shown in Eq. (2.15) that the vector appearing repeatedly in Eq. (2.20), namely $(\mu_0 J)^{-1}\mathbf{F}\cdot\widehat{\mathbf{B}} - \rho\mathbf{M} = \widehat{\mathbf{b}}/\mu_0 - \mathbf{m} = \widehat{\mathbf{h}}$ for points $\mathbf{X}\in V$. Similarly $(\mu_0 J)^{-1}\mathbf{F}\cdot\widehat{\mathbf{B}} = \widehat{\mathbf{h}}$ for points $\mathbf{X}\in\mathbf{R}^3\backslash V$ (since $\mathbf{M} = 0$ outside the magnetoelastic solid). Integration of Eq. (2.20) by parts for the terms involving $\delta\mathbf{u}\boldsymbol{\nabla}$ and subsequent application of Gauss' divergence theorem (assuming adequate continuity of the field quantities involved and recalling that $\widehat{\mathbf{b}}\to 0$ as $\|\mathbf{X}\|\to\infty$) yields, in view of the arbitrariness of $\delta\mathbf{u}$, the following Euler–Lagrange differential equations:

$$\begin{aligned}&\mathbf{X}\in V: \left[J\left(\rho\frac{\partial\psi}{\partial\mathbf{F}} - \frac{\mu_0}{2}(\widehat{\mathbf{h}}\cdot\widehat{\mathbf{h}})\mathbf{F}^{-\mathrm{T}} + \widehat{\mathbf{h}}\widehat{\mathbf{b}}\cdot\mathbf{F}^{-\mathrm{T}}\right)\right]\cdot\boldsymbol{\nabla} + J[\rho\mathbf{f} + \mu_0\mathbf{m}\cdot(\mathbf{h}_0\boldsymbol{\nabla})\cdot\mathbf{F}^{-1}] = 0,\\ &\mathbf{X}\in\mathbf{R}^3\backslash V: [\mu_0 J(\widehat{\mathbf{h}}\widehat{\mathbf{h}} - \tfrac{1}{2}(\widehat{\mathbf{h}}\cdot\widehat{\mathbf{h}})\mathbf{I})\cdot\mathbf{F}^{-\mathrm{T}}]\cdot\boldsymbol{\nabla} = 0,\end{aligned} \tag{2.21}$$

[3]Here and subsequently $[\![f]\!]$ denotes jump of $f$ across a surface of discontinuity.

plus the boundary condition on $\partial V$

$$\mathbf{X} \in \partial V : \left[\!\left[ J\left(\rho \frac{\partial \psi}{\partial \mathbf{F}} - \frac{\mu_0}{2}(\widehat{\mathbf{h}}\cdot\widehat{\mathbf{h}})\mathbf{F}^{-\mathrm{T}}) + \widehat{\mathbf{h}}\widehat{\mathbf{b}}\cdot\mathbf{F}^{-\mathrm{T}}\right) \right]\!\right] \cdot\mathbf{N} = \mathbf{T}. \tag{2.22}$$

Given the following identity from continuum mechanics (e.g. see Chadwick, 1976, p. 59, Eq. (19)) valid for any arbitrary rank two tensor $\mathbf{\Pi}$

$$\nabla\cdot\mathbf{\Pi} = J(\mathbf{\nabla}\cdot\boldsymbol{\sigma}), \quad \boldsymbol{\sigma} \equiv \frac{1}{J}\,\mathbf{F}\cdot\mathbf{\Pi}, \tag{2.23}$$

one can identify $\mathbf{\Pi}$ with the perturbation first Piola–Kirchhoff stress $\widehat{\mathbf{\Pi}}$, i.e. the additional (reference configuration) stress due to the presence of the magnetoelastic solid

$$\widehat{\mathbf{\Pi}}^{\mathrm{T}} \equiv J\left[\rho \frac{\partial \psi}{\partial \mathbf{F}}\cdot\mathbf{F}^{\mathrm{T}} - \mu_0(\widehat{\mathbf{h}}\cdot\widehat{\mathbf{h}})\mathbf{I} + \widehat{\mathbf{h}}\widehat{\mathbf{b}}\right]\cdot\mathbf{F}^{-\mathrm{T}}. \tag{2.24}$$

From Eq. $(2.23)_2$ the corresponding $\boldsymbol{\sigma}$ is identified with the Cauchy stress perturbation $\widehat{\boldsymbol{\sigma}}$, namely

$$\widehat{\boldsymbol{\sigma}} \equiv \left[\rho \frac{\partial \psi}{\partial \mathbf{F}}\cdot\mathbf{F}^{\mathrm{T}} - \mu_0(\widehat{\mathbf{h}}\cdot\widehat{\mathbf{h}})\mathbf{I} + \widehat{\mathbf{h}}\widehat{\mathbf{b}}\right]^{\mathrm{T}}. \tag{2.25}$$

When the total $\mathbf{h}$ and $\mathbf{b}$ fields are substituted for their perturbed counterparts $\widehat{\mathbf{h}}$ and $\widehat{\mathbf{b}}$, Eq. (2.25) yields the total Cauchy stress expression

$$\boldsymbol{\sigma}^{\mathrm{T}} = \rho \frac{\partial \psi}{\partial \mathbf{F}}\cdot\mathbf{F}^{\mathrm{T}} + \mathbf{h}\mathbf{b} - \frac{\mu_0}{2}(\mathbf{h}\cdot\mathbf{h})\,\mathbf{I}. \tag{2.26}$$

Converting the interface condition from the reference to the current configuration requires again the definitions in Eqs. (2.24) and (2.25) plus Nanson's relation ($\mathbf{n}\,\mathrm{d}a = J\mathbf{F}^{-\mathrm{T}}\cdot\mathbf{N}\,\mathrm{d}A$) to give

$$\mathbf{n}\cdot[\![\,\boldsymbol{\sigma}\,]\!] = \mathbf{t}. \tag{2.27}$$

The assertion that for stable equilibrium solutions, the extremization of the potential energy corresponds to a local minimum is better seen from the first of the two equivalent expressions for $\mathscr{P}$ in Eq. (2.9).[4] Notice that the magnetic field's energy over the entire space $\mathbf{R}^3$ is always positive and it depends on $\mathbf{u}, \widehat{\mathbf{A}}, \mathbf{M}$ which are independent variables. Ignoring the linear terms of the potential energy and assuming a positive Helmholtz free energy $\psi$ with reasonable growth conditions and noticing that in the absence of external forces and magnetic fields $\mathscr{P} \geqslant 0$, one can see how for stable solutions the extremization of $\mathscr{P}$ corresponds to a local minimum.

### 2.2. Problem description and principal solution

Consider a two-dimensional magnetoelastic rectangular block subject to a transverse (i.e. $X_2$-direction) magnetic field $h_0$ as depicted in Fig. 1. The reference configuration of the magnetoelastic solid is its stress-free configuration with an aspect ratio of $r \equiv 2L_1/2L_2$. The block deforms under finite plane strain conditions due to the action of the imposed magnetic field.

The rectangular block is made of a magnetoelastic, isotropic, compressible material with a two-dimensional free energy, $\psi(I, J, J_1, J_2)$ that can be readily obtained from its three-dimensional counterpart. Due to isotropy, it can be shown that (Kankanala and Triantafyllidis, 2004) the free energy depends on the two invariants $I$ and $J$ of the rank two left Cauchy–Green deformation tensor $\mathbf{B}$, the invariant $J_1$ of the magnetization vector $\mathbf{M}$, and the invariant $J_2$ (which depends on $\mathbf{B}$ and $\mathbf{M}$), namely[5]:

$$I = B_{ii}, \quad J = \det F_{ij}, \quad J_1 \equiv M_i M_i, \quad J_2 \equiv M_i B_{ij} M_j. \tag{2.28}$$

[4] This assertion can be shown to be valid for small strains but does not, in general, hold for arbitrary strains.

[5] $\delta_{ij}$ is the Kronecker delta.

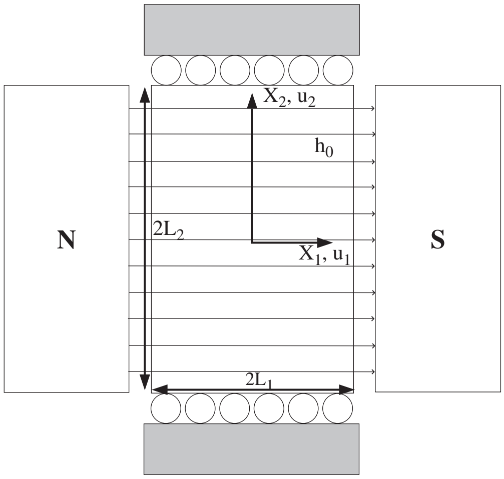


Fig. 1. Schematic representation of a rectangular block subject to a transverse magnetic field $h_0$. The $2L_1 \times 2L_2$ block, made of a slightly compressible magnetoelastic material, is traction free at the ends by virtue of the rollers.

Adopted throughout this paper is the Einstein's summation convention with repeated Latin indices, which range from 1 to 2, as well as the use of a comma to denote partial differentiation with respect to the corresponding Cartesian coordinate, i.e. $f_{,i} \equiv \partial f/\partial x_i$.[6]

To avoid boundary layer effects near the ends, $X_2 = \pm L_2$, the two ends remain flat and free of shear tractions. Consequently, the admissible displacement field and the (magnetic) vector potential must also satisfy the essential boundary conditions[7,8]:

$$u_{2,1}(x_1, -\lambda_2 L_2) = u_{2,1}(x_1, \lambda_2 L_2), \quad u_1(0,0) = u_2(0,0) = 0,$$

$$\alpha_{,2}(\pm\lambda_1 L_1, x_2) = \alpha^{\text{out}}_{,2}(\pm\lambda_1 L_1, x_2), \quad \alpha(x_1, \pm\lambda_2 L_2) = \alpha^{\text{out}}(x_1, \pm\lambda_2 L_2), \tag{2.29}$$

where the second set of constraints eliminate rigid body translations along $x_i$ and where continuity of the magnetic flux perturbation scalar potential and its derivative $(2.29)_{3,4}$ is a consequence of the reference form of the jump condition Eq. $(2.7)_2$.

For the sake of algebraic simplicity, the current configuration of the magnetoelastic block is used in obtaining the principal solution. Due to the (transverse) nature of the applied field, the vertical components of the magnetic quantities are nonexistent, i.e. $b_2 = h_2 = M_2 = 0$ leaving $\lambda_1, \lambda_2, M_1$ as the three unknown quantities. As detailed below, the unknown magnetization and stretch ratios may be obtained from a simultaneous solution of the $m-h$ constitutive equation (2.15) and traction condition (2.27) applied at the sides and at the ends of the block. Substituting (2.28) in the constitutive equation (2.15) for the magnetic field

$$\mu_0 \mathbf{h} = 2\left[\frac{\partial\psi}{\partial J_1}\mathbf{I} + \frac{\partial\psi}{\partial J_2}\mathbf{B}\right]\cdot\mathbf{M}. \tag{2.30}$$

Using the expression for the stress $\boldsymbol{\sigma}$ in Eq. (2.26), noting the objectivity and isotropy of $\psi$ and the symmetry of $\boldsymbol{\sigma}$, using the constitutive relation for $\mathbf{h}$ Eq. (2.15), one can show that the surface traction $\mathbf{t}$ in Eq. (2.27) is given by the following expression:

$$\mathbf{t} = \left\{\rho\left[\frac{\partial\psi}{\partial\mathbf{B}}\cdot\mathbf{B} + \mathbf{B}\cdot\frac{\partial\psi}{\partial\mathbf{B}} + \mu_0\mathbf{M}\mathbf{h}\right]\right\}\cdot\mathbf{n} - \frac{\mu_0}{2}(\mathbf{m}\cdot\mathbf{n})^2\,\mathbf{n}. \tag{2.31}$$

[6]For sake of convenience, we use small case indices for both current and reference configuration coordinates.

[7]Since $\widehat{\mathbf{B}} = \boldsymbol{\nabla} \times \widehat{\mathbf{A}}$ and we ignore $X_3$ dependence of the quantities and $\widehat{\mathbf{B}}_3 = 0$, the only nonzero component of $\widehat{\mathbf{A}}$ is $\widehat{A}_3$. For the sake of simplicity in notation, $\alpha \equiv \widehat{A}_3$.

[8]Unless otherwise indicated, $\alpha$ denotes the component of the vector potential inside the solid.

With the deformation gradient, $\mathbf{F}$, expressed in terms of the principal stretch ratios, the nontrivial component of the magnetic constitutive equation (2.30) becomes

$$\mu_0(h_0 - \rho M_1) = 2\left(\frac{\partial\psi}{\partial J_1} + \frac{\partial\psi}{\partial J_2}\,\lambda_1^2\right)M_1. \tag{2.32}$$

The traction boundary condition (2.31) requires that at $x_1 = \pm\lambda_1 L_1$

$$t_1 = 0 = 2\rho\left[\left(\frac{\partial\psi}{\partial I}\lambda_1^2 + \frac{J}{2}\frac{\partial\psi}{\partial J}\right) + \frac{\partial\psi}{\partial J_2}\lambda_1^2 M_1^2 + \frac{1}{2}\,\mu_0 h_0 M_1\right] - \frac{3}{2}\mu_0(\rho M_1)^2, \tag{2.33}$$

and at $x_2 = \pm\lambda_2 L_2$

$$t_2 = 0 = 2\rho\left[\frac{\partial\psi}{\partial I}\lambda_2^2 + \frac{J}{2}\frac{\partial\psi}{\partial J}\right]. \tag{2.34}$$

Noting that the invariants of $\psi$ are expressed as

$$I = \lambda_1^2 + \lambda_2^2, \quad J = \lambda_1\lambda_2, \quad J_1 = M_1^2, \quad J_2 = (\lambda_1 M_1)^2. \tag{2.35}$$

The principal solution (i.e. $\lambda_1, \lambda_2, M_1$), that corresponds to the uniform strain and magnetization equilibrium of the rectangular block, is obtained by numerically solving Eqs. (2.32)–(2.34) using the straightforward Newton–Raphson approach, as discussed in Section 3.

### 2.3. Bifurcation analysis

For every value of the applied magnetic field $h_0$ in the $x_2$-direction, the corresponding values of the equilibrium displacement, magnetization, and magnetic flux potential can be found, according to the general theory in Section 2.1, by extremizing the potential energy $\mathscr{P}(\mathbf{g})$, with respect to the three independent fields $\mathbf{g} \equiv \mathbf{u}, \mathbf{M}, \widehat{\mathbf{A}}$, i.e.:

$$\mathscr{P}_{,\mathbf{g}}(\mathbf{g})\delta\mathbf{g} = 0. \tag{2.36}$$

One obvious solution to Eq. (2.36), denoted by $\mathbf{g}_0$, which corresponds to zero displacement and magnetization for zero applied magnetic field is the "*Principal Solution*" derived in Section 2.2. At small values of the magnetic field, the principal solution is stable, i.e. it is a local minimizer of the potential energy satisfying $(\mathscr{P}_{,\mathbf{gg}}(\mathbf{g}_0)\delta\mathbf{g})\delta\mathbf{g} > 0$, for arbitrary perturbation, $\delta\mathbf{g}$. As $h_0$ increases, it reaches a value $h_c$ where the principal solution $g_0(h_c)$ is no longer a minimizer of the potential energy, but where the energy vanishes along a particular direction $\Delta g$, called the "*critical mode*" which satisfies the condition[9]:

$$(\mathscr{P}_{,\mathbf{gg}}(\mathbf{g}_0(h_c), h_c)\Delta\mathbf{g})\delta\mathbf{g} = 0, \tag{2.37}$$

The objective of this work is to determine the critical buckling load $h_c$ for the rectangular block as a function of the aspect ratio $r$ for different magnetoelastic materials. Taking the second Frechet derivatives of Eqs. (2.14), (2.16), and (2.20) with respect to the independent variables $(\mathbf{u}, \mathbf{M}, \widehat{\mathbf{A}})$, and recalling that $\widehat{A}_3 = \alpha$, the block's eigenvalue problem in Eq. (2.37) may be rewritten as (the interested reader is referred to the Appendix A for details of the derivation):

$$\int_A [\mathscr{L}^{uu}_{ijkl}\Delta u_{k,l} + \mathscr{L}^{uM}_{ijk}\Delta M_k + \mathscr{L}^{u\alpha}_{ijk}\Delta\alpha_{,k}]\delta u_{i,j}\,\mathrm{d}A = 0,$$

$$\int_A [\mathscr{L}^{Mu}_{kij}\Delta u_{i,j} + \mathscr{L}^{MM}_{kl}\Delta M_l + \mathscr{L}^{M\alpha}_{kl}\Delta\alpha_{,l}]\delta M_k\,\mathrm{d}A = 0,$$

$$\int_{\mathbf{R}^2} [\mathscr{L}^{\alpha u}_{kij}\Delta u_{i,j} + \mathscr{L}^{\alpha M}_{kl}\Delta M_l + \mathscr{L}^{\alpha\alpha}_{kl}\Delta\alpha_{,l}]\delta\alpha_{,k}\,\mathrm{d}A = 0, \tag{2.38}$$

[9]Henceforth, for the sake of convenience, the subscript '0' will be used only when required and it is noted that all quantities, $J, \lambda_i, h_i, M_i, F_{ij}$, are only defined in the principal solution.

where the coefficients appearing in Eq. (2.38) are given in Appendix A. Using Eq. (2.38) (noting that $\delta u$, $\delta M$, and $\delta\alpha$ are arbitrary), and upon application of standard integration by parts and the elimination of $\Delta M_k$ from Eq. $(2.38)_2$, the pointwise form of the governing equations for the eigenmode are obtained:

$$\mathbf{X} \in A, \quad \begin{cases} [L^{uu}_{ijkl}\Delta u_{k,l} + L^{u\alpha}_{ijk}\Delta\alpha_{,k}]_{,j} = 0, \\ [L^{\alpha u}_{kij}\Delta u_{i,j} + L^{\alpha\alpha}_{kl}\Delta\alpha_{,l}]_{,k} = 0, \end{cases}$$

$$\mathbf{X} \in \mathbf{R}^2\backslash A, \quad [\mathscr{L}^{\alpha\alpha}_{ij}\Delta\alpha_{,j}]_{,i} = 0, \tag{2.39}$$

with the corresponding natural boundary conditions:

$$X_1 = \pm L_1, \quad \left[L^{uu}_{i1kl}\Delta u_{k,l} + L^{u\alpha}_{i1k}\Delta\alpha_{,k}\right] = 0 \ \ (i = 1, 2), \quad [L^{\alpha u}_{1ij}\Delta u_{i,j} + L^{\alpha\alpha}_{1l}\Delta\alpha_{,l}] = \mathscr{L}^{\alpha\alpha}_{1i}\Delta\alpha_{,i}$$

$$X_2 = \pm L_2, \quad [L^{uu}_{i2kl}\Delta u_{k,l} + L^{u\alpha}_{i2k}\Delta\alpha_{,k}] = 0 \ \ (i = 1), \tag{2.40}$$

since $\delta u_i, \delta\alpha$ are arbitrary on $X_1 = \pm L_1$ but $\delta u_2 = \delta\alpha = 0$ on $X_2 = \pm L_2$. The definitions of the coefficients $L^{\cdot\cdot}_{\cdot\cdot\cdot\cdot}$ defined for $\mathbf{X} \in A$ are also given in Appendix A.

The symmetries in the problem allow for the following Fourier decomposition of the eigenmodes solution of Eq. (2.39), (2.40) (see also Triantafyllidis et al., 2007 for the analogous case of a hyperelastic block) for $\Delta u_i(\mathbf{X}), \Delta\alpha(\mathbf{X})$:

$$\mathscr{S}^1 : \begin{Bmatrix} \Delta u_1 = v_1(X_1)\cos(p_2X_2) - v_1(0) \\ \Delta u_2 = -v_2(X_1)\sin(p_2X_2) \\ \Delta\alpha = \alpha(X_1)\sin(p_2X_2) \\ p_2 = n\pi/L_2 \end{Bmatrix}, \quad \mathscr{A}^1 : \begin{Bmatrix} \Delta u_1 = v_1(X_1)\sin(p_2X_2) \\ \Delta u_2 = v_2(X_1)\cos(p_2X_2) - v_2(0) \\ \Delta\alpha = -\alpha(X_1)\cos(p_2X_2) \\ p_2 = \left(n - \dfrac{1}{2}\right)\pi/L_2 \end{Bmatrix} \tag{2.41}$$

where the symbols $\mathscr{S}^1$ and $\mathscr{A}^1$ denote the symmetric and antisymmetric modes with respect to $X_1$.

Upon substitution of the eigenmode expression (2.41) into the governing equations (2.39) one obtains the following expressions for $v_i(X_1), \alpha(X_1)$:

$$\mathscr{S}^2 : \begin{Bmatrix} v_1 = V_1\sinh(\xi p_2X_1) \\ v_2 = V_2\cosh(\xi p_2X_1) \\ \alpha = A\cosh(\xi p_2X_1) \\ \alpha^{\text{out}} = A_{\text{s}}\exp(\mp p_2\lambda_1/\lambda_2X_1) \\ A_{\text{s}} \equiv A\dfrac{\cosh(\xi p_2L_1)}{\exp(-p_2\lambda_1/\lambda_2L_1)} \end{Bmatrix}, \quad \mathscr{A}^2 : \begin{Bmatrix} v_1 = V_1\cosh(\xi p_2X_1) \\ v_2 = V_2\sinh(\xi p_2X_1) \\ \alpha = A\sinh(\xi p_2X_1) \\ \alpha^{\text{out}} = A_a\exp(\mp p_2\lambda_1/\lambda_2X_1) \\ A_a \equiv \pm A\dfrac{\sinh(\xi p_2L_1)}{\exp(-p_2\lambda_1/\lambda_2L_1)} \end{Bmatrix}, \tag{2.42}$$

where the symbols $\mathscr{S}^2$ and $\mathscr{A}^2$ denote the symmetric and antisymmetric modes with respect to the reference coordinate $X_2$. Expressions given by $(2.42)_4$ are obtained from the solution of Eq. $(2.39)_3$ subject to the far-field condition $\alpha^{\text{out}}(X_1) \to 0$ as $X_1 \to \infty$ and the symmetry condition $\alpha^{\text{out}}(X_1) = \alpha^{\text{out}}(-X_1)$.

The constants $\xi$ and $V_1, V_2, A$ entering (2.42) are related by

$$\mathbf{Q}(\xi)\cdot\mathbf{V} = 0, \tag{2.43}$$

where the $3 \times 3$ matrix of coefficients $\mathbf{Q}$ and the 3 vector $\mathbf{V}$ are defined by

$$\mathbf{Q} \equiv \begin{bmatrix} \xi^2L^{uu}_{1111} - L^{uu}_{1212} & -\xi(L^{uu}_{1221} + L^{uu}_{1122}) & \xi(L^{u\alpha}_{121} + L^{u\alpha}_{112}) \\ -\xi(L^{uu}_{2211} + L^{uu}_{2112}) & -\xi^2L^{uu}_{2121} + L^{uu}_{2222} & \xi^2L^{u\alpha}_{211} - L^{u\alpha}_{222} \\ \xi(L^{\alpha u}_{112} + L^{\alpha u}_{211}) & \xi^2L^{\alpha u}_{121} - L^{\alpha u}_{222} & -\xi^2L^{\alpha\alpha}_{11} + L^{\alpha\alpha}_{22} \end{bmatrix}, \quad \mathbf{V} \equiv \begin{bmatrix} V_1 \\ V_2 \\ A \end{bmatrix}, \tag{2.44}$$

where $\xi$ is the solution of the bi-cubic:

$$\det \mathbf{Q}(\xi) = 0, \tag{2.45}$$

with $\Re(\xi) \neq 0$ to ensure solution of the system of differential equations (2.39) lies in the elliptic domain.[10]

Having established the general expressions for the eigenmode, the critical field $h_c$ can be found by enforcing the boundary conditions Eq. (2.40), at $X_1 = \pm L_1$ (automatically satisfied for $X_2 = \pm L_2$) which in view of Eq. (2.41) become:

$$\begin{aligned}
&L^{uu}_{1111} v_{1,1} - L^{uu}_{1122} v_2 p_2 + L^{u\alpha}_{112} \alpha p_2 = 0 \\
&L^{uu}_{2112} v_1 p_2 + L^{uu}_{2121} v_{2,1} - L^{u\alpha}_{211} \alpha_{,1} = 0 \\
&L^{\alpha u}_{112} v_1 p_2 + L^{\alpha u}_{121} v_{2,1} - L^{\alpha\alpha}_{11} \alpha_{,1} = -\mathscr{L}^{\alpha\alpha}_{11} \alpha^{\text{out}}_{,1}
\end{aligned} \tag{2.46}$$

The solution to the boundary conditions Eq. (2.46) depending on the symmetry/antisymmetry of the mode with respect to $X_2$ requires the following linear combinations of the eigenmodes in Eq. (2.42) where $H_\beta$ are the yet to be specified amplitudes of each mode component:

$$\mathscr{S}^2 : \left\{ \begin{aligned}
v_1 &= \sum_{\beta=1}^{3} H_\beta V_1^\beta \frac{\sinh(\xi_\beta p_2 X_1)}{\cosh(\xi_\beta p_2 L_1)} \\
v_2 &= \sum_{\beta=1}^{3} H_\beta V_2^\beta \frac{\cosh(\xi_\beta p_2 X_1)}{\cosh(\xi_\beta p_2 L_1)} \\
\alpha &= \sum_{\beta=1}^{3} H_\beta A^\beta \frac{\cosh(\xi_\beta p_2 X_1)}{\cosh(\xi_\beta p_2 L_1)} \\
\alpha^{\text{out}} &= \sum_{\beta=1}^{3} H_\beta A_s^\beta \frac{\exp(\mp p_2 \lambda_1/\lambda_2 X_1)}{\cosh(\xi_\beta p_2 L_1)} \\
A_s^\beta &\equiv A^\beta \frac{\cosh(\xi_\beta p_2 L_1)}{\exp(-p_2 \lambda_1/\lambda_2 L_1)}
\end{aligned} \right\}, \quad
\mathscr{A}^2 : \left\{ \begin{aligned}
v_1 &= \sum_{\beta=1}^{3} H_\beta V_1^\beta \frac{\cosh(\xi_\beta p_2 X_1)}{\cosh(\xi_\beta p_2 L_1)} \\
v_2 &= \sum_{\beta=1}^{3} H_\beta V_2^\beta \frac{\sinh(\xi_\beta p_2 X_1)}{\cosh(\xi_\beta p_2 L_1)} \\
\alpha &= \sum_{\beta=1}^{3} H_\beta A_a^\beta \frac{\sinh(\xi_\beta p_2 X_1)}{\cosh(\xi_\beta p_2 L_1)} \\
\alpha^{\text{out}} &= \sum_{\beta=1}^{3} H_\beta A_a^\beta \frac{\exp(\mp p_2 \lambda_1/\lambda_2 X_1)}{\cosh(\xi_\beta p_2 L_1)} \\
A_a^\beta &\equiv \pm A^\beta \frac{\sinh(\xi_\beta p_2 L_1)}{\exp(-p_2 \lambda_1/\lambda_2 L_1)}
\end{aligned} \right\}. \tag{2.47}$$

The critical buckling field $h_c$ is found from the requirement of a nontrivial solution of the $3 \times 3$ system resulting from substituting Eq. (2.47) into Eq. (2.46), namely:

$$\sum_{\beta=1}^{3} D_{\alpha\beta}(h_c) H_\beta = 0, \quad \alpha = 1, 2, 3, \tag{2.48}$$

where for the symmetric $\mathscr{S}^2$ mode the coefficients $D_{\alpha\beta}$ are

$$\begin{aligned}
D_{1\beta} &\equiv L^{uu}_{1111} \xi_\beta V_1^\beta - L^{uu}_{1122} V_2^\beta + \mu_0 \rho M_1 L^{u\alpha}_{112}, \\
D_{2\beta} &\equiv [L^{uu}_{2112} V_1^\beta + L^{uu}_{2121} \xi_\beta V_2^\beta - \mu_0 \rho M_1 L^{u\alpha}_{211} \xi_\beta] \tanh(\xi_\beta p_2 L_1), \\
D_{3\beta} &\equiv (L^{\alpha u}_{112} V_1^\beta + L^{\alpha u}_{121} \xi_\beta V_2^\beta - \mu_0 \rho M_1 L^{\alpha\alpha}_{11} \xi_\beta) \tanh(\xi_\beta p_2 L_1) + \rho M_1,
\end{aligned} \tag{2.49}$$

and for the antisymmetric $\mathscr{A}^2$ mode the coefficients $D_{\alpha\beta}$ are

$$\begin{aligned}
D_{1\beta} &\equiv [L^{uu}_{1111} \xi_\beta V_1^\beta - L^{uu}_{1122} V_2^\beta + \mu_0 \rho M_1 L^{u\alpha}_{112}] \tanh(\xi_\beta p_2 L_1), \\
D_{2\beta} &\equiv L^{uu}_{2112} V_1^\beta + L^{uu}_{2121} \xi_\beta V_2^\beta - \mu_0 \rho M_1 L^{u\alpha}_{211} \xi_\beta, \\
D_{3\beta} &\equiv L^{\alpha u}_{112} V_1^\beta + L^{\alpha u}_{121} \xi_\beta V_2^\beta - \mu_0 \rho M_1 L^{\alpha\alpha}_{11} \xi_\beta + \rho M_1 \tanh(\xi_\beta p_2 L_1).
\end{aligned} \tag{2.50}$$

[10]Note that although the material does not have to be rank 1 convex (see Kankanala and Triantafyllidis, 2004), its bifurcated solution is always expected to be in the elliptic regime of the governing equations (2.39). This condition is numerically verified in all the calculations reported here.

Note that, for scaling purposes, the constant $A$ in Eq. (2.48) is taken as $A = \mu_0 \rho M_1$. Numerical solution of Eq. (2.48) gives the critical field, $h_c$ as the lowest root of $\det[D_{\alpha\beta}(h)] = 0$. Details of the corresponding calculations are given in the next section.

## 3. Results and discussion

The general theory presented in the Section 2 is now applied to three different materials. Following the introduction of their constitutive laws in the first subsection, the second subsection deals with the principal solution while the third subsection pertains to the critical buckling field, as a function of the block's slenderness ratio, for each of these materials. Finally, the last subsection pertains to an asymptotic solution of the magnetoelastic buckling problem for a special material and the result is compared and contrasted to the magnetoelastic beam buckling papers of Moon and Pao (1968) and Pao and Yeh (1973) which employ classical structural approximations.

### 3.1. Material selection

Three different isotropic material models are used to represent a slightly compressible, magnetoelastic elastomer. The material models are constructed such that the three solids' response coincide asymptotically for small strains and arbitrary magnetization. For plane strain deformations, their Helmholtz free energy per unit reference volume takes the form

$$\rho_0 \psi = \frac{G}{2} \left\{ \left( C_{10} + C_{11} \frac{J_1}{M_s^2} \right) \int_0^{\gamma} 2s(\gamma')\,\mathrm{d}\gamma' + \left( C_{20} + C_{21} \frac{J_1}{M_s^2} \right) \left( J - \frac{1}{J} \right)^2 + C_{01} \frac{J_1}{M_s^2} + C_{02} \frac{J_2}{M_s^2} + C_{01}^{*} \left[ \frac{1}{2} \ln \left( 1 - \frac{J_1^2}{M_s^4} \right) + \frac{J_1}{M_s^2} \tanh^{-1} \left( \frac{J_1}{M_s^2} \right) \right] \right\}, \tag{3.1}$$

where the shear strain function, used to differentiate the three cases of material behavior, is given by

$$\gamma^2 = I - 2J, \quad s(\gamma) = \begin{cases} \gamma_m \tanh^{-1}\left(\dfrac{\gamma}{\gamma_m}\right) & \text{“\textit{Strain-Saturated}”}, \\ \gamma & \text{“\textit{Neo-Hookean}”}, \\ \tau_m \tanh\left(\dfrac{\gamma}{\tau_m}\right) & \text{“\textit{Stress-Saturated}”}. \end{cases} \tag{3.2}$$

In the absence of magnetostriction, the first model, which uses the shear strain term $(3.2)_1$, and henceforth referred to as the "*strain-saturated*" model, simulates the behavior of a natural rubber in which $\gamma_m$ determines the locking strain in a simple-shear test. The model using the shear strain function given by Eq. $(3.2)_2$, is a compressible neo-Hookean solid. Finally, the third model is for a compressible foam type rubber whose shear strain term is given by $(3.2)_3$, in which $\tau_m$ determines the saturation stress of the material in a simple shear test. It can be shown that as $\gamma_m, \tau_m \to \infty$ the "*strain-saturated*" and "*stress-saturated*" models, respectively, approach a "*neo-Hookean*" solid. The values of the coefficients in Eq. (3.1) and used in these calculations are obtained from experiments with a class of Magnetorheological Elastomers (Kankanala et al., 2007) and are given in Tables 1 and 2.

Table 1
Chosen material constants for the different constitutive models

| $G$ (MPa) | $\mu_0 \rho_0 M_s$ (T) | $v$ | $\gamma_m$ | $\tau_m$ |
|---|---|---|---|---|
| 1.0 | 0.45 | 0.4286 | 0.05 | 0.075 |

Table 2
Coefficients chosen for the free energy based on magnetization, uniaxial and simple shear response observed for a typical magnetorheological elastomer (Kankanala et al., 2007)

| $C_{10}$ | $C_{20}$ | $C_{11}$ | $C_{21}$ | $C_{01}$ | $C_{02}$ | $C^{*}_{01}$ |
|---|---|---|---|---|---|---|
| 1.0 | 0.625 | 0.0791 | 0 | $\beta/6$ | $\beta/2$ | 0.05 |

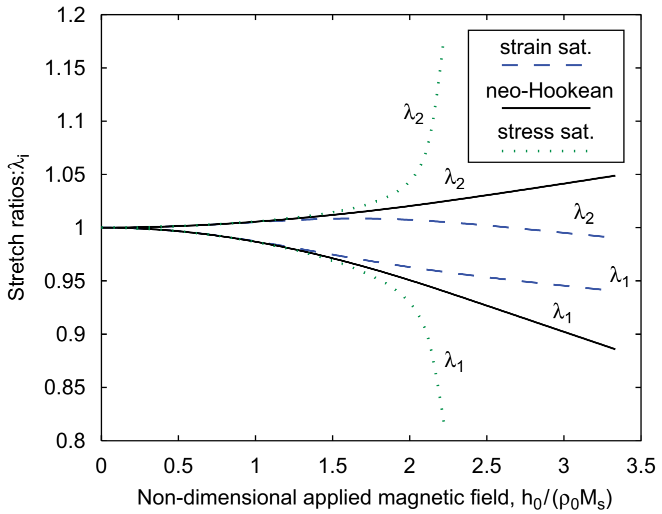


Fig. 2. Comparison of the stretch ratios versus dimensionless external magnetic field $h_0/(\rho_0 M_s)$ for a two-dimensional magnetoelastic block subject to a transverse magnetic field. Notice the strain behavior of the three different constitutive laws, (i) strain saturated (dashed line), (ii) neo-Hookean (solid line), and (iii) stress-saturated (dotted line), coincides as $h_0/(\rho_0 M_s) \to 0$. Note that all three constitutive laws used have the same initial shear modulus $G$ and Poisson's ratio $\nu$.

The nondimensional constant $\beta$ appearing in Table 2 is defined by

$$\beta \equiv \mu_0(\rho_0 M_s)^2/G. \tag{3.3}$$

### 3.2. Principal solution

The principal (uniform strain and magnetization) solution of the rectangular block subjected to a transverse magnetic field, for the three different constitutive laws just introduced is obtained by numerically solving the system of Eqs. (2.32)–(2.34) using an incremental Newton–Raphson method (e.g. Press, 1986) with the load parameter being the applied magnetic field $h_0$.

The effect of the nondimensional applied magnetic field $h_0/(\rho_0 M_s)$ on the stretch ratios, $\lambda_1, \lambda_2$, of the block is plotted in Fig. 2 for the three materials introduced in Section 3.1. In all the plots, the response of the strain-saturated model is denoted by a dashed line, the neo-Hookean response is shown using a solid line and the response of the stress-saturated model is depicted using a dotted line.

The magnetostrictive response of the three materials (Fig. 2) is indistinguishable at dimensionless magnetic fields below 1 while differing remarkably at fields above 1.5. All materials are seen to constrict in the direction of the applied field (i.e. $\lambda_1 < 1$). In the case of the strain-saturated model, magnetostrictive strains in direction perpendicular to applied field initially increase, subsequently decrease and finally reverse sign as the material stiffens significantly due to strain locking. Strains in direction of applied field increase monotonically, albeit at a slower rate, with increasing applied field. For the stress-saturated model, with the significantly reduced stiffness near saturation stress, at $h_0/(\rho_0 M_s) \cong 2.25$, the magnetostrictive strains increase without bound due to the material's loss of stiffness. Finally, for the neo-Hookean solid, magnetostrictive

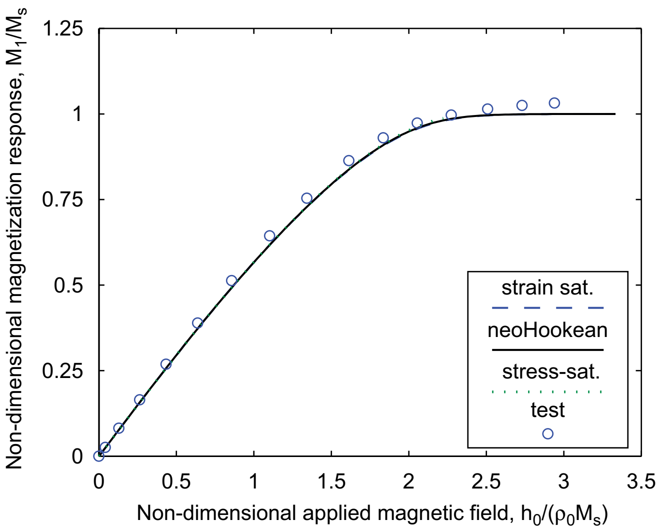


Fig. 3. Magnetization response for three different constitutive laws. Plotted is the dimensionless magnetization $M_1/M_s$ versus the dimensionless external magnetic field $h_0/(\rho_0 M_s)$. Parameters $C_{01}, C_{02}$, and $C_{01}^*$ are adjusted so a close fit with experimental data (symbol 'o') may be obtained. We point out that $C_{01}, C_{02}$ mainly control the initial slope of the curve while the parameter $C_{01}^*$ affects near saturation behavior. Note the virtually identical magnetization behavior of the constitutive laws considered.

strains increase monotonically, bounded above and below by strain- and stress-saturated models, respectively, as expected.

The nondimensional magnetization ($M_1/M_s$) response of the block is shown in Fig. 3 with the test data (symbol "o") from the magnetorheological elastomer of Kankanala et al. (2007). The magnetization response of all three materials considered (Fig. 3) is essentially identical and remains practically unaffected by the different stretch ratios of each model. It thus seems that the strains need to be significantly larger than those seen in the principal solution (Fig. 1), i.e. much larger than 18%, to yield a discernable difference in the magnetization response of the materials. It can also be seen that the magnetization begins to saturate at $h_0/(\rho_0 M_s) \cong 2.25$.

### 3.3. Critical loads and modes

The critical buckling fields as functions of the block's aspect ratio, $r$, for the three constitutive laws in Section 3.1 are plotted in Fig. 4. More specifically, plotted in Fig. 4 are the absolute value of the lowest buckling field, $h_c$, that satisfies the bifurcation equations for the antisymmetric mode (2.50) and the corresponding results for the symmetric (2.49) mode. In each case the lowest critical field always occurs for an eigenmode with the lowest wave number, i.e. $n = 1$ in Eq. (2.41).

As expected the critical magnetic field increases monotonically with $r$ for each material and for both mode types. For very low aspect ratios, beam bending (i.e. antisymmetric) mode is expected from existing structural models to be critical. Indeed, an examination of Fig. 4 shows that for slender beams with aspect ratios $r<0.235$, only antisymmetric buckling modes are found, while symmetric buckling modes also exist for $r>0.235$.[11] For stubby beams in the narrow aspect ratio range approximately $0.235<r<0.25$, the critical field for symmetric buckling is actually lower than what is required for the antisymmetric counterpart. A physical explanation for this somewhat unexpected result is not immediately apparent. For even stubbier blocks with aspect ratios $r>0.25$, buckling is predicted to initiate in an antisymmetric mode for the neo-Hookean and

[11]Our inability to obtain a solution for the symmetric mode as $r \to 0$ is in contrast to the results obtained by Pao and Yeh (1973) who derive an expression that seems to be valid for very slender beams as well. A brief overview of their method will be presented at the end of this section.

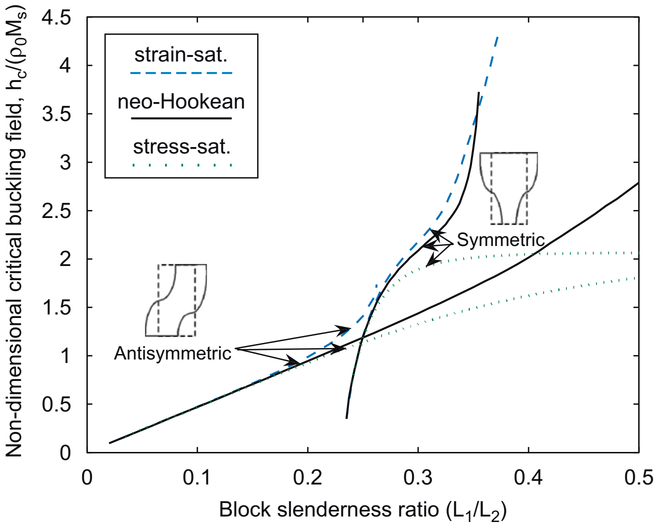


Fig. 4. Critical dimensionless buckling field, $h_c/(\rho_0 M_s)$, as a function of the block's aspect ratio $r$ under plane-strain conditions (for symmetric and antisymmetric bifurcation modes with $n = 1$) and for three different constitutive laws.

stress-saturated materials. For the strain-saturated material, however, only symmetric buckling modes may be predicted for the higher aspect ratios.

For each case (symmetric and antisymmetric), and for $h_c/\rho_0 M_s < 0.75$, no appreciable difference in $r$ is seen among the materials, as expected from Fig. 2. An inspection of the nondimensional magnetization response as a function of the nondimensional applied magnetic field (Fig. 3) shows that magnetic saturation is approached (i.e. $M_1/M_s \to 1$) for $h_0/\rho_0 M_s > 2.25$ in all the materials chosen. Hence, the magnetization corresponding to the critical buckling field, for the antisymmetric case, approaches saturation levels only for a very stubby ($r > 0.45$) neo-Hookean material. For the symmetric case, however, the magnetization levels at criticality approach saturation for both strain-saturated and neo-Hookean materials for less stubby beams ($r > 0.33$). Finally, it seems a stress-saturated block buckles (symmetrically or antisymmetrically) before magnetic saturation is achieved, for even very stubby blocks ($r \approx 0.5$).

Antisymmetric buckling of the strain-saturated block was not found at aspect ratios $r > 0.263$ (for $r \doteq 0.263$, $h_c/(\rho_0 M_s) \doteq 1.734$). From Fig. 2, this is likely due to the decrease in the axial stretch, $\lambda_2$, for values of the nondimensional applied field of $h_0/(\rho_0 M_s) > 1.688$. In the cases of the neo-Hookean and stress-saturated blocks, critical fields for symmetric buckling are shown for aspect ratios up to $r \doteq 0.35$.

### 3.4. Asymptotic solution

It is interesting to compare the present exact continuum solution to existing structural approximations that are based on thin beam/plate models (e.g. Pao and Yeh, 1973). To this end and in order to alleviate the admittedly long calculations, and without loss of generality, a special neo-Hookean material of the general form (3.1) will be used which exhibits no magnetostriction in its principal solution. Moreover, since buckling is expected to occur in the linear regime of the material's elastic and magnetic response, the constants $C_{11} = C_{21} = C_{01}^*$ are neglected.

To eliminate the magnetostriction effect, i.e. to guarantee that $\lambda_1(h_0) = \lambda_2(h_0) = 1$ in the principal solution, requires (see definition of the dimensionless parameter $\beta$ in Eq. (3.3)):

$$C_{01} + 2C_{02} = \tfrac{1}{2}\beta, \tag{3.4}$$

to satisfy the traction condition (2.33) at the sides (the traction condition (2.34) at the ends is automatically satisfied). Separately, the slope of the magnetization curve (see Fig. 3) is taken as $[\rho_0 M_1/h_0]_{h_0=0} \doteq 0.6$, which from Eq. (2.32) and the simplified version of Eq. (3.1), requires that

$$C_{01} + C_{02} = \tfrac{2}{3}\beta. \tag{3.5}$$

Hence from Eqs. (3.4) and (3.5), one has $C_{01} = 5\beta/6$ and $C_{02} = -\beta/6$. To further simplify the lengthy asymptotic calculations[12] a numerical value of $v = \frac{3}{7}$ is adopted for the Poisson's ratio thus resulting in

$$\rho_0\psi = \frac{G}{2}\left[ I_1 - 2J + \frac{5}{8}\left(J - \frac{1}{J}\right)^2 + \frac{5\beta}{6}\frac{J_1}{M_s^2} - \frac{\beta}{6}\frac{J_2}{M_s^2} \right]. \tag{3.6}$$

Using the small parameter $\varepsilon$ (for convenience defined as the ratio $M_1/M_s$), the moduli from the general expression (A.11) in Appendix A can be expanded as

$$L^{uu}_{ijkl} = \overset{0}{L}{}^{uu}_{ijkl} + \varepsilon^2 \overset{2}{L}{}^{uu}_{ijkl}, \quad L^{u\alpha}_{ijk} = \varepsilon \overset{1}{L}{}^{u\alpha}_{ijk} = L^{\alpha u}_{kij}, \quad L^{\alpha\alpha}_{ij} = \overset{0}{L}{}^{\alpha\alpha}_{ij} = L^{\alpha\alpha}_{ji}, \quad \varepsilon \equiv \frac{M_1}{M_s} \tag{3.7}$$

which, on the account of Eqs. (3.1) and (A.11)–(A.15) (from Appendix A), for this special case the nonzero (fourth order tensor) elastic moduli reduce to

$$\overset{0}{L}{}^{uu}_{1111} = \overset{0}{L}{}^{uu}_{2222} = \frac{2G}{1-v}, \quad \overset{0}{L}{}^{uu}_{1122} = v\overset{0}{L}{}^{uu}_{1111},$$

$$\overset{2}{L}{}^{uu}_{1111} = -\frac{7}{30}G\beta, \quad \overset{2}{L}{}^{uu}_{2222} = \frac{2}{5}G\beta, \quad \overset{2}{L}{}^{uu}_{1122} = -\frac{3}{10}G\beta,$$

$$\overset{0}{L}{}^{uu}_{2112} = \overset{0}{L}{}^{uu}_{2121} = \overset{0}{L}{}^{uu}_{1212} = G, \quad \overset{2}{L}{}^{uu}_{2112} = \frac{29}{60}G\beta, \quad \overset{2}{L}{}^{uu}_{2121} = -\frac{1}{60}G\beta, \quad \overset{2}{L}{}^{uu}_{1212} = -\frac{11}{60}G\beta,$$

$$L^{uu}_{2211} = L^{uu}_{1122}, \quad L^{uu}_{1221} = L^{uu}_{2112}, \tag{3.8}$$

while the nonzero (third order tensor) elasto-magnetic coupling moduli are given by

$$\overset{1}{L}{}^{u\alpha}_{112} = -\frac{4}{5}\rho_0 M_s, \quad \overset{1}{L}{}^{u\alpha}_{211} = -\frac{1}{10}\rho_0 M_s, \quad \overset{1}{L}{}^{u\alpha}_{121} = \frac{11}{10}\rho_0 M_s, \quad \overset{1}{L}{}^{u\alpha}_{222} = \frac{2}{5}\rho_0 M_s,$$

$$L^{\alpha u}_{211} = L^{u\alpha}_{112}, \quad L^{\alpha u}_{121} = L^{u\alpha}_{211}, \quad L^{\alpha u}_{112} = L^{u\alpha}_{121}, \quad L^{\alpha u}_{222} = L^{u\alpha}_{222}, \tag{3.9}$$

and finally the nonzero (second order tensor) magnetic moduli take the simple form

$$\overset{0}{L}{}^{\alpha\alpha}_{11} = \overset{0}{L}{}^{\alpha\alpha}_{22} = \frac{8}{5\mu_0}. \tag{3.10}$$

The characteristic equation from Eq. (2.45) becomes

$$(\xi^2 - 1)(d_0(\beta,\varepsilon) + d_1(\beta,\varepsilon)\xi^2 + d_2(\beta,\varepsilon)\xi^4) + \cdots = 0, \tag{3.11}$$

where $d_i$ are functions of $\beta$ and $\varepsilon$, as indicated.

By assuming an expansion for the roots $\xi_\beta$ as

$$\xi_\beta = 1 + \varepsilon^2 \overset{2}{\xi}_\beta + \varepsilon^4 \overset{4}{\xi}_\beta + \varepsilon^6 \overset{6}{\xi}_\beta + \cdots, \tag{3.12}$$

and noting $\xi_3 = 1$ on account of Eq. (3.11), $\overset{j}{\xi}_\beta$ (and hence the roots $\xi_\beta$) in Eq. (3.12) are obtained by equating like powers in $\varepsilon$ upon substitution of Eq. (3.12) in Eq. (3.11).

The corresponding amplitudes, $V_1^\beta, V_2^\beta$ of the eigenmodes in Eqs. (2.43), (2.44) are obtained from the solution to the $2\times 2$ system

$$\begin{bmatrix} (\xi_\beta^2 L^{uu}_{1111} - L^{uu}_{1212}) & -\xi_\beta(L^{uu}_{1221} + L^{uu}_{1122}) \\ -\xi_\beta(L^{uu}_{2211} + L^{uu}_{2112}) & (-\xi_\beta^2 L^{uu}_{2121} + L^{uu}_{2222}) \end{bmatrix} \begin{bmatrix} V_1^\beta \\ V_2^\beta \end{bmatrix} = \begin{bmatrix} -\xi_\beta(L^{u\alpha}_{121} + L^{u\alpha}_{112})\mu_0\rho M_s\varepsilon \\ (L^{u\alpha}_{222} - \xi_\beta^2 L^{u\alpha}_{211})\mu_0\rho M_s\varepsilon \end{bmatrix}, \tag{3.13}$$

[12] As it turns out asymptotic results depend on $v$ in a much more complicated fashion than in the case of structural models.

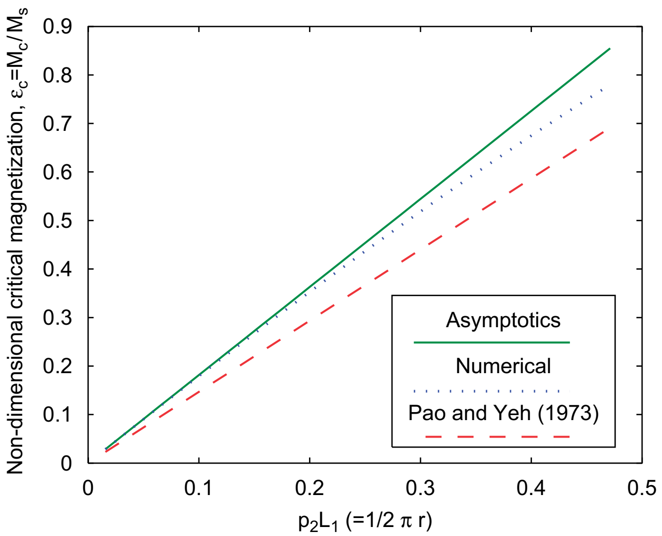


Fig. 5. Critical nondimensional magnetization, $\varepsilon_c = M_c/M_s$, as a function of the block's aspect ratio $r$ under plane-strain conditions for antisymmetric bifurcation modes with $n = 1$ using, without loss of generality, a special neo-Hookean material. To avoid magnetostriction (i.e. $\lambda_1 = \lambda_2 = 1$, $\forall h_0 \neq 0$), a material parameter constraint $C_{01} + 2C_{02} = \mu_0(\rho_0 M_s)^2/G$ is imposed. Further, $C_{11} = C_{21} = C^*_{01} = 0$. The asymptotic solution (solid line) is compared with (i) the numerical solution (dotted line) and (ii) the result (dashed line) obtained used the formula of Pao and Yeh (1973). Notice the convergence of the asymptotic solution to the numerical solution for small aspect ratios ($r < 0.1$).

where as mentioned earlier, for scaling purposes the constant $A$ in Eq. (2.42) $A = \mu_0 \rho M_s \varepsilon$, while the constants $V^\beta_{1,2}$ are assumed to take the expansion:

$$V^\beta_{1,2} = \overset{0}{V}{}^\beta_{1,2} + \varepsilon^2 \overset{2}{V}{}^\beta_{1,2} + \varepsilon^4 \overset{4}{V}{}^\beta_{1,2} + \cdots. \tag{3.14}$$

Substitution of expressions for $\xi_\beta$ (3.12) and $V^\beta_{1,2}$ (3.14) in $\det[D_{\alpha\beta}(h_c)] = 0$ (from Eqs. (2.48), (2.50)) results in a (purely imaginary) polynomial in $\varepsilon$:

$$\varepsilon^3 (L_1 p_2)^3 \left[a_0 + a_1(L_1 p_2) + a_2 (L_1 p_2)^3\right] + \beta \varepsilon^5 (L_1 p_2)[b_0 + b_1(L_1 p_2) + b_2(L_1 p_2)^2 + b_3(L_1 p_2)^3 + b_3(L_1 p_2)^5] + \mathcal{O}(\varepsilon^7) = 0.$$

As $\varepsilon \to 0$, and with $p_2 = \pi/(2L_2)$ (i.e. for $n = 1$ in Eq. $(2.41)_4$), then retaining the dominant terms yields

$$\varepsilon_c \cong \frac{1}{2}\left(\frac{a_0}{\beta b_0}\right)^{1/2} (\pi r), \tag{3.16}$$

where, $a_0/b_0 \doteq 0.5302$, is calculated for the assumed Poisson's ratio ($\nu = \frac{3}{7}$). A corresponding solution for the critical magnetization for the symmetric mode, however, could not be obtained suggesting that the block buckles in a symmetric manner only for finite slenderness ratios, a conclusion in contrast to the result of Pao and Yeh (1973).

The asymptotic solution for the nondimensional critical magnetization (solid line) is plotted against a measure of the aspect ratio, $p_2 L_1$, in Fig. 5. For sake of comparison, the corresponding numerical solution (dotted line) and the solution obtained by using the result of Pao and Yeh (1973) (dashed line) are also plotted in Fig. 5. As expected, the asymptotic solution serves as a tangent for the numerical solution and the two converge for vanishing aspect ratios. There is a noticeable difference, however, between the asymptotic result and the result obtained using Pao and Yeh's (1973) formula (Eq. (8.13)). The highlights of their paper, and the eventual result, are given in the next subsection.

### 3.5. A brief note on Pao and Yeh's (1973) solution

To understand the possible sources for the difference, in the antisymmetric mode, between our results and those of Pao and Yeh (1973), we take a closer look at their model for the buckling of an elastic plate under a transverse magnetic field.

For sake of clarity and to the extent possible, we use our notation to represent the major equations in their paper. Based on the assumption of small strains and a negligible magnetostrictive effect, they obtain for thin (i.e. $p_2L_2 \ll 1$) plates:

$$\frac{B_c^2}{\mu_0 G} \cong \frac{2\mu_r(\mu_r(p_2L_1)+1)(p_2L_1)^2}{3(1-v)\chi^2}, \tag{3.17}$$

where $\mu_r$ is the relative permeability given by $\mu_r = 1+\chi$, where $h_i = \chi^{-1}m_i$. So in our notation:

$$\chi^{-1} = \frac{1}{\beta}(C_{01}+C_{02}) = \frac{2}{3}, \quad \mu_r = 1+\chi = \frac{5}{2}, \quad B_1 = \mu_0(\chi^{-1}+1)\rho M_1. \tag{3.18}$$

It is important to note that for most ferromagnetic materials, the magnetic susceptibilities are of the order $10^4$, and on this basis Pao and Yeh (1973) argue that $\mu_r p_2 L_1 \gg 1$ and $\mu_r \cong \chi$. For the materials of interest to us, i.e. magnetorheological elastomers, the magnetic susceptibilities are much smaller so that $\mu_r p_2 L_1 \ll 1$ and (3.17) becomes

$$\frac{B_c^2}{\mu_0 G} \cong \frac{2\mu_r}{3(1-v)\chi^2}(p_2L_1)^2. \tag{3.19}$$

For sake of comparison with the asymptotic solution, (3.17) is first converted to a thin beam solution. Taking $v = \frac{3}{10}$ (which is the 3D value of $v_{2d} = \frac{3}{7}$ used in the current work), Pao and Yeh's result in terms of critical nondimensional magnetization may be written as

$$\varepsilon^c \cong \frac{1}{2}\left(\frac{c_{PY}}{\beta}\right)^{1/2}(\pi r), \tag{3.20}$$

where, $c_{PY} \doteq 0.3467$ compared to $a_0/b_0 \doteq 0.5302$ from the asymptotic solution. The slight difference in the coefficients from the asymptotic solution and the published structural model is likely due to the complicated $v$ dependence that is not captured by the structural models.

## 4. Conclusion

In this work we present a continuum formulation to the stability problem of a homogeneous, compressible, magnetoelastic rectangular block in finite plane strain subjected to a uniform transverse magnetic field. This boundary value problem is motivated by the classical problem of magnetoelastic buckling of a thin beam. The benefits of the continuum approach over traditionally employed structural models are in (a) the ability to assess effect of different nonlinear material responses and (b) the validity of the formulation for a wide range of block aspect ratios.

Critical magnetic fields, i.e. those corresponding to the onset of a bifurcation buckling, in the form of symmetric and antisymmetric modes, are obtained for three different constitutive laws. In general, the critical magnetic field is shown to increase monotonically with the block's aspect ratio for each material and mode type. For most aspect ratios, antisymmetric modes are always the critical buckling modes for stress saturated and neo-Hookean materials. In the narrow range of moderate aspect ratios (about 0.25) symmetric modes become critical. For strain-saturated solids no buckling is possible above a maximum aspect ratio. As expected, the results for stubby blocks are found to be very sensitive to the nonlinearity of the governing constitutive laws.

Furthermore, an asymptotic solution is obtained for slender beams that shows a linear relationship between the critical buckling field and the block's slenderness ratio. This result is found to agree reasonably well with the formula obtained from structural models.

The general methodology described here covers only the onset of magnetoelastic instability (analogous to the works of, for example, Rivlin and Sawyers, 1974; Ogden, 1984 for hyperelastic materials). Of particular interest would be the study of the post-buckling behavior in magnetoelastic solids corresponding to the recent article of Triantafyllidis et al. (2007), also for hyperelastic materials.

## Appendix A. Detailed derivations of bifurcation equations

### A.1. Bifurcation equations for rectangular block

The eigenvalue problem of the magnetoelastic block is according to Eq. (2.37)

$$(\mathscr{P}_{,\mathbf{gg}}(\mathbf{g}_0(h_c), h_c)\Delta\mathbf{g})\delta\mathbf{g} = 0. \tag{A.1}$$

The above bilinear (i.e. linear in $\Delta\mathbf{g}$ and $\delta\mathbf{g}$, where $\mathbf{g} \equiv (\mathbf{u}, \mathbf{M}, \alpha)$) equation is expanded as

$$[\mathscr{P}_{,\mathbf{uu}}\Delta\mathbf{u} + \mathscr{P}_{,\mathbf{uM}}\Delta\mathbf{M} + \mathscr{P}_{,\mathbf{u}\alpha}\Delta\alpha]\delta\mathbf{u} = 0,$$

$$[\mathscr{P}_{,\mathbf{Mu}}\Delta\mathbf{u} + \mathscr{P}_{,\mathbf{MM}}\Delta\mathbf{M} + \mathscr{P}_{,\mathbf{M}\alpha}\Delta\alpha]\delta\mathbf{M} = 0,$$

$$[\mathscr{P}_{,\alpha\mathbf{u}}\Delta\mathbf{u} + \mathscr{P}_{,\alpha\mathbf{M}}\Delta\mathbf{M} + \mathscr{P}_{,\alpha\alpha}\Delta\alpha]\delta\alpha = 0. \tag{A.2}$$

Prior to taking the second variations of the potential energy, $\mathscr{P}$, with respect to $\mathbf{u}, \mathbf{M}$ and $\alpha$, the following intermediate relations—based on the definitions of $\mathbf{F}$ and mass conservation given in Eq. (2.11)—are noted:

$$\Delta F_{ij}^{-\mathrm{T}} = -F_{kj}^{-\mathrm{T}} F_{il}^{-\mathrm{T}} \Delta u_{k,l}, \quad \Delta\rho = -\rho F_{ij}^{-1} \Delta u_{i,j}. \tag{A.3}$$

Using the above results, the three terms of the $\delta\mathbf{u}$ component of the bifurcation equations (A.2) are:

$$(\mathscr{P}_{,\mathbf{uu}}\Delta\mathbf{u})\delta\mathbf{u} = \int_A \rho J \left\{ \frac{\partial^2\psi}{\partial F_{ij}\partial F_{kl}} + \frac{\mu_0\rho}{2} M_m M_m [F_{ij}^{-\mathrm{T}} F_{kl}^{-\mathrm{T}} + F_{il}^{-\mathrm{T}} F_{kj}^{-\mathrm{T}}] \right\} \Delta u_{k,l} \delta u_{i,j} \, \mathrm{d}A,$$

$$(\mathscr{P}_{,\mathbf{uM}}\Delta\mathbf{M})\delta\mathbf{u} = \int_A \rho J \left\{ \frac{\partial^2\psi}{\partial F_{ij}\partial M_k} - \mu_0 \rho M_k F_{ij}^{-\mathrm{T}} \right\} \Delta M_k \delta u_{i,j} \, \mathrm{d}A,$$

$$(\mathscr{P}_{,\mathbf{u}\alpha}\Delta\alpha)\delta\mathbf{u} = \int_A \{\rho M_m (F_{ml} F_{ij}^{-\mathrm{T}} - \delta_{im}\delta_{jl})\varepsilon_{lk}\} \Delta\alpha_{,k} \delta u_{i,j} \, \mathrm{d}A. \tag{A.4}$$

Similarly, the three terms of the $\delta\mathbf{M}$ components of the bifurcation equations (A.2) are

$$(\mathscr{P}_{,\mathbf{Mu}}\Delta\mathbf{u})\delta\mathbf{M} = \int_A \rho J \left\{ \frac{\partial^2\psi}{\partial M_k \partial F_{ij}} - \mu_0 \rho M_k F_{ij}^{-\mathrm{T}} \right\} \Delta u_{i,j} \delta M_k \, \mathrm{d}A,$$

$$(\mathscr{P}_{,\mathbf{MM}}\Delta\mathbf{M})\delta\mathbf{M} = \int_A \left\{ \rho J \frac{\partial^2\psi}{\partial M_i \partial M_j} + \mu_0 \rho \delta_{ij} \right\} \Delta M_j \delta M_i \, \mathrm{d}A,$$

$$(\mathscr{P}_{,\mathbf{M}\alpha}\Delta\alpha)\delta\mathbf{M} = \int_A \{-\rho F_{il}\varepsilon_{lj}\} \Delta\alpha_{,j} \delta M_i \, \mathrm{d}A, \tag{A.5}$$

while the three terms of the $\delta\alpha$ component of the bifurcation equation (A.2) are

$$(\mathscr{P}_{,\alpha\mathbf{u}}\Delta\mathbf{u})\delta\alpha = \int_A \{\rho M_p \varepsilon_{qk} (F_{pq} F_{ij}^{-\mathrm{T}} - \delta_{ip}\delta_{jq})\} \Delta u_{i,j} \delta\alpha_{,k} \, \mathrm{d}A,$$

$$(\mathscr{P}_{,\alpha\mathbf{M}}\Delta\mathbf{M})\delta\alpha = \int_A \{-\rho F_{ik}\varepsilon_{kj}\} \Delta M_i \delta\alpha_{,j} \, \mathrm{d}A,$$

$$(\mathscr{P}_{,\alpha\alpha}\Delta\alpha)\delta\alpha = \int_{\mathbf{R}^2} \left\{ \frac{1}{\mu_0 J} \varepsilon_{lj}\varepsilon_{ki} C_{kl} \right\} \Delta\alpha_{,i} \delta\alpha_{,j} \, \mathrm{d}A. \tag{A.6}$$

Upon substitution of Eqs. (A.3)–(A.5) into Eq. (A.2) one rewrites (A.2) in the form of Eq. (2.38) with the following definitions[13] for the coefficients $L^{\cdot\cdot}_{\cdots\cdot}$ in $\mathbf{R}^2$:

$$\mathscr{L}^{uu}_{ijkl} \equiv \rho J\left[\frac{\partial^2\psi}{\partial F_{ij}\partial F_{kl}} + \frac{\mu_0}{2}(\rho M_q M_q)(F^{-\mathrm{T}}_{ij}F^{-\mathrm{T}}_{kl} + F^{-\mathrm{T}}_{kj}F^{-\mathrm{T}}_{il})\right] = \mathscr{L}^{uu}_{klij},$$

$$\mathscr{L}^{uM}_{ijk} \equiv \rho J\left[\frac{\partial^2\psi}{\partial F_{ij}\partial M_k} - \mu_0\rho M_k F^{-\mathrm{T}}_{ij}\right] = \mathscr{L}^{Mu}_{kij},$$

$$\mathscr{L}^{u\alpha}_{ijk} \equiv [\rho M_p(F_{pq}F^{-\mathrm{T}}_{ij} - \delta_{ip}\delta_{jq})\varepsilon_{qk}] = \mathscr{L}^{\alpha u}_{kij},$$

$$\mathscr{L}^{MM}_{kl} \equiv \rho J\left[\frac{\partial^2\psi}{\partial M_k\partial M_l} + \mu_0\rho\delta_{kl}\right] = \mathscr{L}^{MM}_{lk},$$

$$\mathscr{L}^{M\alpha}_{kl} \equiv [-\rho F_{kj}\varepsilon_{jl}] = \mathscr{L}^{\alpha M}_{lk},$$

$$\mathscr{L}^{\alpha\alpha}_{kl} \equiv \left[\frac{1}{\mu_0 J}\varepsilon_{jl}\varepsilon_{ik}C_{ij}\right] = \mathscr{L}^{\alpha\alpha}_{lk}. \tag{A.7}$$

From the second equation in Eq. (A.2) one can express $\Delta M_k$ in terms of $\Delta u_{i,j}$ and $\Delta\alpha_k$ at each point of the block thus rewriting (A.2) as

$$\int_A (L^{uu}_{ijkl}\Delta u_{k,l} + L^{u\alpha}_{ijk}\Delta\alpha_{,k})\delta u_{i,j} = 0,$$

$$\int_{\mathbf{R}^2} (L^{\alpha u}_{kij}\Delta u_{i,j} + L^{\alpha\alpha}_{kl}\Delta\alpha_{,l})\delta\alpha_{,k} = 0, \tag{A.8}$$

where the following definitions of the coefficients $L^{\cdot\cdot}_{\cdots\cdot}$ are used for $\mathbf{X} \in A$:

$$L^{uu}_{ijkl} \equiv \mathscr{L}^{uu}_{ijkl} - \mathscr{L}^{uM}_{ijp}[\mathscr{L}^{MM}_{pq}]^{-1}\mathscr{L}^{Mu}_{qkl}, \quad L^{u\alpha}_{ijk} \equiv \mathscr{L}^{u\alpha}_{ijk} - \mathscr{L}^{uM}_{ijp}[\mathscr{L}^{MM}_{pq}]^{-1}\mathscr{L}^{M\alpha}_{qk},$$

$$L^{\alpha u}_{kij} \equiv \mathscr{L}^{\alpha u}_{kij} - \mathscr{L}^{\alpha M}_{kp}[\mathscr{L}^{MM}_{pq}]^{-1}\mathscr{L}^{Mu}_{qij}, \quad L^{\alpha\alpha}_{kl} \equiv \mathscr{L}^{\alpha\alpha}_{kl} - \mathscr{L}^{\alpha M}_{kp}[\mathscr{L}^{MM}_{pq}]^{-1}\mathscr{L}^{M\alpha}_{ql}. \tag{A.9}$$

### *A.2. Coefficients for final Euler–Lagrange equations for the bifurcation problem*

The coefficients of the final bifurcation equations (A.8) can be further detailed for the case of isotropic materials considered here. Since $\psi(I_1, J, J_1, J_2)$ is a function of invariants, the following identities are recorded:

$$\frac{\partial I_1}{\partial F_{ij}} = 2F_{ij}, \quad \frac{\partial J}{\partial F_{ij}} = JF^{-\mathrm{T}}_{ij}, \quad \frac{\partial J_2}{\partial F_{ij}} = 2M_iM_pF_{pj},$$

$$\frac{\partial J_1}{\partial M_k} = 2M_k, \quad \frac{\partial J_2}{\partial M_k} = 2B_{kr}M_r,$$

$$\frac{\partial^2 I_1}{\partial F_{ij}\partial F_{kl}} = 2\delta_{ik}\delta_{jl}, \quad \frac{\partial^2 J}{\partial F_{ij}\partial F_{kl}} = J(F^{-\mathrm{T}}_{ij}F^{-\mathrm{T}}_{kl} - F^{-\mathrm{T}}_{il}F^{-\mathrm{T}}_{kj}),$$

$$\frac{\partial^2 J_2}{\partial F_{ij}\partial F_{kl}} = 2M_iM_k\delta_{jl}, \quad \frac{\partial^2 J_2}{\partial F_{ij}\partial M_k} = 2(\delta_{ik}M_rF_{rj} + M_iF_{kj}). \tag{A.10}$$

[13] $\varepsilon_{ij}$ denotes the alternating symbol in $\mathbf{R}^2$.

The nonzero components of the fourth order incremental moduli are listed as

$$L^{uu}_{1111} = \rho J\left\{\frac{\partial^2\psi}{\partial F_{11}\partial F_{11}} + \mu_0\rho\left(\frac{M_1}{\lambda_1}\right)^2 - \left(\frac{\partial^2\psi}{\partial F_{11}\partial M_1} - \mu_0\rho\frac{M_1}{\lambda_1}\right)\mu_{11}\left(\frac{\partial^2\psi}{\partial F_{11}\partial M_1} - \mu_0\rho\frac{M_1}{\lambda_1}\right)\right\},$$

$$L^{uu}_{1122} = \rho J\left\{\frac{\partial^2\psi}{\partial F_{11}\partial F_{22}} + \frac{1}{2}\mu_0\rho\frac{M_1^2}{\lambda_1\lambda_2} - \left(\frac{\partial^2\psi}{\partial F_{11}\partial M_1} - \mu_0\rho\frac{M_1}{\lambda_1}\right)\mu_{11}\left(\frac{\partial^2\psi}{\partial F_{22}\partial M_1} - \mu_0\rho\frac{M_1}{\lambda_2}\right)\right\} = L^{uu}_{2211},$$

$$L^{uu}_{1221} = \rho J\left\{\frac{\partial^2\psi}{\partial F_{21}\partial F_{12}} + \frac{1}{2}\mu_0\rho\frac{M_1^2}{\lambda_1\lambda_2} - \left(\frac{\partial^2\psi}{\partial F_{21}\partial M_2}\right)\mu_{22}\left(\frac{\partial^2\psi}{\partial F_{12}\partial M_2}\right)\right\} = L^{uu}_{2112},$$

$$L^{uu}_{2121} = \rho J\left\{\frac{\partial^2\psi}{\partial F_{21}\partial F_{21}} - \left(\frac{\partial^2\psi}{\partial F_{21}\partial M_2}\right)\mu_{22}\left(\frac{\partial^2\psi}{\partial F_{21}\partial M_2}\right)\right\},$$

$$L^{uu}_{1212} = \rho J\left\{\frac{\partial^2\psi}{\partial F_{12}\partial F_{12}} - \left(\frac{\partial^2\psi}{\partial F_{12}\partial M_2}\right)\mu_{22}\left(\frac{\partial^2\psi}{\partial F_{12}\partial M_2}\right)\right\},$$

$$L^{uu}_{2222} = \rho J\left\{\frac{\partial^2\psi}{\partial F_{22}\partial F_{22}} + \mu_0\rho\left(\frac{M_1}{\lambda_2}\right)^2 - \left(\frac{\partial^2\psi}{\partial F_{22}\partial M_1} - \mu_0\rho\frac{M_1}{\lambda_2}\right)\mu_{11}\left(\frac{\partial^2\psi}{\partial F_{22}\partial M_1} - \mu_0\rho\frac{M_1}{\lambda_2}\right)\right\},$$

$$\mu_{ij} \equiv \left\{\left(\frac{\partial^2\psi}{\partial M_i\partial M_j} + \mu_0\rho\delta_{ij}\right)^{-1}\right\}, \quad \begin{cases}\mu_{ij} = 0, & i \neq j,\\ \mu_{ij} \neq 0, & i = j,\end{cases} \tag{A.11}$$

while the non-zero components of the third order incremental moduli are given by

$$L^{u\alpha}_{112} = \left(\frac{\partial^2\psi}{\partial F_{11}\partial M_1} - \mu_0\rho\frac{M_1}{\lambda_1}\right)\mu_{11}\rho\lambda_1 = L^{\alpha u}_{211},$$

$$L^{u\alpha}_{211} = -\left(\frac{\partial^2\psi}{\partial F_{21}\partial M_2}\right)\mu_{22}\rho\lambda_2 = L^{\alpha u}_{121},$$

$$L^{u\alpha}_{222} = \left\{\left(\rho M_1\frac{\lambda_1}{\lambda_2}\right) + \left(\frac{\partial^2\psi}{\partial F_{22}\partial M_1} - \mu_0\rho\frac{M_1}{\lambda_2}\right)\mu_{11}\rho\lambda_1\right\} = L^{\alpha u}_{222},$$

$$L^{u\alpha}_{121} = \rho M_1 - \left(\frac{\partial^2\psi}{\partial F_{12}\partial M_2}\right)\mu_{22}\rho\lambda_2 = L^{\alpha u}_{112}, \tag{A.12}$$

and finally the second order moduli are expressed as

$$L^{\alpha\alpha}_{11} = \frac{1}{\mu_0 J}\lambda_2^2 + \left(\frac{\rho}{\rho J}\right)^2\lambda_2^2\mu_{22},$$

$$L^{\alpha\alpha}_{22} = \frac{1}{\mu_0 J}\lambda_1^2 + \left(\frac{\rho}{\rho J}\right)^2\lambda_1^2\mu_{11},$$

$$L^{\alpha\alpha}_{12} = L^{\alpha\alpha}_{21} = 0. \tag{A.13}$$

It can be shown that the moduli as functions of the principal stretches, $\lambda_i$ and specific magnetization $M_i$ $(M_1 M_2 = 0)$ are given by

$$\frac{\partial^2\psi}{\partial F_{11}\partial F_{11}} = 4\frac{\partial^2\psi}{\partial I_1\partial I_1}\lambda_1^2 + 4\frac{\partial^2\psi}{\partial I_1\partial J}\lambda_1\lambda_2 + 8\frac{\partial^2\psi}{\partial I_1\partial J_2}\lambda_1^2 M_1^2 + \frac{\partial^2\psi}{\partial J\partial J}\lambda_2^2 + 4\frac{\partial^2\psi}{\partial J\partial J_2}\lambda_1\lambda_2 M_1^2 + 4\frac{\partial^2\psi}{\partial J_2\partial J_2}\lambda_1^2 M_1^4 + 2\frac{\partial\psi}{\partial I_1} + 2\frac{\partial\psi}{\partial J_2}M_1^2,$$

$$\frac{\partial^2\psi}{\partial F_{11}\partial F_{22}} = 4\frac{\partial^2\psi}{\partial I_1\partial I_1}\lambda_1\lambda_2 + 2\frac{\partial^2\psi}{\partial I_1\partial J}(\lambda_1^2+\lambda_2^2) + 4\frac{\partial^2\psi}{\partial I_1\partial J_2}\lambda_1\lambda_2(M_1^2+M_2^2) + \frac{\partial^2\psi}{\partial J\partial J}\lambda_1\lambda_2 + 2\frac{\partial^2\psi}{\partial J\partial J_2}(\lambda_1^2 M_1^2+\lambda_2^2 M_2^2) + 4\frac{\partial^2\psi}{\partial J_2\partial J_2}\lambda_1\lambda_2 M_1^2 M_2^2 + \frac{\partial\psi}{\partial J} = \frac{\partial^2\psi}{\partial F_{22}\partial F_{11}},$$

$$\frac{\partial^2\psi}{\partial F_{22}\partial F_{22}} = 4\frac{\partial^2\psi}{\partial I_1\partial I_1}\lambda_2^2 + 4\frac{\partial^2\psi}{\partial I_1\partial J}\lambda_1\lambda_2 + 8\frac{\partial^2\psi}{\partial I_1\partial J_2}\lambda_2^2 M_2^2 + \frac{\partial^2\psi}{\partial J\partial J}\lambda_1^2 + 4\frac{\partial^2\psi}{\partial J\partial J_2}\lambda_1\lambda_2 M_2^2 + 4\frac{\partial^2\psi}{\partial J_2\partial J_2}\lambda_2^2 M_2^4 + 2\frac{\partial\psi}{\partial I_1} + 2\frac{\partial\psi}{\partial J_2}M_2^2,$$

$$\frac{\partial^2\psi}{\partial F_{12}\partial F_{12}} = 2\frac{\partial\psi}{\partial I_1} + 2\frac{\partial\psi}{\partial J_2}M_1^2,$$

$$\frac{\partial^2\psi}{\partial F_{21}\partial F_{21}} = 2\frac{\partial\psi}{\partial I_1} + 2\frac{\partial\psi}{\partial J_2}M_2^2,$$

$$\frac{\partial^2\psi}{\partial F_{12}\partial F_{21}} = -\frac{\partial\psi}{\partial J} = \frac{\partial^2\psi}{\partial F_{21}\partial F_{12}} \tag{A.14}$$

and

$$\frac{\partial^2\psi}{\partial F_{11}\partial M_1} = 4\frac{\partial^2\psi}{\partial I_1\partial J_1}\lambda_1 M_1 + 4\frac{\partial^2\psi}{\partial I_1\partial J_2}\lambda_1^3 M_1 + 2\frac{\partial^2\psi}{\partial J\partial J_1}\lambda_2 M_1 + 2\frac{\partial^2\psi}{\partial J\partial J_2}\lambda_1^2\lambda_2 M_1 + 4\frac{\partial^2\psi}{\partial J_2\partial J_1}\lambda_1 M_1^3 + 4\frac{\partial^2\psi}{\partial J_2\partial J_2}\lambda_1^3 M_1^3 + 4\frac{\partial\psi}{\partial J_2}\lambda_1 M_1,$$

$$\frac{\partial^2\psi}{\partial F_{11}\partial M_2} = 4\frac{\partial^2\psi}{\partial I_1\partial J_1}\lambda_1 M_2 + 4\frac{\partial^2\psi}{\partial I_1\partial J_2}\lambda_1\lambda_2^2 M_2 + 2\frac{\partial^2\psi}{\partial J\partial J_1}\lambda_2 M_2 + 2\frac{\partial^2\psi}{\partial J\partial J_2}\lambda_2^3 M_2,$$

$$\frac{\partial^2\psi}{\partial F_{12}\partial M_1} = 2\frac{\partial\psi}{\partial J_2}\lambda_2 M_2, \quad \frac{\partial^2\psi}{\partial F_{12}\partial M_2} = 2\frac{\partial\psi}{\partial J_2}\lambda_2 M_1,$$

$$\frac{\partial^2\psi}{\partial F_{21}\partial M_1} = 2\frac{\partial\psi}{\partial J_2}\lambda_1 M_2, \quad \frac{\partial^2\psi}{\partial F_{21}\partial M_2} = 2\frac{\partial\psi}{\partial J_2}\lambda_1 M_1,$$

$$\frac{\partial^2\psi}{\partial F_{22}\partial M_1} = 4\frac{\partial^2\psi}{\partial I_1\partial J_1}\lambda_2 M_1 + 4\frac{\partial^2\psi}{\partial I_1\partial J_2}\lambda_1^2\lambda_2 M_1 + 2\frac{\partial^2\psi}{\partial J\partial J_1}\lambda_1 M_1 + 2\frac{\partial^2\psi}{\partial J\partial J_2}\lambda_1^3 M_1,$$

$$\frac{\partial^2\psi}{\partial F_{22}\partial M_2} = 4\frac{\partial^2\psi}{\partial I_1\partial J_1}\lambda_2 M_2 + 4\frac{\partial^2\psi}{\partial I_1\partial J_2}\lambda_2^3 M_2 + 2\frac{\partial^2\psi}{\partial J\partial J_1}\lambda_1 M_2 + 2\frac{\partial^2\psi}{\partial J\partial J_2}\lambda_1\lambda_2^2 M_2 + 4\frac{\partial^2\psi}{\partial J_1\partial J_2}\lambda_2 M_2^3 + 4\frac{\partial^2\psi}{\partial J_2\partial J_2}\lambda_2^3 M_2^3 + 4\frac{\partial\psi}{\partial J_2}\lambda_2 M_2,$$

$$\frac{\partial^2\psi}{\partial M_1\partial M_1} = 4\frac{\partial^2\psi}{\partial J_1\partial J_1}\, M_1^2 + 8\frac{\partial^2\psi}{\partial J_1\partial J_2}\,\lambda_1^2 M_1^2 + 4\frac{\partial^2\psi}{\partial J_2\partial J_2}\,\lambda_1^4 M_1^2 + 2\frac{\partial\psi}{\partial J_1} + 2\frac{\partial\psi}{\partial J_2}\,\lambda_1^2,$$

$$\frac{\partial^2\psi}{\partial M_1\partial M_2} = \frac{\partial^2\psi}{\partial M_2\partial M_1} = 0,$$

$$\frac{\partial^2\psi}{\partial M_2\partial M_2} = 4\frac{\partial^2\psi}{\partial J_1\partial J_1}\, M_2^2 + 8\frac{\partial^2\psi}{\partial J_1\partial J_2}\,\lambda_2^2 M_2^2 + 4\frac{\partial^2\psi}{\partial J_2\partial J_2}\,\lambda_2^4 M_2^2 + 2\frac{\partial\psi}{\partial J_1} + 2\frac{\partial\psi}{\partial J_2}\,\lambda_2^2. \tag{A.15}$$

### *A.3. Detailed expressions of bifurcation equations and boundary conditions for isotropic block subjected to a transverse magnetic field*

The governing equations (2.39) reduce to

$$\mathbf{X}\in A: \begin{array}{l} L^{uu}_{1111}\Delta u_{1,11} + L^{uu}_{1122}\Delta u_{2,21} + L^{uu}_{1212}\Delta u_{1,22} + L^{uu}_{1221}\Delta u_{2,12} + L^{u\alpha}_{112}\Delta\alpha_{,21} + L^{u\alpha}_{121}\Delta\alpha_{,12} = 0, \\ L^{uu}_{2112}\Delta u_{1,21} + L^{uu}_{2121}\Delta u_{2,11} + L^{uu}_{2222}\Delta u_{2,22} + L^{uu}_{2211}\Delta u_{1,12} + L^{u\alpha}_{211}\Delta\alpha_{,11} + L^{u\alpha}_{222}\Delta\alpha_{,22} = 0, \\ L^{\alpha u}_{121}\Delta u_{2,1} + L^{\alpha u}_{112}\Delta u_{1,2} + L^{\alpha\alpha}_{11}\Delta\alpha_{,1} = \dfrac{1}{\mu_0}\Delta\alpha^{\text{out}}_{,1}, \end{array}$$

$$\mathbf{X}\in \mathbf{R}^2\backslash A: \ \lambda_2^2\Delta\alpha^{\text{out}}_{,11} + \lambda_1^2\Delta\alpha^{\text{out}}_{,22} = 0 \tag{A.16}$$

and the resulting boundary conditions (2.40):

$$L^{uu}_{1111}\Delta u_{1,1} + L^{uu}_{1122}\Delta u_{2,2} + L^{u\alpha}_{112}\Delta\alpha_{,2} = \mathscr{L}^{u\alpha}_{112}\Delta\alpha^{\text{out}}_{,2},$$

$$L^{uu}_{2112}\Delta u_{1,2} + L^{uu}_{2121}\Delta u_{2,1} + L^{u\alpha}_{211}\Delta\alpha_{,1} = \mathscr{L}^{u\alpha}_{211}\Delta\alpha^{\text{out}}_{,1},$$

$$L^{\alpha u}_{112}\Delta u_{1,2} + L^{\alpha u}_{121}\Delta u_{2,1} + L^{\alpha\alpha}_{11}\Delta\alpha_{,1} = \mathscr{L}^{\alpha\alpha}_{11}\Delta\alpha^{\text{out}}_{,1},$$

$$\Delta\alpha_{,2} = \Delta\alpha^{\text{out}}_{,2}, \quad \lim_{\|\mathbf{X}\|\to\infty}\Delta\alpha_{,i} = 0, \tag{A.17}$$

of the buckling eigenmode. Substitution of the expression for the eigenmode (2.41) leads to the following ODE form:

$$\mathbf{X}\in A \begin{cases} L^{uu}_{1111}v_{1,1} - (L^{uu}_{1122} + L^{uu}_{1221})v_{2,1}p_2 - L^{uu}_{1212}v_1 p_2^2 + (L^{u\alpha}_{112} + L^{u\alpha}_{121})\alpha_{,1}p_2 = 0, \\ (L^{uu}_{2112} + L^{uu}_{2211})v_{1,1}p_2 + L^{uu}_{2121}v_{2,11} - L^{uu}_{2222}v_2 p_2^2 - L^{u\alpha}_{211}\alpha_{,11} + L^{u\alpha}_{222}\alpha p_2^2 = 0, \\ L^{\alpha u}_{121}v_{2,11} + (L^{\alpha u}_{112} + L^{\alpha u}_{211})p_2 v_{1,1} - L^{\alpha u}_{222}p_2^2 v_2 - L^{\alpha\alpha}_{11}\alpha_{,11} + L^{\alpha\alpha}_{22}\alpha p_2^2 = 0, \end{cases} \tag{A.18}$$

in which use has been made of the orthotropy of the incremental moduli tensor **L** with respect to the coordinate axes. The expressions for $v_i(X_1), \alpha(X_1)$ in Eq. (A.18) are obtained from Eq. (2.42).

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