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On finitely strained magnetorheological elastomers

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Abstract

Magnetorheological elastomers (MREs) are ferromagnetic particle impregnated rubbers whose mechanical properties are altered by the application of external magnetic fields. Due to their strong magnetoelastic coupling response MREs are finding an increasing number of engineering applications, thus necessitating appropriate theoretical descriptions which is the objective of this work.

Two different continuum formulations for MREs are presented: an Eulerian (current configuration) based approach using the second law of thermodynamics plus the conservation laws method of mechanics and a new, Lagrangian (reference configuration) based formulation based on the unconstrained minimization of a potential energy functional. It is shown that both approaches yield the same governing equations and boundary conditions. Following a discussion of general properties of the free energy function of MREs, we use a particular such function to illustrate the magnetoelastic coupling phenomena in a cylinder subjected to traction or torsion under the presence of external magnetic fields.

Keywords: A. Electromechanical processes; B. Finite strain; B. Particulate reinforced material; C. Conservation integrals; C. Energy methods

1. Introduction and motivation

Magnetorheological elastomers (MREs) are a class of solids that consist of a rubber matrix filled with magnetizable particles, typically sub-micron sized iron particles (see Rigbi and Jilkén, 1983; Ginder, 1996). The interest in these materials stems from their

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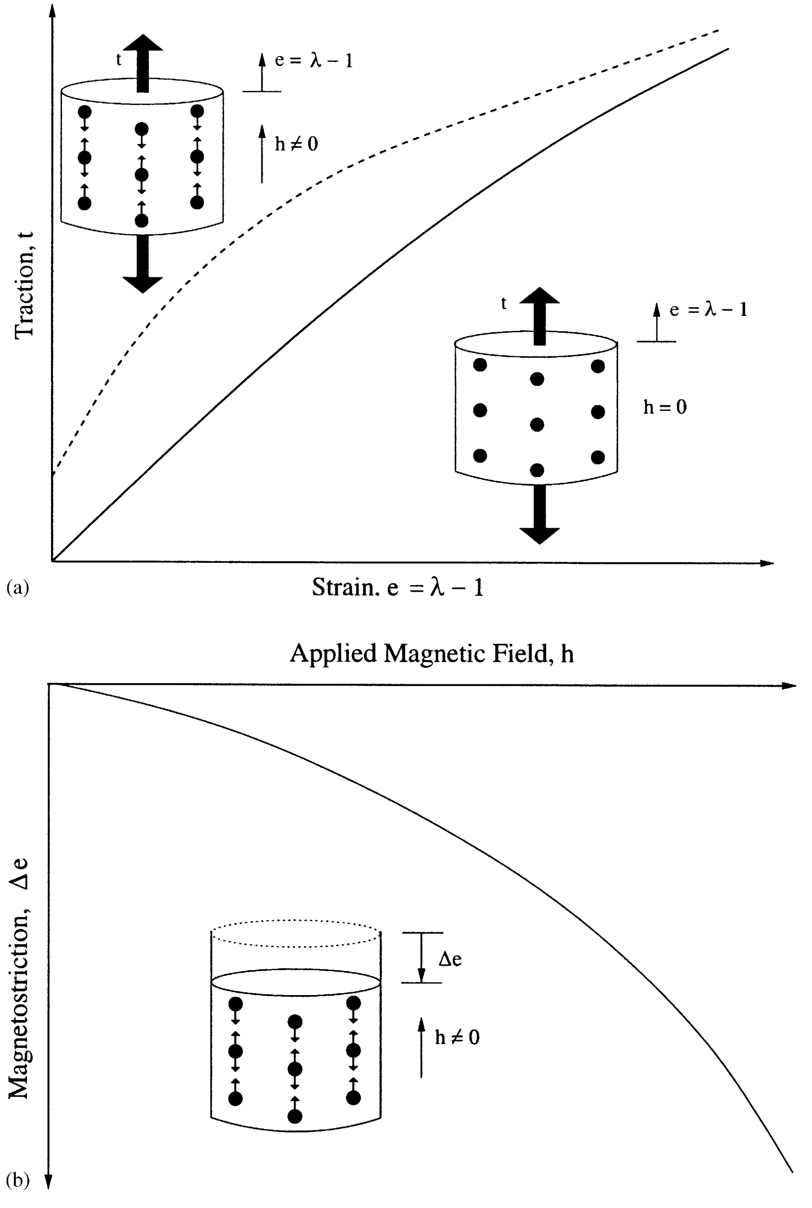

Fig. 1. Schematic diagrams of magnetoelastic coupling effects in MREs. The mechanical stiffening of a cylindrical specimen subjected to a magnetic field h is shown in the traction-strain diagram in (a) and the magnetostrictive shortening of the traction free specimen is shown in (b).

strong magnetoelastic coupling properties, as depicted in Fig. 1. The application of an external magnetic field h, tends to align the initially random magnetization vectors of the particles with the applied external field. As a result the interparticle attractive magnetic forces shorten the average particle distance, thus stiffening the material. The macroscopic manifestation of the above explained mechanism in a traction-engineering strain response is depicted in Fig. 1a, where the application of an external magnetic field $h \neq 0$ requires a higher traction (dotted line) for the same strain level as compared to its unmagnetized, $h = 0$, counterpart (solid line). The same mechanism is responsible for the specimen's "magnetostriction" depicted in Fig. 1b, according to which the length of a cylindrical specimen is reduced upon the application of an external magnetic field along the specimen's axial direction.

Since the mechanical properties of MREs can be altered rapidly and reversibly, they have been proposed and tested for a variety of applications in which it is desirable to continuously and controllably vary the effective stiffness of a device under different operating conditions. More specifically, MREs have been manufactured and studied as adaptive engine mounts and tunable shock absorbers by Ginder et al. (1999). A significant obstacle for the efficient design of devices employing these materials is the lack of a proper modeling for their coupled, finite strain magnetoelastic response, which is the main motivation of the present study.

While the majority of engineering applications of MREs are relatively recent, the theoretical foundations for the magnetoelastic response of solids go back to the 1950s and 1960s. The modeling approaches adopted can be broadly classified into two categories: (i) based on the "direct" method which uses conservation laws of continuum mechanics, e.g. Truesdell and Toupin (1960), Tiersten (1964), Maugin and Eringen (1972), Pao and Yeh (1973) and Pao (1978) and (ii) based on the "energy" method which uses the calculus of variations to extremize an appropriate potential energy, e.g. Tiersten (1965), Brown (1966) and Maugin and Eringen (1972). Recently macroscopic constitutive MRE models have been derived from micromechanical considerations by Borcea and Bruno (2001) for the small strain linearly elastic response of isotropic elastomers filled with rigid ferromagnetic spherical particles. Although the latter approach is useful to understand the connections between the MRE's microstructure and the resulting macroscopic properties, attention is here on restricted to macroscopic phenomena that can be reasonably captured by continuum theories. To help the reader put this work into perspective a brief review of the various continuum descriptions using both the "direct" and the "energy" approach is given below.

One of the first and elegant presentations of the direct approach is given in the classical review article of Truesdell and Toupin (1960) who use conservation laws for the derivation of Maxwell's equations for deformable media subjected to electrical and magnetic fields, but who do not give the general equations for finitely strained magnetoelastic solids. Subsequent work by Tiersten (1964) on elastically deformable but magnetically saturated insulators provides a continuum theory based on the assumption of two superposed, interacting continua, a "lattice continuum" and an "electronic spin continuum". His theory has (non-symmetric) macroscopic stresses as well as couples per unit volume and considers a free energy that depends, in addition to the deformation gradient, on magnetization and the spatial gradient of the magnetization. The stress measure he employs is assumed to contain an electromagnetic part, justified on assumptions for the interaction between his two continua, which depends on magnetization and the applied magnetic field. Jordan and Eringen (1964) derive the governing equations and boundary conditions for an isotropic solid under the influence of electromagnetic and thermoelastic effects. In contrast to Tiersten (1964) and other later works, the constitutive equations are obtained from invariance principles and the theory of matrix polynomials (as in Pipkin and Rivlin, 1960a,b), without any restrictions imposed by the entropy inequality. A detailed account of the state of the art up to the late 1980s in the continuum description of magnetoelastic materials can be found in the textbook by Eringen and Maugin (1990). It should be noted here that in addition to the purely magnetoelastic case, the direct method has been employed for solids exhibiting a more general thermomechanical and electromagnetic coupling, as for example in Tiersten and Tsai (1972) and Thurston (1974), which include magnetoelastic solids as a special case. It should also be pointed out that the direct method of derivation is always based on a current configuration description.

Concurrently with derivations based on the direct approach, energy methods have also been used to derive the governing equations of magnetoelastic solids. Tiersten (1965) uses Hamilton's principle to rederive the equations he obtained earlier by a direct approach (Tiersten, 1964). Brown (1966) uses a constrained minimization approach to derive the governing equations for finite strain magnetoelasticity in an approach analogous to Toupin's (1956) variational treatment of an elastic dielectric. Tiersten's (1965) formulation uses a current configuration description, while Brown (1966) employs an easier to use reference configuration formulation. Following Tiersten (1965), Maugin and Eringen (1972) also derive, using a variational approach, their governing equations obtained earlier through a direct method.

In contrast to the steady stream of publications in the field between the mid-1950s to the early 1970s, the number of publications in magnetoelasticity subsequently reduced to a trickle. In the 1990s, a renewed interest in magnetoelasticity was motivated by novel applications involving sensors and actuators made of solids exhibiting coupled thermomechanical and electromagnetic properties. Among these recent publications in continuum magnetoelasticity we cite the work of James and Kinderlehrer (1993), DeSimone and Podio-Guidugli (1996), DeSimone and James (2002) and James (2002). The first work along with the last two use Brown's energetic approach to examine magnetoelastic solids that undergo phase transformations; a judicious choice of a free energy with several local minima is used to model the experimentally observed fine microstructures in these solids. The second work uses the direct method of deriving continuum micro- and macro-mechanical equations of magnetoelasticity by improving and clarifying Tiersten's approach which involves two interacting continua.

In summarizing the current state of affairs in the available continuum formulations of magnetoelasticity, there are several points that in our opinion could be simplified and/or improved. (a) Several authors require two different interacting continua to formulate the problem. This approach, although physically motivated is rather cumbersome for a continuum theory. A theory based on only one underlying continuum is easier to present and makes a more concise formulation. (b) Related to the above comment is the need for some theories to introduce body couples and use non-symmetric stresses. A theory that is based on a symmetric macroscopic stress measure simplifies the problem formulation. (c) Most authors need to assume ab initio expressions for the electromagnetic body forces and for the electromagnetic part of the stress (the so-called "Maxwell stress"). An approach that derives the electromagnetic body forces and also gives the relation between a symmetric macroscopic stress measure and its mechanical and electromagnetic parts is more appealing. (d) Different authors use different arguments for the solid's Helmholtz free energy. These different choices lead to different expressions for the electromagnetic part of the body force, a source of endless confusion in the magnetoelasticity literature. In our opinion the best choice of variables for the free energy is the one proposed by Brown (1966) and are: the right (or left) Cauchy–Green tensor $\mathbf{C}$ (or $\mathbf{B}$) and the magnetization per unit mass $\mathbf{M}$, since it vanishes outside the solid (unlike the magnetic field $\mathbf{h}$ or the flux $\mathbf{b}$). (e) All the variational approaches presented thus far are either constrained minimizations of the potential energy or saddle points in an unconstrained extremization problem. Having in mind the solution of stability problems of MREs and stable numerical algorithms for boundary value problems, we seek an energy minimization formulation with no added constraints, i.e., all variables are independent of each other. (f) All direct formulations use an Eulerian (current configuration) description while most energy formulations use a full Lagrangian (reference configuration) description. With few exceptions, authors follow either one or the other approach but rarely do authors follow both approaches to show that they give the same results. The consistent formulation that we seek should give the same governing equations and boundary conditions when both a direct—Eulerian based—and energy—Lagrangian based—approaches are used.

In our quest for a consistent formulation of the MRE problem that requires a minimum number of assumptions we found the recent book by Kovetz (2000) and the book by Brown (1966) to be extremely useful. The appeal of Kovetz's (2000) presentation, who uses the direct approach and an Eulerian formulation, lies in the minimal number—compared to previous works—of required assumptions. More specifically he assumes the existence of a total stress measure (which satisfies Cauchy's tetrahedron relation) and an initially unspecified external body force.[2] The coupled mechanical electromagnetic nature of the theory is introduced through the energy balance law, in which an electromagnetic energy flux term is added (flow of Poynting vector) to the standard mechanical and thermal flux terms.[3] A subsequent application of Coleman–Noll's (1963) method based on an ingenious use of the entropy production inequality gives the total stress (symmetric due to angular momentum balance) in terms of mechanical and electromagnetic quantities, i.e. gives as a result the expression of the Maxwell stress tensor. The expressions for the electromagnetic body force and of the traction boundary condition are results of the equilibrium equation and Cauchy tetrahedron relation which are expressed in terms of the total stress. Instead of adopting Kovetz's use of magnetic flux as one of the independent variables of the free energy, we followed Brown's (1966) example by using the magnetization per unit mass, which, unlike the flux, vanishes outside the solid. Brown's (1966) energy formulation, which is based on a full Lagrangian description, uses a constrained minimization formulation and also suggests in the appendix an unconstrained formulation which is not an energy

[2] In the same manner as suggested by Coleman and Noll (1963).

[3] That this Poynting term represents an energy flux is a point made by Truesdell and Toupin (1960) based partly on a dimensionality argument.

minimizer. The approach presented here, suggested in a comment by Brown (1966) but not followed up, is based on an unconstrained energy minimization approach which is a preferable formulation for the study of stability problems.

The present work derives the governing equations for MREs in Section 2 using the direct approach and in Section 3 using the energy approach and shows how they end up with the same result. Section 4 discusses some important properties of the solid's free energy and introduces the notion of quasiconvexity for magnetoelastic solids. An example based on an experimentally obtained energy density is given in Section 5 which shows various aspects of the coupled magnetoelastic response in uniaxial stretching and pure torsion tests and the work is concluded with some general comments.

2. Direct approach (current configuration formulation)

The goal of Section 2 is the derivation of the differential equations and boundary/interface conditions of magnetoelastic continua using the conservation law (or direct) approach of continuum mechanics. As discussed in more detail in the Introduction, the approach presented here, which is based on Kovetz (2000), uses the current configuration for the definition of all the required field quantities. The different conservation laws are presented below in a standard dyadic notation [4] for a two- or three-dimensional Euclidean space.

2.1. Ampère's law

For the case of time-independent problems and in the absence of external currents, which is of interest here, Ampère's law takes the following form:

$$\int_{\partial S} \mathbf{h} \bullet \mathbf{s} \, \mathrm{d}l = 0, \tag{2.1}$$

where $\mathbf{h}$ is the magnetic field, ∂S any arbitrary but smooth and closed curve in the solid, $\mathbf{s}$ its unit tangent at the same material point as $\mathbf{h}$ and $\mathrm{d}l$ the corresponding element of length (see Fig. 2a). Assuming adequate smoothness of $\mathbf{h}$, Stokes' theorem dictates that for any smooth surface S whose boundary is ∂S:

$$\int_{S} (\nabla \times \mathbf{h}) \bullet \mathbf{n} \, \mathrm{d}a = 0, \tag{2.2}$$

where $\nabla \equiv \partial(\;)/\partial \mathbf{x}$ is the gradient operator in the current configuration, $\mathbf{n}$ the outward normal to S and $\mathrm{d}a$ the corresponding surface element (see Fig. 2a). The arbitrariness of the surface S results in the following pointwise equation for $\mathbf{h}$:

$$\nabla \times \mathbf{h} = \mathbf{0}. \tag{2.3}$$

In the case of a surface of discontinuity in $\mathbf{h}$, a standard application of Eq. (2.1) (e.g. Kovetz, 2000) in a closed loop with parts just above and just below the surface of

[4] Bold letters represent tensors while normal script is used for scalars.

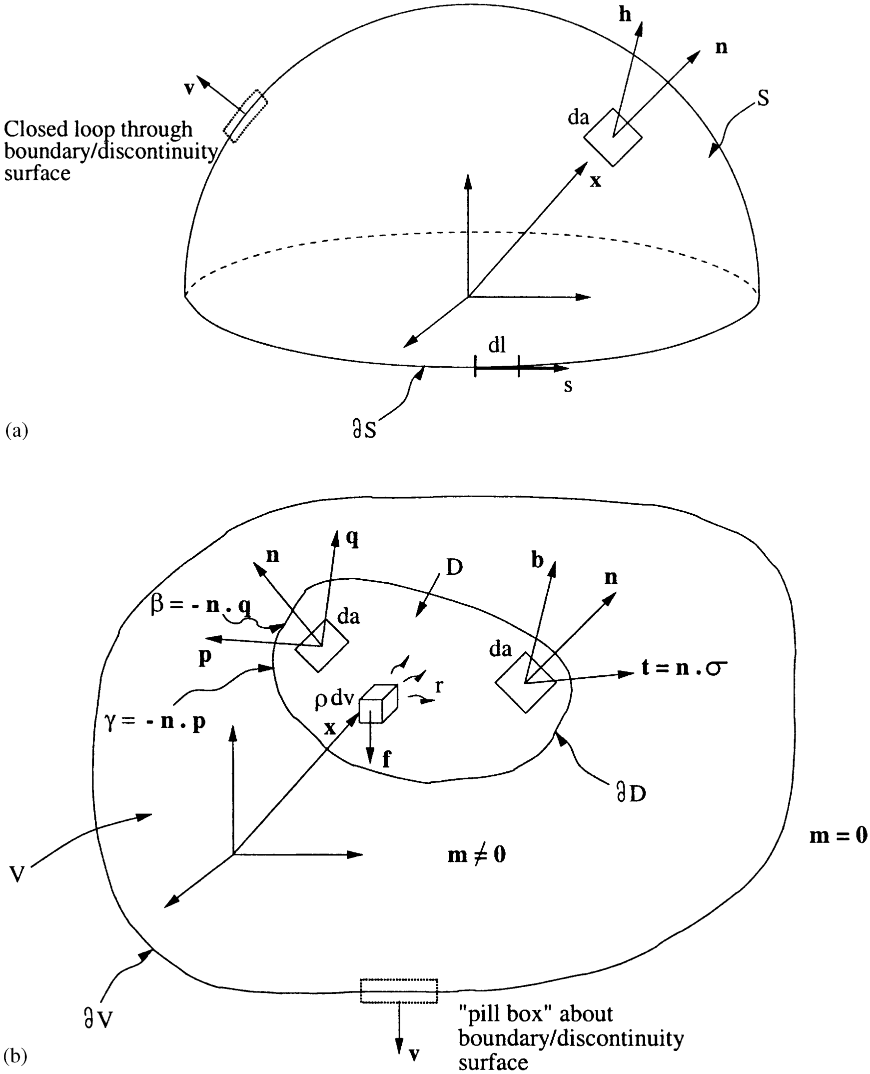

Fig. 2. Schematic diagrams for the balance laws. A surface S bounded by a closed curve ∂S is shown in (a) while an arbitrary volume V bounded by a closed surface ∂V is shown in (b). An arbitrary discontinuity surface moves through the material with velocity $\mathbf{v}$, and on the solid's boundary $\mathbf{v} = \dot{\mathbf{x}}$ the velocity of the surface material points.

discontinuity gives the following interface discontinuity/boundary condition:

$$\mathbf{n} \times [\![\mathbf{h}]\!] = \mathbf{0}, \tag{2.4}$$

where $[\![f]\!] \equiv f_+ - f_-$, is the difference of the field quantity f evaluated at either side of the discontinuity surface. In the derivation of Eq. (2.4) it is tacitly assumed that the surface of discontinuity has no surface currents and it does not move with respect to the material (or equivalently, that there is no electric displacement in the solid).

2.2. Absence of magnetic monopole

In the case of magnetostatics, i.e. in the absence of time varying electromagnetic fields, this condition cannot be derived, but has to be postulated ab initio. Assuming that D is an arbitrary closed volume with surface ∂D, the absence of magnetic monopole assumption states that

$$\int_{\partial D} \mathbf{b} \bullet \mathbf{n}\, \mathrm{d}a = 0, \tag{2.5}$$

$\mathbf{b}$ is the magnetic flux, $\mathbf{n}$ the outward normal and $\mathrm{d}a$ the corresponding surface area at a material point on ∂D (see Fig. 2b). Assuming adequate smoothness on $\mathbf{b}$, Gauss' theorem dictates that

$$\int_{D} \nabla \bullet \mathbf{b}\, \mathrm{d}v = 0. \tag{2.6}$$

The arbitrariness of the volume D results in the following pointwise equation for $\mathbf{b}$

$$\nabla \bullet \mathbf{b} = 0. \tag{2.7}$$

In the case of a surface of discontinuity in $\mathbf{b}$, a standard application of Eq. (2.5) (e.g. Kovetz, 2000) in a "pill box" with parts just above and just below the surface of discontinuity (see Fig. 2b) gives the following interface discontinuity/boundary condition:

$$\mathbf{n} \bullet [\![\mathbf{b}]\!] = 0. \tag{2.8}$$

To (pointwise) Eqs. (2.3) and (2.7) and interface discontinuity/boundary conditions (2.4) and (2.8), one must also add the $\mathbf{b}$–$\mathbf{h}$ relationship:

$$\mathbf{b} = \mu_0(\mathbf{h} + \mathbf{m}), \tag{2.9}$$

where $\mathbf{m}$ is the magnetization per unit current volume (in $\mathrm{A\,m^{-1}}$), a material property, and μ_0 the magnetic permeability of vacuum ($\mu_0 = 4\pi 10^{-7}\ \mathrm{N\ A^{-2}}$). The relation between $\mathbf{h}$ and $\mathbf{m}$ depends on the properties of the solid under investigation. For the path-independent, magnetoelastic solids of interest here, the general form of the constitutive relation which gives $\mathbf{h}$ in terms of $\mathbf{m}$ and the deformation gradient $\mathbf{F}$ will be derived in the sequel.

2.3. Mass conservation

The next postulate pertains to the conservation of mass $\mathscr{M}$ of an arbitrary body of volume D (see Fig. 2b) namely

$$\dot{\mathscr{M}} = 0, \quad \mathscr{M} \equiv \int_D \rho\, \mathrm{d}v, \tag{2.10}$$

where $\dot{f}$ denotes the material time derivative (total time derivative associated with the change of f defined over a fixed material point) of the quantity f. To be able to take the time derivative inside the integrand, we must first rewrite $\mathscr{M}$ in the reference configuration (recall $\mathrm{d}v = J\,\mathrm{d}V$) thus transforming Eq. (2.10) into

$$\begin{aligned}\dot{\mathscr{M}} &= \int_D (\rho J)^{\bullet}\,\mathrm{d}V \\ &= \int_D [\dot{\rho} + \rho(\nabla \bullet \dot{\mathbf{x}})] J\,\mathrm{d}V \\ &= \int_D [\dot{\rho} + \rho(\nabla \bullet \dot{\mathbf{x}})]\,\mathrm{d}v = 0,\end{aligned} \tag{2.11}$$

where the following kinematic relations were also used:

$$\dot{J} = J(\nabla \bullet \dot{\mathbf{x}}), \quad J \equiv \det \mathbf{F}, \quad \mathbf{F} \equiv \mathbf{x}\boldsymbol{\nabla}, \tag{2.12}$$

where $\boldsymbol{\nabla} \equiv \partial(\,)/\partial\mathbf{X}$ is the gradient operator in the reference configuration. From here on, the usual continuum mechanics convention is employed according to which all field quantities in capital letters are associated with the reference configuration while their counterparts in small letters are associated with the current configuration. Due to the arbitrariness of volume D, the first and the last terms in Eq. (2.11) result in the following pointwise form of the mass conservation equation:

$$\dot{\boldsymbol{\rho}} = \dot{\rho} + \rho(\nabla \bullet \dot{\mathbf{x}}) = 0, \quad \boldsymbol{\rho} \equiv \rho J, \tag{2.13}$$

with $\boldsymbol{\rho}$ denoting the reference density of the solid. The surface discontinuity condition associated with mass conservation is again found by the application of (2.10) on a thin “pill box” volume constructed on both sides of the discontinuity surface, which moves at a velocity $\mathbf{v}$ (see Fig. 2b), yielding the following interface discontinuity/boundary condition:

$$\mathbf{n} \bullet [\![\rho(\dot{\mathbf{x}} - \mathbf{v})]\!] = 0. \tag{2.14}$$

2.4. Balance of linear momentum

This postulate states that the time rate of change of linear momentum of an arbitrary body is equal to the sum of all the forces exerted inside it (sum of body forces) and on its surface (sum of surface tractions). For the case of continuum magnetoelasticity (or electrodynamics more generally) there are several different ways of defining stresses, tractions and body forces, leading to a somewhat confusing state of affairs. To this end, we adopt the approach of Kovetz (2000) who defines a total Cauchy stress measure $\boldsymbol{\sigma}$ which includes both mechanical and electromagnetic contributions, a body force $\mathbf{f}$ per unit mass, which is not specified at this stage, but it will turn out to be non-electromagnetic in nature (e.g. gravitational or chemical) and a surface traction vector $\mathbf{t}$ per unit current area (see Fig. 2b). Since $\boldsymbol{\sigma}$ includes electromagnetic contributions, there is no need to introduce electromagnetic body forces and couples. The balance of linear momentum $\mathscr{L}$ of a volume D subjected to an external force $\mathscr{F}$, which is the sum of body forces $\mathbf{f}$ and surface tractions $\mathbf{t}$, reads

$$\dot{\mathscr{L}} = \mathscr{F}, \quad \mathscr{L} \equiv \int_D \rho \dot{\mathbf{x}} \, \mathrm{d}v, \quad \mathscr{F} \equiv \int_D \rho \mathbf{f} \, \mathrm{d}v + \int_{\partial D} \mathbf{t} \, \mathrm{d}a. \tag{2.15}$$

To find the pointwise form of the linear momentum balance one must postulate in addition to Eq. (2.15) the relation between the current traction $\mathbf{t}$ at a surface element with normal $\mathbf{n}$ and the total (Cauchy) stress $\boldsymbol{\sigma}$, namely the well known Cauchy tetrahedron relation:

$$\mathbf{t} = \mathbf{n} \bullet \boldsymbol{\sigma}. \tag{2.16}$$

By exploiting the mass conservation results in Eq. (2.13) to bring the time derivative inside the integrand in $\mathscr{L}$ as well as Gauss' divergence theorem to convert—with the help of Eq. (2.16)—the surface term in $\mathscr{F}$ into a volume term (assuming of course adequate continuity in $\boldsymbol{\sigma}$), one can rewrite Eq. (2.15) as follows:

$$\dot{\mathscr{L}} = \int_D \rho \ddot{\mathbf{x}} \, \mathrm{d}v = \int_D \rho \mathbf{f} \, \mathrm{d}v + \int_D (\boldsymbol{\nabla} \bullet \boldsymbol{\sigma}) \, \mathrm{d}v = \mathscr{F}. \tag{2.17}$$

Due to the arbitrariness of D, the pointwise form of the linear momentum balance is

$$\rho \ddot{\mathbf{x}} = \boldsymbol{\nabla} \bullet \boldsymbol{\sigma} + \rho \mathbf{f}. \tag{2.18}$$

Once more by now familiar application of Eq. (2.15) on a "pill box" volume about the surface of discontinuity (see Fig. 2b), which moves with velocity $\mathbf{v}$ and is subjected to a traction $\mathbf{t}$ at the point of interest, yields the following interface discontinuity/boundary condition: [5]

$$\mathbf{n} \bullet [\![\rho(\dot{\mathbf{x}} - \mathbf{v})\dot{\mathbf{x}} - \boldsymbol{\sigma}]\!] + \mathbf{t} = 0. \tag{2.19}$$

2.5. Balance of angular momentum

This postulate states that the time rate of change of angular momentum of an arbitrary body is equal to the moment of all the forces, both body and surface, exerted on it. The angular momentum of the body as well as the moments of all the forces are taken with respect to a fixed point, which without loss of generality is taken to be the origin of the coordinate system. Consequently, the balance of angular momentum $\mathscr{A}$ of a volume D, subjected to external moment $\mathscr{M}$, which is the sum of a body force term and a surface traction term reads

$$\dot{\mathscr{A}} = \mathscr{M}, \quad \mathscr{A} \equiv \int_D \mathbf{x} \wedge (\rho \dot{\mathbf{x}}) \, \mathrm{d}v, \quad \mathscr{M} \equiv \int_D \mathbf{x} \wedge (\rho \mathbf{f}) \, \mathrm{d}v + \int_{\partial D} \mathbf{x} \wedge \mathbf{t} \, \mathrm{d}a, \tag{2.20}$$

where $\mathbf{a} \wedge \mathbf{b} \equiv \mathbf{ab} - \mathbf{ba}$ denotes the exterior product of two arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$. To find the pointwise form of the angular momentum balance, one again needs the mass conservation equation (2.13) to bring the time derivative inside the integrand in $\mathscr{A}$ and Gauss' divergence theorem to convert—with the help of Cauchy tetrahedron relation

[5] Here the interface discontinuity and boundary conditions are treated in a uniform manner. When a boundary condition is concerned $\dot{\mathbf{x}} = \mathbf{v}$ and the traction on the boundary $\mathbf{t} \neq \mathbf{0}$. In the case of a propagating discontinuity surface, $\dot{\mathbf{x}} \neq \mathbf{v}$ but no traction is applied (i.e. $\mathbf{t} = \mathbf{0}$).

(2.16)—the surface term in $\mathscr{M}$ into a volume term (assuming of course adequate continuity in $\boldsymbol{\sigma}$), thus obtaining

$$\dot{\mathscr{A}} = \int_D [\dot{\mathbf{x}} \wedge (\rho\dot{\mathbf{x}}) + \mathbf{x} \wedge (\rho\ddot{\mathbf{x}})]\, \mathrm{d}v = \int_D \mathbf{x} \wedge (\rho\mathbf{f})\, \mathrm{d}v + \int_D [\mathbf{x} \wedge (\boldsymbol{\nabla} \bullet \boldsymbol{\sigma}) + \boldsymbol{\sigma}^{\mathrm{T}} - \boldsymbol{\sigma}]\, \mathrm{d}v = \mathscr{M}. \tag{2.21}$$

Using in Eq. (2.21) the pointwise linear balance law (2.18) and noticing that $\dot{\mathbf{x}} \wedge (\rho\dot{\mathbf{x}}) = 0$, one obtains because of the arbitrariness of D

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}. \tag{2.22}$$

The symmetry of the current stress measure $\boldsymbol{\sigma}$ used in this approach is a consequence of the assumption that it contains the mechanical as well as the electromagnetic contributions, and is one of the reasons for our preference of Kovetz's (2000) elegant approach (in contrast to other theories which result in non-symmetric stress measures).

2.6. Energy balance

This postulate states that the rate of change of the total energy $\mathscr{E}$, defined as the integral of the total specific energy (per mass) ε, contained in an arbitrary body of volume D is the sum of three contributions: a mechanical power $\mathscr{P}$, due to body forces $\mathbf{f}$ and surface tractions $\mathbf{t}$, a thermal power $\mathscr{Q}$ due to internal heat sources r and surface heat flux β and an electromagnetic contribution $\mathscr{R}$ due to an electromagnetic surface energy flux γ, i.e.

$$\begin{aligned}
&\dot{\mathscr{E}} = \mathscr{P} + \mathscr{Q} + \mathscr{R}, \quad \mathscr{E} \equiv \int_D \rho\varepsilon\, \mathrm{d}v,\\
&\mathscr{P} \equiv \int_D \dot{\mathbf{x}} \bullet (\rho\mathbf{f})\, \mathrm{d}v + \int_{\partial D} \dot{\mathbf{x}} \bullet \mathbf{t}\, \mathrm{d}a,\\
&\mathscr{Q} \equiv \int_D \rho r\, \mathrm{d}v + \int_{\partial D} \beta\, \mathrm{d}a, \quad \beta \equiv -\mathbf{n} \bullet \mathbf{q},\\
&\mathscr{R} \equiv \int_{\partial D} \gamma\, \mathrm{d}a, \quad \gamma \equiv -\mathbf{n} \bullet \mathbf{p}, \quad \mathbf{p} \equiv (\dot{\mathbf{x}} \times \mathbf{b}) \times \mathbf{h}.
\end{aligned} \tag{2.23}$$

Both the heat flux β and the electromagnetic energy flux γ are taken positive when the energy flows into the volume D (see Fig. 2b). The vector $\mathbf{q}$ denotes the heat flux vector and the vector $\mathbf{p}$ denotes the electromagnetic energy flux, also called "Poynting vector", whose expression for the case of the absence of an electric field is given in Eq. $(2.23)_8$ (see Kovetz, 2000). It is also assumed that the time dependence of all field quantities is due to the time dependence of their position vector which, even in the absence of time varying fields, is why $\mathbf{p} \neq \mathbf{0}$. Both energy flux vectors $\mathbf{p}$ and $\mathbf{q}$ are assumed positive when energy flows out of the body (see Fig. 2b) thus justifying the negative sign in definitions $(2.23)_5$ and $(2.23)_7$.

Using the mass conservation relation (2.13) to pass the time derivative inside the integrand of $\mathscr{E}$, the Cauchy relation (2.16) for rewriting the surface term in the mechanical power contribution in Eq. $(2.23)_2$ and subsequently applying Gauss' divergence theorem in all the surface integrals appearing in Eq. (2.23) one has

$$\int_D \rho\dot{\varepsilon}\,\mathrm{d}v = \int_D [\dot{\mathbf{x}} \bullet (\rho\mathbf{f}) + \rho r]\,\mathrm{d}v + \int_D [\boldsymbol{\nabla} \bullet (\boldsymbol{\sigma} \bullet \dot{\mathbf{x}} - \mathbf{q} - \mathbf{p})]\,\mathrm{d}v. \tag{2.24}$$

In view of the arbitrariness of D, Eq. (2.24) yields the following pointwise form of the energy balance:

$$\rho\dot{\varepsilon} = \rho(\mathbf{b} \bullet \dot{\mathbf{x}} + r) + \boldsymbol{\nabla} \bullet (\boldsymbol{\sigma} \bullet \dot{\mathbf{x}} - \mathbf{q} - \mathbf{p}). \tag{2.25}$$

For the case of a surface of discontinuity moving with velocity $\mathbf{v}$, by applying again Eq. (2.23) in a standard "pillbox" volume about this surface (see Fig. 2b), one finds the following interface discontinuity/boundary condition:

$$\mathbf{n} \bullet [\![\rho\varepsilon(\dot{\mathbf{x}} - \mathbf{v}) - \boldsymbol{\sigma} \bullet \dot{\mathbf{x}} + \mathbf{q} + \mathbf{p}]\!] = 0. \tag{2.26}$$

As it turns out, the pointwise energy balance can be put in a more convenient form for subsequent use. To this end, with the help of the vector identities:

$$\begin{aligned}
&\boldsymbol{\nabla} \bullet (\boldsymbol{\sigma} \bullet \dot{\mathbf{x}}) = (\boldsymbol{\nabla} \bullet \boldsymbol{\sigma}) \bullet \dot{\mathbf{x}} + \boldsymbol{\sigma} \bullet\bullet (\dot{\mathbf{x}}\boldsymbol{\nabla}),\\
&\boldsymbol{\nabla} \bullet (\mathbf{c} \times \mathbf{d}) = \mathbf{d} \bullet (\boldsymbol{\nabla} \times \mathbf{c}) - \mathbf{c} \bullet (\boldsymbol{\nabla} \times \mathbf{d}), \quad \mathbf{c} \equiv \dot{\mathbf{x}} \times \mathbf{b}, \quad \mathbf{d} \equiv \mathbf{h},\\
&\boldsymbol{\nabla} \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{c}\boldsymbol{\nabla}) \bullet \mathbf{d} - (\mathbf{d}\boldsymbol{\nabla}) \bullet \mathbf{c} + \mathbf{c}(\boldsymbol{\nabla} \bullet \mathbf{d}) - \mathbf{d}(\boldsymbol{\nabla} \bullet \mathbf{c}),\\
&\quad \mathbf{c} \equiv \dot{\mathbf{x}}, \quad \mathbf{d} \equiv \mathbf{b}
\end{aligned} \tag{2.27}$$

and recalling the pointwise expressions of the equilibrium equation (2.18) and Ampère's law (2.3), the energy balance (2.25) can be rewritten as [6]

$$\begin{aligned}
\rho\dot{\varepsilon} = {} & [\boldsymbol{\sigma} - \mathbf{bh} + (\mathbf{h} \bullet \mathbf{b})\mathbf{I}] \bullet\bullet (\dot{\mathbf{x}}\boldsymbol{\nabla}) + \rho(\dot{\mathbf{x}} \bullet \ddot{\mathbf{x}} + r)\\
& - \boldsymbol{\nabla} \bullet \mathbf{q} + \mathbf{h} \bullet (\mathbf{b}\boldsymbol{\nabla}) \bullet \dot{\mathbf{x}},
\end{aligned} \tag{2.28}$$

where $\mathbf{I}$ is the identity tensor.

2.7. Entropy production inequality

The second law of thermodynamics states that the entropy $\mathscr{H}$ of an arbitrary body of volume D, defined as the integral of the specific entropy (per mass) η increases at least as rapidly as the sum of the volume and surface heating, each divided by the absolute temperature θ at which it takes place, namely

$$\dot{\mathscr{H}} \geqslant \int_D \rho r\theta^{-1}\,\mathrm{d}v + \int_{\partial D} \beta\theta^{-1}\,\mathrm{d}a, \quad \mathscr{H} \equiv \int_D \rho\eta\,\mathrm{d}v. \tag{2.29}$$

[6] From here on the double contraction notation will be used $\mathbf{A} \bullet\bullet \mathbf{B} \equiv A_{ij}B_{ji}$.

Recalling the definition for the surface heat flux β in Eq. $(2.23)_5$ and the mass conservation (2.13) to pass the entropy derivative inside the integral and then invoking Gauss' divergence theorem to convert the surface to a volume term, Eq. (2.29) is rewritten as

$$\dot{\mathscr{H}} = \int_D \rho\dot{\eta}\,\mathrm{d}v \geqslant \int_D \rho r\theta^{-1}\,\mathrm{d}v - \int_D [\boldsymbol{\nabla}\bullet(\mathbf{q}\theta^{-1})]\,\mathrm{d}v. \tag{2.30}$$

The arbitrariness of volume D leads to the following pointwise form of the entropy inequality, in which we have also used the pointwise form of the energy balance (2.28) (to eliminate the heat source term r) plus the identity $\dot{\mathbf{b}} = (\mathbf{b}\boldsymbol{\nabla})\bullet\dot{\mathbf{x}}$ (since $\partial\mathbf{b}/\partial t = 0$, see remark following Eq. (2.23)):

$$\begin{aligned}&\rho\ (\theta\dot{\eta} - \dot{\varepsilon} + \dot{\mathbf{x}}\bullet\ddot{\mathbf{x}}) + [\boldsymbol{\sigma} - \mathbf{bh} + (\mathbf{h}\bullet\mathbf{b})\mathbf{I}]\bullet\bullet(\dot{\mathbf{x}}\boldsymbol{\nabla}) + \mathbf{h}\bullet\dot{\mathbf{b}}\\ &\quad -[\mathbf{q}\bullet(\boldsymbol{\nabla}\theta)]\theta^{-1} \geqslant 0.\end{aligned} \tag{2.31}$$

Once more, in case of a discontinuous surface moving with velocity $\mathbf{v}$, the application of Eq. (2.29) to a "pillbox" volume about this surface results in the following interface discontinuity/boundary condition:

$$\mathbf{n}\bullet[\![\rho\eta(\dot{\mathbf{x}} - \mathbf{v}) + \mathbf{q}\theta^{-1}]\!] \geqslant 0. \tag{2.32}$$

At this point all the ingredients have been assembled to derive the constitutive equations of the magnetoelastic solid using the method introduced by Coleman and Noll (1963) for thermoelastic solids and extended by Kovetz (2000) to the continuum electro-magneto-thermoelastic case.

The specific total energy ε consists of two parts: the specific internal energy u plus the macroscopic kinetic energy $1/2\dot{\mathbf{x}}\bullet\dot{\mathbf{x}}$:

$$\varepsilon = u + \tfrac{1}{2}\,\dot{\mathbf{x}}\bullet\dot{\mathbf{x}}. \tag{2.33}$$

The specific internal energy u of a magnetoelastic material is made up of the following contributions: a thermal contribution $\eta\theta$, a magnetic energy contribution $(\mu_0/2\rho)\ \mathbf{h}\bullet\mathbf{h}$ and a Helmholtz free energy $\psi(\mathbf{F},\mathbf{m},\theta)$[7] is a function of the deformation gradient $\mathbf{F}$ (see definition in Eq. (2.12)) and the magnetization per unit current volume $\mathbf{m}$ (see definition in Eq. (2.9)), i.e.

$$u = \psi(\mathbf{F},\mathbf{m},\theta) + \theta\eta + \frac{\mu_0}{2\rho}\,\mathbf{h}\bullet\mathbf{h}. \tag{2.34}$$

The existence of a Helmholtz free energy ψ is based on the assumption that there are no hysteretic or rate effects in the magnetoelastic solid and that there is no energy dissipation in a closed loading loop in strain and magnetization space under fixed temperature. By substituting the above expressions (2.33) and (2.34) for the total specific energy ε into the entropy inequality (2.31) and recalling the mass conservation (2.13),

[7] In the context of MREs, we ignore the dependence of the free energy on magnetization gradients (see Borcea and Bruno, 2001).

one obtains

$$\left[\left(\frac{\partial\psi}{\partial\mathbf{F}}\bullet\mathbf{F}^{\mathrm{T}}\right)^{\mathrm{T}}+\boldsymbol{\sigma}-\mathbf{bh}+\mu_0\left(\mathbf{h}\bullet\mathbf{m}+\frac{1}{2}\mathbf{h}\bullet\mathbf{h}\right)\mathbf{I}\right]\bullet\bullet(\dot{\mathbf{x}}\boldsymbol{\nabla})$$
$$-\left[\rho\frac{\partial\psi}{\partial\theta}+\eta\right]\dot{\theta}+\left[-\rho\frac{\partial\psi}{\partial\mathbf{m}}+\mu_0\mathbf{h}\right]\bullet\dot{\mathbf{m}}-[\mathbf{q}\bullet(\boldsymbol{\nabla}\theta)]\theta^{-1}\geqslant 0. \tag{2.35}$$

Given the arbitrariness in the choice of the velocity gradient $\dot{\mathbf{x}}\boldsymbol{\nabla}$, the temperature change $\dot{\theta}$ and the magnetization change $\dot{\mathbf{m}}$, the entropy inequality dictates that the coefficients of $\dot{\mathbf{x}}\boldsymbol{\nabla}$, $\dot{\theta}$, and $\dot{\mathbf{m}}$ must vanish, giving the following results for the total stress, the magnetic field and specific entropy:

$$\boldsymbol{\sigma}^{\mathrm{T}}=\rho\frac{\partial\psi}{\partial\mathbf{F}}\bullet\mathbf{F}^{\mathrm{T}}+\mathbf{hb}-\mu_0\left(\mathbf{h}\bullet\mathbf{m}+\frac{1}{2}\mathbf{h}\bullet\mathbf{h}\right)\mathbf{I},$$
$$\mu_0\mathbf{h}=\rho\frac{\partial\psi}{\partial\mathbf{m}},$$
$$\eta=-\frac{\partial\psi}{\partial\theta}. \tag{2.36}$$

The contribution to the general stress measure that does not depend on the $\mathbf{F}$ derivative of the free energy is termed by some authors (e.g. Tiersten, 1964) as the "Maxwell stress", a definition that will also be adopted here. Different choices of arguments of the free energy result in different Maxwell stresses as subsequently discussed. [8]

At first sight it seems that the result for the stress measure in Eq. $(2.36)_1$ is in contradiction with the balance of angular momentum (2.22) that dictates a symmetric total stress. Fortunately, the material frame indifference (objectivity) of ψ comes to the rescue. Indeed, the invariance of ψ under any orthogonal transformation $\mathbf{Q}$, namely

$$\psi(\mathbf{Q}\bullet\mathbf{F},\mathbf{Q}\bullet\mathbf{m})=\psi(\mathbf{F},\mathbf{m}), \tag{2.37}$$

leads, with the help of the polar decomposition theorem by taking $\mathbf{F}=\mathbf{Q}\bullet\mathbf{U}$ ($\mathbf{U}$ is the right stretch tensor) to the following result for ψ:

$$\psi=\psi(\mathbf{C},\mathbf{F}^{\mathrm{T}}\bullet\mathbf{m}),\qquad \mathbf{C}\equiv\mathbf{F}^{\mathrm{T}}\bullet\mathbf{F}, \tag{2.38}$$

which, as it can be easily checked, automatically satisfies the objectivity relation (2.37). ($\mathbf{C}$ is the right "Cauchy–Green" tensor of continuum mechanics). From Eq. (2.38), we obtain two useful identities:

$$\frac{\partial\psi}{\partial\mathbf{F}}=2\mathbf{F}\bullet\frac{\partial\psi}{\partial\mathbf{C}}+\mathbf{m}\frac{\partial\psi}{\partial(\mathbf{F}^{\mathrm{T}}\bullet\mathbf{m})},\qquad \frac{\partial\psi}{\partial\mathbf{m}}=\frac{\partial\psi}{\partial(\mathbf{F}^{\mathrm{T}}\bullet\mathbf{m})}\bullet\mathbf{F}^{\mathrm{T}}. \tag{2.39}$$

[8] Notice that in vacuum ($\psi=0$, $\mathbf{m}=\mathbf{0}$), the total stress is non-zero and equals the Maxwell stress $\mu_0[\mathbf{hh}-(1/2)(\mathbf{h}\bullet\mathbf{h})\mathbf{I}]$.

With the help of Eqs. (2.39) and $(2.36)_2$, expression $(2.36)_1$ for the total stress $\boldsymbol{\sigma}$ becomes

$$\boldsymbol{\sigma}^{\mathrm{T}} = \rho \left(2\mathbf{F} \bullet \frac{\partial \psi}{\partial \mathbf{C}} \bullet \mathbf{F}^{\mathrm{T}} \right) + \mu_0 \left[\mathbf{mh} + \mathbf{hm} + \mathbf{hh} - \left(\mathbf{h} \bullet \mathbf{m} + \frac{1}{2} \mathbf{h} \bullet \mathbf{h} \right) \mathbf{I} \right] = \boldsymbol{\sigma}, \tag{2.40}$$

which is obviously a symmetric rank two tensor as expected (recall that the right Cauchy–Green tensor is symmetric, as easily follows from its definition in Eq. $(2.38)_2$).

At this point all the differential equations and their corresponding boundary/interface conditions in the current configuration for a temperature-dependent magnetoelastic solid have been presented. As it turns out, in the variational formulation of the magnetoelastic boundary value problem, it is convenient to use the magnetization per unit mass $\mathbf{M}$ instead of the magnetization per unit volume $\mathbf{m}$ which has been employed up to this point. In the interest of comparing the governing equations of the direct and energy approach, it is necessary to present our previous results in terms of the Helmholtz free energy $\widehat{\psi}$ expressed this time in terms of the specific magnetization $\mathbf{M}$:

$$\widehat{\psi}(\mathbf{F}, \mathbf{M}) = \widehat{\psi}(\mathbf{F}, \rho^{-1}\mathbf{m}) = \psi(\mathbf{F}, \mathbf{m}), \qquad \mathbf{m} = \rho \mathbf{M}. \tag{2.41}$$

The constitutive relation for the magnetic field $\mathbf{h}$ $(2.36)_2$ can be thus rewritten in terms of $\widehat{\psi}$ as

$$\mu_0 \mathbf{h} = \frac{\partial \widehat{\psi}}{\partial \mathbf{M}}. \tag{2.42}$$

The derivation of the constitutive relation for the stress $\boldsymbol{\sigma}$ in terms of $\widehat{\psi}$ requires first the help of mass conservation $\rho J = \mathcal{P}$ (see Eq. (2.13)) and the kinematics relation $J = \det \mathbf{F}$ (see Eq. (2.12)) to establish the intermediate result

$$\frac{\partial \rho^{-1}}{\partial \mathbf{F}} = \rho^{-1} \mathbf{F}^{-\mathrm{T}}. \tag{2.43}$$

With the help of Eqs. (2.42) and (2.43), the constitutive equation for $\boldsymbol{\sigma}$ in Eq. $(2.36)_1$ is rewritten as

$$\boldsymbol{\sigma}^{\mathrm{T}} = \rho \frac{\partial \widehat{\psi}}{\partial \mathbf{F}} \bullet \mathbf{F}^{\mathrm{T}} + \mathbf{hb} - \frac{\mu_0}{2} (\mathbf{h} \bullet \mathbf{h}) \mathbf{I}. \tag{2.44}$$

The pointwise equation of motion for the magnetoelastic solid (2.18) can thus be written in terms of $\widehat{\psi}$, with the help of Eqs. (2.44), (2.7) and (2.3):

$$\rho \ddot{\mathbf{x}} = \left(\rho \frac{\partial \widehat{\psi}}{\partial \mathbf{F}} \bullet \mathbf{F}^{\mathrm{T}} \right) \bullet \boldsymbol{\nabla} + \mu_0 (\mathbf{h} \boldsymbol{\nabla}) \bullet \mathbf{m} + \rho \mathbf{f}. \tag{2.45}$$

Note the appearance of the second term $\mu_0(\mathbf{h}\boldsymbol{\nabla}) \bullet \mathbf{m}$ in addition to the body force $\rho \mathbf{f}$ postulated ab initio. Several authors postulate the existence of a magnetic body force term $\mu_0(\mathbf{h}\boldsymbol{\nabla}) \bullet \mathbf{m}$ (e.g. Tiersten, 1964) but the approach followed here gives this term from the divergence of the general stress measure $\boldsymbol{\sigma}$ which contains magnetic contributions. We can now support our earlier assertion that in view of the assumption made, the body force $\mathbf{f}$ does not contain magnetic contributions.

Also of interest is the surface traction boundary condition corresponding to the above equation, expressed in terms of $\widehat{\psi}$. To this end, one notices that the boundary of a magnetoelastic solid is a surface of discontinuity moving with a velocity $\mathbf{v} = \dot{\mathbf{x}}$, the material velocity of its surface points, thus obtaining from Eq. (2.19):

$$\mathbf{n} \bullet [\![\boldsymbol{\sigma}]\!] = \mathbf{t}. \tag{2.46}$$

Since the discontinuity surface coincides with the boundary ∂D of the solid, on the material side $\widehat{\psi} \neq 0$, $\mathbf{m} \neq 0$ while on the free space side $\widehat{\psi} = 0$, $\mathbf{m} = 0$. Using this information plus the interface conditions (2.4) and (2.8) one obtains the following expression for $[\![\mathbf{h}]\!]$:

$$[\![\mathbf{h}]\!] = (\mathbf{m} \bullet \mathbf{n})\mathbf{n}, \tag{2.47}$$

which in conjunction with (2.44) allows (2.46) to be expressed in terms of $\widehat{\psi}$ as

$$\mathbf{t} = \rho \left(\frac{\partial \widehat{\psi}}{\partial \mathbf{F}} \bullet \mathbf{F}^{\mathrm{T}} \right) \bullet \mathbf{n} - \frac{\mu_0}{2} (\mathbf{m} \bullet \mathbf{n})^2 \mathbf{n}, \tag{2.48}$$

thus completing the task of expressing the equations of motion and accompanying surface traction boundary conditions in terms of $\widehat{\psi}(\mathbf{F}, \mathbf{M})$.

Table 1 is presented to help the reader see the influence of the choice of arguments in the free energy on the Maxwell stress, magnetic body forces and traction boundary conditions. It is assumed that $\mathbf{n}$ is the outward normal acting on the boundary of the solid; the $(-)$ subscript indicates interior of the solid while $(+)$ subscript is used to indicate the surrounding space (e.g. $\mathbf{m}_+ = \mathbf{0}$, $\mathbf{m}_- = \mathbf{m}$).

3. Energy approach (reference configuration formulation)

The goal of this section is the alternative derivation of differential equations and boundary conditions for a magnetoelastic solid as the Euler–Lagrange equations of an energy minimization principle. In addition it will be proved that they coincide with their counterparts obtained by the direct approach in the previous section. For simplicity (and without loss in generality) kinetic energy and thermal effects are ignored in this section.

In contrast to finite elasticity, a magnetoelastic solid not only stores energy inside the volume V it occupies but its presence changes the magnetic field of the free space around it. Hence the total energy, $\mathscr{E}$ (see definitions $(2.23)_2$, (2.33) and (2.34)) is the sum of the solid's free energy plus the magnetic energy of the entire space:

$$\mathscr{E} = \int_{\mathbb{R}^3} \rho u \, \mathrm{d}v = \int_V \rho \widehat{\psi} \, \mathrm{d}v + \int_{\mathbb{R}^3} \frac{\mu_0}{2} (\mathbf{h} \bullet \mathbf{h}) \, \mathrm{d}v, \tag{3.1}$$

since only quasistatic and isothermal processes are considered here.

At this stage it is important to separate the magnetic field $\mathbf{h}$ into the externally applied field $\mathbf{h}_0$ plus the perturbation field $\mathbf{h}_1$ due to the presence of the magnetoelastic solid, namely

$$\mathbf{h} = \mathbf{h}_0 + \mathbf{h}_1. \tag{3.2}$$

Table 1
Summary of possible forms of constitutive equations, body forces, and traction conditions

Form of free energy	"Maxwell" stress (part of total stress measure $\boldsymbol{\sigma}^{\mathrm{T}}$)	Magnetic body forces	Traction boundary conditions	Magnetic constitutive equations
$\psi(\mathbf{F},\mathbf{m})$	$\mathbf{hb}-\mu_0\left(\mathbf{h}\bullet\mathbf{m}+\dfrac{h^2}{2}\right)\mathbf{I}$	$-\mu_0\mathbf{h}\bullet(\mathbf{m}\boldsymbol{\nabla})$	$\mathbf{t}=\rho\left(\dfrac{\partial\psi}{\partial\mathbf{F}}\bullet\mathbf{F}^{\mathrm{T}}\right)\bullet\mathbf{n}-$ $\mu_0\left(\dfrac{1}{2}(\mathbf{m}\bullet\mathbf{n})^2+\mathbf{h}_-\bullet\mathbf{m}\right)\mathbf{n}$	$\mu_0\mathbf{h}=\rho\,\dfrac{\partial\psi}{\partial\mathbf{m}}$
$\widehat{\psi}(\mathbf{F},\mathbf{M})$[1]	$\mathbf{hb}-\dfrac{\mu_0}{2}\,h^2\mathbf{I}$	$\mu_0(\mathbf{h}\boldsymbol{\nabla})\bullet\mathbf{m}$	$\mathbf{t}=\rho\left(\dfrac{\partial\widehat{\psi}}{\partial\mathbf{F}}\bullet\mathbf{F}^{\mathrm{T}}\right)\bullet\mathbf{n}-$ $\dfrac{\mu_0}{2}(\mathbf{m}\bullet\mathbf{n})^2\mathbf{n}$	$\mu_0\mathbf{h}=\dfrac{\partial\widehat{\psi}}{\partial\mathbf{M}}$
$\phi(\mathbf{F},\mathbf{b})$[2]	$\mathbf{hb}-\dfrac{\mu_0}{2}(h^2-m^2)\mathbf{I}$	$\mathbf{m}\bullet(\mathbf{b}\boldsymbol{\nabla})$	$\mathbf{t}=\rho\left(\dfrac{\partial\phi}{\partial\mathbf{F}}\bullet\mathbf{F}^{\mathrm{T}}\right)\bullet\mathbf{n}-$ $\mu_0\left(\dfrac{1}{2}(\mathbf{m}\bullet\mathbf{n})^2+\mathbf{m}\bullet\mathbf{m}\right)\mathbf{n}$	$\mathbf{m}=-\rho\,\dfrac{\partial\phi}{\partial\mathbf{b}}$
$\widetilde{\psi}(\mathbf{F},\mathbf{h})$	$\mathbf{hb}-\dfrac{\mu_0}{2}\,h^2\mathbf{I}$	$\mu_0(\mathbf{h}\boldsymbol{\nabla})\bullet\mathbf{m}$	$\mathbf{t}=\rho\left(\dfrac{\partial\widetilde{\psi}}{\partial\mathbf{F}}\bullet\mathbf{F}^{\mathrm{T}}\right)\bullet\mathbf{n}-$ $\dfrac{\mu_0}{2}(\mathbf{m}\bullet\mathbf{n})^2\mathbf{n}$	$\mu_0\mathbf{m}=-\rho\,\dfrac{\partial\widetilde{\psi}}{\partial\mathbf{h}}$

The total stress ($\boldsymbol{\sigma}^{\mathrm{T}}$) is the sum of a free energy part ($\partial f/\partial\mathbf{F}\bullet\mathbf{F}^{\mathrm{T}}$, where f is the chosen free energy) and the Maxwell stress. Results using [1] are in agreement with Tiersten (1964), Brown (1966) and Pao and Yeh (1973); results using [2] are in agreement with Eringen and Maugin (1990) and Kovetz (2000).

According to Ampère's law in Eq. (2.3) for $\mathbf{h}$ and $\mathbf{h}_0$, both $\mathbf{h}_0$ and $\mathbf{h}_1$ are gradients of scalar functions, say α_0 and α_1. In addition since the perturbation field must vanish far away from the solid, i.e. $\|\mathbf{h}_1\|\to 0$ at $\|\mathbf{x}\|\to\infty$, it is reasonable to assume that α_1 does the same, i.e.

$$\mathbf{h}_0=-\boldsymbol{\nabla}\alpha_0,\quad \mathbf{h}_1=-\boldsymbol{\nabla}\alpha_1,\quad \alpha_1\to 0\quad \text{as } \|\mathbf{x}\|\to\infty. \tag{3.3}$$

Thus, the magnetic energy of the entire space can be rewritten as

$$\int_{\mathbb{R}^3}\frac{\mu_0}{2}(\mathbf{h}\bullet\mathbf{h})\,\mathrm{d}v=\int_{\mathbb{R}^3}\frac{\mu_0}{2}(\mathbf{h}_1\bullet\mathbf{h}_1)\,\mathrm{d}v+\int_{\mathbb{R}^3}\frac{\mu_0}{2}(\mathbf{h}_0\bullet\mathbf{h}_0)\,\mathrm{d}v, \tag{3.4}$$

since the integral of $\mathbf{h}_1\bullet\mathbf{h}_0$ over $\mathbb{R}^3$ can be shown to vanish. Indeed if $B(\|\mathbf{x}\|)$ is a sphere of radius $\|\mathbf{x}\|$ centered at the coordinate origin, using Eq. $(3.3)_2$

$$\begin{aligned}\int_{B(\|\mathbf{x}\|)}\mathbf{h}_0\bullet\mathbf{h}_1\,\mathrm{d}v&=-\int_{B(\|\mathbf{x}\|)}\mathbf{h}_0\bullet(\boldsymbol{\nabla}\alpha_1)\,\mathrm{d}v\\&=\int_{B(\|\mathbf{x}\|)}[-(\mathbf{h}_0\alpha_1)\bullet\boldsymbol{\nabla}+(\mathbf{h}_0\bullet\boldsymbol{\nabla})\alpha_1]\,\mathrm{d}v.\end{aligned} \tag{3.5}$$

Since $\mathbf{h}_0$ is the magnetic field existing in the absence of the solid ($\mathbf{m}=\mathbf{0}$) from Eqs. (2.7) and (2.9) we find in this case that $\nabla\bullet\mathbf{h}_0=0$, which from Eq. (3.5) in conjunction with the application of Gauss' divergence theorem yields

$$\int_{B(\|\mathbf{x}\|)} \mathbf{h}_0\bullet\mathbf{h}_1\,\mathrm{d}v = \int_{B(\|\mathbf{x}\|)} [-(\mathbf{h}_0\alpha_1)\bullet\nabla]\,\mathrm{d}v = -\int_{\partial B(\|\mathbf{x}\|)} [(\mathbf{h}_0\alpha_1)\bullet\mathbf{n}]\,\mathrm{d}a. \tag{3.6}$$

The boundary term in Eq. (3.6) $\to 0$ as $\|\mathbf{x}\|\to\infty$ according to Eq. (3.3) (assuming an adequately rapid decay rate for α_1 far away from the origin) thus completing the proof of Eq. (3.4).

To find the potential energy Π of the magnetoelastic solid, one has to subtract from $\mathscr{E}$ the external work $\mathscr{W}$ of the applied loads

$$\mathscr{W} = \int_V [\mu_0(\mathbf{h}_0\bullet\mathbf{m}) + \rho(\mathbf{f}\bullet\mathbf{u})]\,\mathrm{d}v + \int_{\partial V} \mathbf{t}\bullet\mathbf{u}\,\mathrm{d}a, \tag{3.7}$$

where the $\mathbf{h}_0\bullet\mathbf{m}$ term is the contribution of the applied external magnetic field $\mathbf{h}_0$, $\rho(\mathbf{f}\bullet\mathbf{u})$ is the body force contribution ($\mathbf{u}\equiv\mathbf{x}-\mathbf{X}$ denotes the displacement field) and the $\mathbf{t}\bullet\mathbf{u}$ term is the surface force contribution. Hence the potential energy Π of the system (solid plus surrounding free space) is from Eqs. (3.1), (3.4), and (3.7):

$$\begin{aligned}\Pi \equiv \mathscr{E}-\mathscr{W} = {} & \int_V (\rho\widehat{\psi} - \mu_0\mathbf{h}_0\bullet\mathbf{m} - \rho\mathbf{f}\bullet\mathbf{u})\,\mathrm{d}v + \int_{\mathbb{R}^3} \frac{\mu_0}{2}(\mathbf{h}_1\bullet\mathbf{h}_1)\,\mathrm{d}v \\ & - \int_{\partial V} \mathbf{t}\bullet\mathbf{u}\,\mathrm{d}a + \int_{\mathbb{R}^3} \frac{\mu_0}{2}(\mathbf{h}_0\bullet\mathbf{h}_0)\,\mathrm{d}v.\end{aligned} \tag{3.8}$$

The last term in the potential energy expression (3.8) is fixed (it depends on the applied external magnetic field $\mathbf{h}_0$ which exists in the absence of the magnetoelastic solid) and as a constant can be omitted from the potential energy. Hence the potential energy depends on the displacement field $\mathbf{u}$, the magnetization per unit mass $\mathbf{M}$ and the magnetic potential $\alpha_1(\mathbf{x})$ (of the perturbation field $\mathbf{h}_1$). There are two ways of obtaining the governing equations of magnetoelasticity from a variational principle based on the extremization of Π. One approach (see Brown, 1966) is by constrained minimization, where the imposed constraint is the combination of Eqs. (2.7) and (2.9). The other approach, also discussed in Brown (1966) and based on Toupin's (1956) work on the elastic dielectric, treats $\mathbf{u}$, $\mathbf{M}$, α_1 as independent variables, but the corresponding extremum principle is not a minimizer of Π, thus complicating the study of magnetoelastic stability problems.

To correct these deficiencies, i.e. to obtain an energy with a zero first variation that corresponds to a local minimum, one must express the potential energy Π not in terms of the potential α_1 of the perturbation magnetic field $\mathbf{h}_1$ but in terms of the potential $\mathbf{a}_1$ (to be defined next) of the perturbation magnetic flux $\mathbf{b}_1$. Recalling relation (2.9) between magnetic flux, magnetic field and magnetization, and noting that the $\mathbf{b}_0=\mu_0\mathbf{h}_0$, one has for the perturbation fields

$$\mathbf{b}_1 = \mu_0(\mathbf{h}_1+\mathbf{m}). \tag{3.9}$$

Since in addition the perturbation flux $\mathbf{b}_1$ has to satisfy the divergence free condition (2.7) (given that $\mathbf{b}_0=\mathbf{b}-\mathbf{b}_1$ is also divergence-free) one can express $\mathbf{b}_1$ in terms of a vector potential $\mathbf{a}_1$:

$$\mathbf{b}_1 = \nabla \times \mathbf{a}_1. \tag{3.10}$$

By using Eqs. (3.9) and (3.10) one can rewrite the potential energy (3.8) (without the constant term representing the magnetic energy of the imposed external field $\mathbf{h}_0$) as

$$\begin{aligned} \Pi = &\int_V (\rho\widehat{\psi} - \mu_0 \mathbf{h}_0 \bullet \mathbf{m} - \rho \mathbf{f} \bullet \mathbf{u})\,\mathrm{d}v \\ &+ \int_{\mathbb{R}^3} \frac{1}{2\mu_0} (\nabla \times \mathbf{a}_1 - \mu_0 \mathbf{m}) \bullet (\nabla \times \mathbf{a}_1 - \mu_0 \mathbf{m})\,\mathrm{d}v - \int_{\partial V} \mathbf{t} \bullet \mathbf{u}\,\mathrm{d}a. \end{aligned} \tag{3.11}$$

It can be shown that by extremizing Π with respect to its independent variables $\mathbf{u}$, $\mathbf{m}$ and $\mathbf{a}_1$ one can obtain the desired governing equations and boundary conditions for the magnetoelastic solid. However, the derivations are easier when a total Lagrangian formulation of the variational principle is considered and a reference configuration is used. Moreover a total Lagrangian formulation of this problem is also advantageous as the starting point for numerical (f.e.m.) algorithms in magnetoelastic solid applications (analytical solutions are not generally feasible—save for some trivial geometries—due to strong nonlinearities of the governing equations) as well as in stability analyses.

Consequently the potential energy (3.11) will be rewritten with respect to the reference configuration where all field variables are functions of $\mathbf{X}$, the reference configuration coordinate of a material point which in its current configuration is at $\mathbf{x} = \mathbf{X} + \mathbf{u}$. We repeat that here the usual continuum mechanics convention is employed, according to which all field quantities in capital letters are associated with the reference configuration while their counterparts in small letters are associated with the current configuration. Thus, the potential energy which is expressed in the current configuration in Eq. (3.11), takes the following form in the reference configuration:

$$\begin{aligned} \Pi = &\int_V \boldsymbol{\rho}\,[\widehat{\psi}(\mathbf{F}, \mathbf{M}) - \mu_0 \mathbf{M} \bullet \mathbf{h}_0 - \mathbf{f} \bullet \mathbf{u}]\,\mathrm{d}V - \int_{\partial V} \mathbf{T} \bullet \mathbf{u}\,\mathrm{d}A \\ &+ \int_{\mathbb{R}^3} \frac{1}{2\mu_0 J} (\mathbf{F} \bullet (\nabla \times \mathbf{A}_1) - \mu_0 \boldsymbol{\rho} \mathbf{M}) \bullet (\mathbf{F} \bullet (\nabla \times \mathbf{A}_1) - \mu_0 \boldsymbol{\rho} \mathbf{M})\,\mathrm{d}V \\ = &\int_V \boldsymbol{\rho} \left[\widehat{\psi}(\mathbf{F}, \mathbf{M}) - \mu_0 \mathbf{M} \bullet \mathbf{h}_0 - \mathbf{f} \bullet \mathbf{u} + \frac{\mu_0}{2J} \boldsymbol{\rho} \mathbf{M} \bullet \mathbf{M} \right. \\ &\left. - \frac{1}{J} \mathbf{M} \bullet \mathbf{F} \bullet (\nabla \times \mathbf{A}_1) \right] \mathrm{d}V + \int_{\mathbb{R}^3} \frac{1}{2\mu_0 J} (\nabla \times \mathbf{A}_1) \bullet \mathbf{C} \bullet (\nabla \times \mathbf{A}_1)\,\mathrm{d}V \\ &- \int_{\partial V} \mathbf{T} \bullet \mathbf{u}\,\mathrm{d}A. \end{aligned} \tag{3.12}$$

In the above expression for the potential energy of the reference configuration $\Pi(\mathbf{u}(\mathbf{X}), \mathbf{M}(\mathbf{X}), \mathbf{A}_1(\mathbf{X}))$, the following quantities have been used: $\boldsymbol{\rho}$ the mass density of the reference configuration defined in Eq. (2.13), $\mathbf{T}$ the reference traction on the boundary (force per unit reference area), $\mathbf{M}$ the current magnetization per unit mass defined in Eq. $(2.41)_2$. Recall also that $\mathbf{C}$ denotes the right Cauchy–Green deformation tensor and is defined in Eq. $(2.38)_2$, while the current magnetic flux perturbation $\mathbf{b}_1$ in

Eq. (3.10) has been replaced in Eq. (3.12) by its reference configuration counterpart $\mathbf{B}_1$, where

$$\mathbf{B}_1 = J\mathbf{F}^{-1} \bullet \mathbf{b}_1, \quad \mathbf{B}_1 = \boldsymbol{\nabla} \times \mathbf{A}_1. \tag{3.13}$$

The above relationship between $\mathbf{b}_1$ and $\mathbf{B}_1$ is deduced from the integral form of the absence of magnetic monopole principle in Eq. (2.5). Indeed by invoking Nanson's relationship between reference and current normals (see for example Chadwick, 1976):

$$\mathbf{n}\,\mathrm{d}a = J\mathbf{F}^{-\mathrm{T}} \bullet \mathbf{N}\,\mathrm{d}A, \tag{3.14}$$

one can rewrite Eq. (2.5) in the reference configuration as

$$\int_{\partial V} J\mathbf{b}_1 \bullet \mathbf{F}^{-\mathrm{T}} \bullet \mathbf{N}\,\mathrm{d}A = 0, \tag{3.15}$$

which upon the application of Gauss' divergence theorem leads to the pointwise form

$$\boldsymbol{\nabla} \bullet (J\mathbf{F}^{-1} \bullet \mathbf{b}_1) = 0, \tag{3.16}$$

thus explaining the definitions of the reference flux perturbation $\mathbf{B}_1$ and its potential $\mathbf{A}_1$ recorded in Eq. $(3.13)_2$.

At this point all the machinery is in place to show that the first variation of Π with respect to the independent variables $\mathbf{u}(\mathbf{X})$, $\mathbf{M}(\mathbf{X})$ and $\mathbf{A}_1(\mathbf{X})$ gives as its Euler–Lagrange equations the linear momentum and Ampère's equations, the corresponding boundary conditions plus the $\mathbf{m}$–$\mathbf{h}$ constitutive relation. It should also be noted here that the field of admissible flux perturbation potentials $\mathbf{A}_1(\mathbf{X})$ is any continuous vector field defined over $\mathbb{R}^3$ while $\mathbf{M}(\mathbf{X})$ is defined only on V and $\mathbf{M}=0$ for $\mathbf{X} \notin V$. The situation of the displacement field $\mathbf{u}(\mathbf{X})$ requires clarification: although only the values of $\mathbf{u}(\mathbf{X})$ for points $\mathbf{X} \in V$ make physical sense, one can without loss of generality extend the admissible displacement fields over $\mathbb{R}^3$. Hence $\delta\mathbf{u}(\mathbf{X})$ is taken to be any continuous function on $\mathbb{R}^3$ which in addition satisfies kinematic admissibility i.e. $\delta\mathbf{u}=0$ on the part of ∂V_u of the boundary where $\mathbf{u}$ is prescribed.

The variation of the potential energy Π with respect to $\mathbf{M}$ is considered first. By taking the extremum of Π in Eq. (3.12) with respect to the magnetization per unit mass $\mathbf{M}$:

$$\Pi_{,\mathbf{M}}\delta\mathbf{M} = \int_V \rho \left\{ \frac{\partial\widehat{\psi}}{\partial\mathbf{M}} \bullet \delta\mathbf{M} - \mu_0\mathbf{h}_0 \bullet \delta\mathbf{M} + \mu_0\rho\mathbf{M} \bullet \delta\mathbf{M} \right.$$
$$\left. - \left[\frac{1}{J}\mathbf{F} \bullet (\boldsymbol{\nabla} \times \mathbf{A}_1)\right] \bullet \delta\mathbf{M} \right\} \mathrm{d}V = 0. \tag{3.17}$$

In view of the arbitrariness of $\delta\mathbf{M}$ one obtains from Eq. (3.17) the following pointwise equation:

$$\frac{\partial\widehat{\psi}}{\partial\mathbf{M}} - \left(\mu_0\mathbf{h}_0 + \frac{1}{J}\mathbf{F} \bullet (\boldsymbol{\nabla} \times \mathbf{A}_1) - \mu_0\rho\mathbf{M}\right) = 0. \tag{3.18}$$

By considering the relations between $\mathbf{M}$ and $\mathbf{m}$ in Eq. $(2.41)_2$ and the perturbed current and reference magnetic fluxes $\mathbf{b}_1$ and $\mathbf{B}_1$ in Eq. (3.13), one can rewrite Eq. (3.18) with the help of Eq. (3.9):

$$\frac{\partial \widehat{\psi}}{\partial \mathbf{M}} = \mu_0 \left[\mathbf{h}_0 + \left(\frac{1}{\mu_0} \mathbf{b}_1 - \mathbf{m} \right) \right] = \mu_0 (\mathbf{h}_0 + \mathbf{h}_1) = \mu_0 \mathbf{h}, \tag{3.19}$$

which coincides, as expected, with the **h**–**M** relation given in Eq. (2.42) and derived by the direct approach.

The next step is to consider the extremum of the potential energy Π with respect to the potential $\mathbf{A}_1$ of the perturbed magnetic flux, namely

$$\Pi_{,\mathbf{A}_1} \delta \mathbf{A}_1 = \int_{\mathbb{R}^3} \left\{ \rho \left[-\frac{1}{J} \mathbf{M} \bullet \mathbf{F} \bullet (\nabla \times \delta \mathbf{A}_1) \right] + \frac{1}{\mu_0 J} (\nabla \times \mathbf{A}_1) \bullet \mathbf{C} \bullet (\nabla \times \delta \mathbf{A}_1) \right\} \mathrm{d}V = 0. \tag{3.20}$$

Note that in the above expression the integral is taken over the entire space $\mathbb{R}^3$. At this stage the following identity of integral calculus is recalled:

$$\int_V \mathbf{c} \bullet (\nabla \times \mathbf{d})\, \mathrm{d}v = \int_V \mathbf{d} \bullet (\nabla \times \mathbf{c})\, \mathrm{d}v + \int_{\partial V} (\mathbf{n} \times \mathbf{c}) \bullet \mathbf{d}\, \mathrm{d}a, \tag{3.21}$$

where $\mathbf{c}$, $\mathbf{d}$ are arbitrary vector fields defined in V (and $\mathbf{n}$ is the outward normal to the boundary ∂V) and where V is any arbitrary, finite subdomain of $\mathbb{R}^3$.

By successively applying Eqs. (3.21) to (3.20), once to V and once to $\mathbb{R}^3 \setminus V$, with $\mathbf{d} \equiv \delta \mathbf{A}_1$ and $\mathbf{c} \equiv J^{-1}(\mu_0^{-1}(\nabla \times \mathbf{A}_1) \bullet \mathbf{C} - \rho \mathbf{M} \bullet \mathbf{F})$ while keeping in mind that the perturbed reference flux $\mathbf{B}_1$, and hence $\|\mathbf{A}_1\| \to 0$ as $\|\mathbf{X}\| \to \infty$, one obtains

$$\Pi_{,\mathbf{A}_1} \delta \mathbf{A}_1 = \int_{\mathbb{R}^3} \left\{ \left[\nabla \times \left[\frac{1}{J} \left(\frac{1}{\mu_0} (\nabla \times \mathbf{A}_1) \bullet \mathbf{C} - \rho \mathbf{M} \bullet \mathbf{F} \right) \right] \right] \bullet \delta \mathbf{A}_1 \right\} \mathrm{d}V + \int_{\partial V} \left\{ \left[\mathbf{N} \times \left[\!\left[\frac{1}{J} \left(\frac{1}{\mu_0} (\nabla \times \mathbf{A}_1) \bullet \mathbf{C} - \rho \mathbf{M} \bullet \mathbf{F} \right) \right]\!\right] \right] \bullet \delta \mathbf{A}_1 \right\} \mathrm{d}A = 0. \tag{3.22}$$

Recalling relation (3.13) between the reference and current magnetic flux perturbations $\mathbf{B}_1$ and $\mathbf{b}_1$, definition $(2.38)_2$ of the right Cauchy–Green tensor $\mathbf{C}$ and definitions (2.13) of the reference density ρ and Eq. (2.41) of the magnetization per unit mass $\mathbf{M}$, the vector field appearing in the volume and surface integrals in Eq. (3.22) is in view of Eq. (3.9)

$$\frac{1}{\mu_0} \left(\frac{1}{J} (\nabla \times \mathbf{A}_1) \bullet \mathbf{C} \right) - \frac{\rho}{J} \mathbf{M} \bullet \mathbf{F} = \left(\frac{1}{\mu_0} \mathbf{b}_1 - \mathbf{m} \right) \bullet \mathbf{F} = \mathbf{h}_1 \bullet \mathbf{F} \equiv \mathbf{H}_1. \tag{3.23}$$

Consequently, and in view of the arbitrariness of the vector field $\mathbf{A}_1$, one can restate Eq. (3.22) in view of Eq. (3.23) as the differential equation

$$\nabla \times \mathbf{H}_1 = 0 \quad \text{in } \mathbb{R}^3 \tag{3.24}$$

and the accompanying boundary condition:

$$\mathbf{N} \times [\![\mathbf{H}_1]\!] = 0 \quad \text{on } \partial V, \tag{3.25}$$

where $\mathbf{H}_1$ is the perturbed magnetic field in the reference configuration which is related to its current configuration counterpart $\mathbf{h}_1$ by the last expression of Eq. (3.23). It can easily be shown that Eqs. (3.24) and (3.25) are the reference configuration counterparts of Ampère's law (2.3) and (2.4), respectively. The proof of this assertion follows easily from kinematics since Eq. (2.1) can be rewritten in the reference configuration as

$$\int_{\partial S} \mathbf{h}_1 \bullet \mathrm{d}\mathbf{x} = \int_{\partial S} (\mathbf{h}_1 \bullet \mathbf{F}) \bullet \mathrm{d}\mathbf{X} = \int_{\partial S} \mathbf{H}_1 \bullet \mathrm{d}\mathbf{X} \quad (\mathrm{d}\mathbf{x} = \mathbf{F} \bullet \mathrm{d}\mathbf{X}) \tag{3.26}$$

from which Eqs. (3.24) and (3.25) follow by application of Stokes' theorem. Thus Ampère's law has also been recovered, as expected, through the variational approach.

The final step in the variational approach is the re-derivation of the equilibrium equations and traction boundary conditions for the magnetoelastic solid by extremizing the potential energy with respect to $\mathbf{u}(\mathbf{X})$. From Eq. (3.12), one has

$$\begin{aligned}
\Pi_{,\mathbf{u}}\delta\mathbf{u} = \int_V \rho \Bigg[&\left(\frac{\partial \widehat{\psi}}{\partial \mathbf{F}}\right)^{\mathrm{T}} \bullet\bullet (\delta\mathbf{u}\boldsymbol{\nabla}) - \mu_0 \delta\mathbf{h}_0 \bullet \mathbf{M} - \mathbf{f} \bullet \delta\mathbf{u} + \frac{\mu_0}{2} \mathbf{M} \bullet \mathbf{M} \delta\rho \\
&- \mathbf{M} \bullet \mathbf{F} \bullet (\boldsymbol{\nabla} \times \mathbf{A}_1) \delta\left(\frac{1}{J}\right) - \frac{1}{J} \mathbf{M} \bullet (\delta\mathbf{u}\boldsymbol{\nabla}) \bullet (\boldsymbol{\nabla} \times \mathbf{A}_1) \Bigg] \mathrm{d}V \\
&+ \int_{\mathbb{R}^3} \frac{1}{2\mu_0} \Bigg[(\boldsymbol{\nabla} \times \mathbf{A}_1) \bullet \mathbf{C} \bullet (\boldsymbol{\nabla} \times \mathbf{A}_1) \delta\left(\frac{1}{J}\right) \\
&+ \frac{1}{J} (\boldsymbol{\nabla} \times \mathbf{A}_1) \bullet \delta\mathbf{C} \bullet (\boldsymbol{\nabla} \times \mathbf{A}_1) \Bigg] \mathrm{d}V \\
&- \int_{\partial V} [\mathbf{T} \bullet \delta\mathbf{u}] \, \mathrm{d}A = 0,
\end{aligned} \tag{3.27}$$

where all the variations are with respect to $\mathbf{u}$ (i.e. $\delta(\ ) \equiv (\ )_{,\mathbf{u}}\delta\mathbf{u}$).

There are several intermediate results that are required in order to transform (3.27) into a more useful expression. We start with the calculation of the $\delta\mathbf{h}_0$ term in the integrand of (3.27) which is

$$\delta\mathbf{h_0} = (\mathbf{h}_0 \bullet \nabla) \bullet \delta\mathbf{u} = (\mathbf{h}_0 \boldsymbol{\nabla}) \bullet \mathbf{F}^{-1} \bullet \delta\mathbf{u}. \tag{3.28}$$

We continue with the expressions for $\delta\rho$ and $\delta(J^{-1})$, which are found with the help of the kinematic relations (2.12) and the mass conservation in Eq. (2.13):

$$\delta\left(\frac{1}{J}\right) = -\frac{1}{J} \mathbf{F}^{-1} \bullet\bullet (\delta\mathbf{u}\boldsymbol{\nabla}), \tag{3.29}$$

$$\delta\rho = -\rho\mathbf{F}^{-1} \bullet\bullet (\delta\mathbf{u}\boldsymbol{\nabla}). \tag{3.30}$$

Using Eqs. $(3.13)_2$ and (3.28)–(3.30) into Eq. (3.27), one obtains the following expression for the first variation of the potential energy with respect to the displacement:

$$
\begin{aligned}
\Pi_{,\mathbf{u}}\,\delta\mathbf{u} = \int_V \Bigg\{ & \rho\,[-\mu_0\mathbf{M}\bullet(\mathbf{h}_0\boldsymbol{\nabla})\bullet\mathbf{F}^{-1} - \mathbf{f}]\bullet\delta\mathbf{u} + \Bigg[\rho\left(\frac{\partial\widehat{\psi}}{\partial\mathbf{F}}\right)^{\mathrm{T}} \\
& -\frac{\mu_0 J}{2}\left(\frac{1}{\mu_0 J}\mathbf{F}\bullet\mathbf{B}_1 - \rho\mathbf{M}\right)\bullet\left(\frac{1}{\mu_0 J}\mathbf{F}\bullet\mathbf{B}_1 - \rho\mathbf{M}\right)\mathbf{F}^{-1} \\
& +\mathbf{B}_1\left(\frac{1}{\mu_0 J}\mathbf{F}\bullet\mathbf{B}_1 - \rho\mathbf{M}\right)\Bigg]\bullet\bullet(\delta\mathbf{u}\boldsymbol{\nabla})\Bigg\}\,\mathrm{d}V \\
& + \int_{\mathbb{R}^3\setminus V}\Bigg\{\Bigg[\frac{1}{\mu_0 J}\mathbf{B}_1(\mathbf{F}\bullet\mathbf{B}_1) - \frac{\mu_0 J}{2}\left(\frac{1}{\mu_0 J}\mathbf{F}\bullet\mathbf{B}_1\right) \\
& \bullet\left(\frac{1}{\mu_0 J}\mathbf{F}\bullet\mathbf{B}_1\right)\mathbf{F}^{-1}\Bigg]\bullet\bullet(\delta\mathbf{u}\boldsymbol{\nabla})\Bigg\}\,\mathrm{d}V \\
& + \int_{\partial V}[\mathbf{T}\bullet\delta\mathbf{u}]\,\mathrm{d}A = 0.
\end{aligned} \tag{3.31}
$$

It has been shown in Eqs. (3.18) and (3.19) that the vector appearing repeatedly in Eq. (3.31), namely $(\mu_0 J)^{-1}\mathbf{F}\bullet\mathbf{B}_1 - \rho\mathbf{M} = \mathbf{b}_1/\mu_0 - \mathbf{m} = \mathbf{h}_1$ for points $\mathbf{X}\in V$. Similarly $(\mu_0 J)^{-1}\mathbf{F}\bullet\mathbf{B}_1 = \mathbf{h}_1$ for points $\mathbf{X}\in\mathbb{R}^3\setminus V$ (since $\mathbf{m}=0$ outside the magnetoelastic solid). Integration of Eq. (3.31) by parts for the terms involving $\delta\mathbf{u}\boldsymbol{\nabla}$ i.e. using the identity $(2.27)_2$, and subsequent application of Gauss' divergence theorem (assuming adequate continuity of the field quantities involved and recalling that $\mathbf{b}_1 \to 0$ as $\|\mathbf{X}\| \to \infty$) yields, in view of the arbitrariness of $\delta\mathbf{u}$, the following differential equations:

$$
\begin{aligned}
\mathbf{X}\in V:\ & \left[J\left(\rho\frac{\partial\widehat{\psi}}{\partial\mathbf{F}} - \frac{\mu_0}{2}(\mathbf{h}_1\bullet\mathbf{h}_1)\mathbf{F}^{-\mathrm{T}} + \mathbf{h}_1\mathbf{b}_1\bullet\mathbf{F}^{-\mathrm{T}}\right)\right]\bullet\boldsymbol{\nabla} \\
& + J[\rho\mathbf{f} + \mu_0\mathbf{m}\bullet(\mathbf{h}_0\boldsymbol{\nabla})\bullet\mathbf{F}^{-1}] = 0,
\end{aligned} \tag{3.32}
$$

$$
\mathbf{X}\in\mathbb{R}^3\setminus V:\ \left[\mu_0 J\left(\mathbf{h}_1\mathbf{h}_1 - \frac{1}{2}(\mathbf{h}_1\bullet\mathbf{h}_1)\mathbf{I}\right)\bullet\mathbf{F}^{-\mathrm{T}}\right]\bullet\boldsymbol{\nabla} = 0, \tag{3.33}
$$

plus the boundary condition on ∂V

$$
\mathbf{X}\in\partial V:\ \left[\!\left[J\left(\rho\frac{\partial\widehat{\psi}}{\partial\mathbf{F}} - \frac{\mu_0}{2}(\mathbf{h}_1\bullet\mathbf{h}_1)\mathbf{F}^{-\mathrm{T}} + \mathbf{h}_1\mathbf{b}_1\bullet\mathbf{F}^{-\mathrm{T}}\right)\right]\!\right]\bullet\mathbf{N} = \mathbf{T}. \tag{3.34}
$$

For the solid it will now be shown that the equilibrium equation in the reference configuration (3.32) and the boundary condition (3.34) obtained by the energy approach are actually the same as their current configuration counterparts (2.45) and (2.48) which were obtained by the direct approach. For the space outside the solid it will also be shown that Eq. (3.33) is identically satisfied.

To prove the correspondence between Eqs. (3.32) and (2.45) one needs the following identity from continuum mechanics (e.g. see Chadwick, 1976, p. 59, Eq. (19))

valid for any arbitrary rank two tensor $\boldsymbol{\Pi}$

$$\boldsymbol{\nabla} \bullet \boldsymbol{\Pi} = J(\nabla \bullet \boldsymbol{\sigma}), \quad \boldsymbol{\sigma} \equiv \frac{1}{J} \mathbf{F} \bullet \boldsymbol{\Pi}. \tag{3.35}$$

By identifying $\boldsymbol{\Pi}$ with the perturbation first Piola–Kirchhoff stress $\boldsymbol{\Pi}_1$, i.e. the additional (reference configuration) stress due to the presence of the magnetoelastic solid

$$\boldsymbol{\Pi}_1^{\mathrm{T}} \equiv J \left[\rho \frac{\partial \widehat{\psi}}{\partial \mathbf{F}} \bullet \mathbf{F}^{\mathrm{T}} - \mu_0 (\mathbf{h}_1 \bullet \mathbf{h}_1) \mathbf{I} + \mathbf{h}_1 \mathbf{b}_1 \right] \bullet \mathbf{F}^{-\mathrm{T}} \tag{3.36}$$

from Eq. $(3.35)_2$ the corresponding $\boldsymbol{\sigma}$ is identified with the Cauchy stress perturbation $\boldsymbol{\sigma}_1$, namely

$$\boldsymbol{\sigma}_1 \equiv \left[\rho \frac{\partial \widehat{\psi}}{\partial \mathbf{F}} \bullet \mathbf{F}^{\mathrm{T}} - \mu_0 (\mathbf{h}_1 \bullet \mathbf{h}_1) \mathbf{I} + \mathbf{h}_1 \mathbf{b}_1 \right]^{\mathrm{T}}. \tag{3.37}$$

Note that Eq. (3.37) is obtained from the Cauchy stress expression (2.44) (derived using the direct approach) when the total $\mathbf{h}$ and $\mathbf{b}$ fields are substituted by their perturbed counterparts $\mathbf{h}_1$ and $\mathbf{b}_1$. Using Eq. (3.35) (together with definitions (3.36) and (3.37)) and with the help of Eq. (3.28) which relates $\mathbf{h}_0 \nabla$ to its reference configuration counterpart $\mathbf{h}_0 \boldsymbol{\nabla}$, the equilibrium equation of the magnetoelastic solid in the reference configuration (3.32) is rewritten in the current configuration as

$$J \left[\left(\rho \frac{\partial \widehat{\psi}}{\partial \mathbf{F}} \bullet \mathbf{F}^{\mathrm{T}} \right) \bullet \nabla + \mu_0 (\mathbf{h} \nabla) \bullet \mathbf{m} + \rho \mathbf{f} \right] = 0, \tag{3.38}$$

thus recovering (term inside brackets) (2.45) as expected, albeit for the special case $\ddot{\mathbf{x}} = 0$. In deriving Eq. (3.38) from Eq. (3.32) use was made of the identity $\mathbf{m} \bullet (\mathbf{h}_0 \nabla) = (\mathbf{h}_0 \nabla) \bullet \mathbf{m}$ (easily shown with the help of Eq. $(3.3)_1$) and the constitutive relation (3.9) for the perturbed fields.

Converting the interface condition from the reference to the current configuration requires again the definitions in Eqs. (3.36) and (3.37) plus Nanson's relation (3.14) to give

$$\mathbf{n} \bullet [\![\boldsymbol{\sigma}_1]\!] = \mathbf{t}. \tag{3.39}$$

At first sight (3.39) is different from its counterpart (2.46) which was derived by the direct approach. Note however that $[\![\mathbf{h}]\!] = [\![\mathbf{h}_1]\!]$ and by following exactly the same steps as in Section 2, one obtains again Eq. (2.46) from Eq. (3.39), exactly as expected. We have thus shown that the equilibrium equations and surface boundary conditions in the reference configuration, obtained by extremizing the potential energy with respect to the displacement, coincide with their current configuration counterparts, obtained in Section 2 by the direct approach.

One last point remains before concluding the part on the potential energy extremization with respect to the displacement, namely proving that Eq. (3.33) is identically satisfied outside the magnetoelastic solid. Indeed using Eq. (3.35) one has, since $\mathbf{b}_1 = \mathbf{h}_1$ ($\mathbf{m} = 0$ in $\mathbb{R}^3 \setminus V$),

$$\mu_0 J \left\{ \left[\mathbf{h}_1 \mathbf{h}_1 - \tfrac{1}{2} (\mathbf{h}_1 \bullet \mathbf{h}_1) \mathbf{I} \right] \bullet \nabla \right\} = 0, \tag{3.40}$$

an easily verifiable identity since $\nabla \times \mathbf{h}_1 = \nabla \bullet \mathbf{h}_1 = 0$ in $\mathbb{R}^3 \setminus V$ (from Ampère's law and absence of magnetic monopole, respectively).

The assertion that the extremization of the potential energy corresponds to a local minimum is better seen from the first of the two equivalent expressions for Π in Eq. (3.12). Note that the magnetic field's energy over the entire space $\mathbb{R}^3$ is always positive and it depends on $\mathbf{u}, \mathbf{A}_1, \mathbf{M}$ which are independent variables. Ignoring the linear terms of the potential energy and assuming a positive Helmholtz free energy $\widehat{\psi}$ with reasonable growth conditions and noting that in the absence of external forces and magnetic fields $\Pi \geqslant 0$, one can see how the extremization of Π corresponds to a local minimum.

In closing this section on the variational derivation of the governing equations and boundary conditions for the magnetoelastic solid, two remarks are in order: the first remark pertains to the derivation of equilibrium equations using the variational approach, as opposed to the equations of motion derived using the direct approach. Had one employed Hamilton's principle by considering also the kinetic energy, i.e.

$$\delta\left\{\int_{t_1}^{t_2} [\mathscr{K} - \Pi]\,\mathrm{d}t\right\}, \qquad \mathscr{K} \equiv \int_V \frac{1}{2}\rho(\dot{\mathbf{x}} \bullet \dot{\mathbf{x}})\,\mathrm{d}V, \tag{3.41}$$

one can derive from the variational approach Eq. (2.45) including the inertia terms. It is of course tacitly assumed that $\|\dot{\mathbf{x}}\| \ll c$ (velocities much lower than the speed of light).

The second remark pertains to the non-uniqueness of the perturbed flux potential $\mathbf{A}_1$ (defined in Eq. $(3.13)_2$). In fact the addition of the gradient of any scalar field Λ to $\mathbf{A}_1$ leaves $\mathbf{B}_1$ unchanged ($\nabla \times \mathbf{A}_1 = \nabla \times (\mathbf{A}_1 + \nabla\Lambda)\,\forall\,\Lambda(\mathbf{X})$). To avoid problems in numerical calculations a constraint on $\mathbf{A}_1$ can be imposed, the simplest one being the Coulomb gauge $\nabla \bullet \mathbf{A}_1 = 0$, which can be achieved by adding to the potential energy Π the penalty term $(1/2\varepsilon)\int_{\mathbb{R}^3}(\nabla \bullet \mathbf{A}_1)^2\,\mathrm{d}V$, where $0 < \varepsilon \ll 1$.

4. Properties of free energy $\widehat{\psi}$ and special cases

In this section some important properties of the free energy $\widehat{\psi}(\mathbf{F}, \mathbf{M})$ are presented and discussed, namely its objectivity, its material symmetry, its time-reversal and finally its quasiconvexity. The resulting expression for $\widehat{\psi}$ for the special case of an isotropic material is also recorded. The section concludes with the constitutive equations of an isotropic magnetoelastic solid in both its compressible and incompressible versions.

(a) *Objectivity*: The objectivity of $\widehat{\psi}(\mathbf{F}, \mathbf{M})$ or equivalently of $\psi(\mathbf{F}, \mathbf{m})$ has already been presented in Section 2 (see Eq. (2.37)) and need not be repeated here; recall simply that frame invariance of the free energy requires

$$\widehat{\psi}(\mathbf{F}, \mathbf{M}) = \widehat{\psi}(\mathbf{C}, \mathbf{F}^{\mathrm{T}} \bullet \mathbf{M}). \tag{4.1}$$

(b) *Material symmetry*: The material symmetry of the magnetoelastic solid requires

$$\widehat{\psi}(\mathbf{F}, \mathbf{M}) = \widehat{\psi}(\mathbf{F} \bullet \mathbf{P}, \mathbf{M} \bullet \mathbf{P}), \quad \forall\ \mathbf{P} \in \mathscr{G}, \tag{4.2}$$

where $\mathbf{P}$ is any orthogonal matrix belonging to the group of symmetry transformations $\mathscr{G}$ of the solid.

(c) *Time reversal invariance*: Time reversal invariance of Maxwell's equations implies the invariance of the free energy on the sign of the magnetization vector (e.g. see Landau and Lifshitz, 1984):

$$\widehat{\psi}(\mathbf{F},\mathbf{M}) = \widehat{\psi}(\mathbf{F},-\mathbf{M}). \tag{4.3}$$

(d) *Quasiconvexity*: For the magnetoelastic solids of interest (ferromagnetic particle impregnated rubbers) no fine structures—or equivalently, no strain and no magnetization discontinuities—have been observed for the applied mechanical and magnetic loads.

In the absence of magnetization, i.e. in the case of finite elasticity, there is a substantial literature on existence and regularity of solutions, with the issue of discontinuous solutions—appearing in boundary value problems with smooth coefficients due to the nonlinearity of the governing equations—occupying a prominent position (see Ball's (1977a,b) review articles). A key concept, due to Morrey (1952), that has been developed to study the issue of regularity of solutions is that of "quasiconvexity", i.e. that the free energy stored in an arbitrary region of a homogeneous elastic solid under constant strain is the lowest possible under a fixed boundary displacement. This condition, under adequate smoothness, implies pointwise "rank one convexity" of the free energy, thus excluding locally discontinuous solutions of the elastostatic problem. We present an extension of these concepts for magnetoelasticity.

Like in the case of finite elasticity, the absence of locally discontinuous solutions is ensured by the assumption that the solid's potential energy is "quasiconvex", i.e. if an arbitrary subregion $D \subset V$ of the solid has homogeneous (i.e. position independent) properties and is under constant strain, constant magnetic field and constant magnetization, then these constant mechanical and magnetic fields minimize the potential energy of D over all other admissible fields which satisfy Dirichlet conditions at the boundary ∂D. More specifically if

$$\begin{aligned}
&\Pi(\mathbf{F},\mathbf{M},\mathbf{A}_1) \equiv \int_D \left\{ \rho[\widehat{\psi} - \mu_0 \mathbf{h}_0 \bullet \mathbf{M}] + \frac{\mu_0 J}{2}\mathbf{h}_1 \bullet \mathbf{h}_1 \right\} \mathrm{d}V, \\
&\mathbf{h}_1 \equiv \frac{1}{\mu_0 J}\mathbf{F} \bullet \mathbf{B}_1 - \rho\mathbf{M},
\end{aligned} \tag{4.4}$$

where ρ and $\widehat{\psi}$ are $\mathbf{X}$-independent and where $\mathbf{F},\mathbf{M},\mathbf{B}_1 = \boldsymbol{\nabla} \times \mathbf{A}_1$ and $\mathbf{h}_0$ are constant then Π satisfies

$$\begin{aligned}
&\Pi(\mathbf{F} + \delta\mathbf{u}\boldsymbol{\nabla}, \mathbf{M} + \delta\mathbf{M}, \mathbf{B}_1 + \boldsymbol{\nabla} \times \delta\mathbf{A}_1) \geqslant \Pi(\mathbf{F},\mathbf{M},\mathbf{B}_1), \\
&\quad \delta\mathbf{u} = \delta\mathbf{A}_1 = 0 \quad \text{on } \partial D,
\end{aligned} \tag{4.5}$$

for all admissible fields $\delta\mathbf{u}(\mathbf{X})$, $\delta\mathbf{M}(\mathbf{X})$ and $\delta\mathbf{A}_1(\mathbf{X})$ such that $\delta\mathbf{u}$ and $\delta\mathbf{A}_1$ vanish on ∂D. Without loss of generality it can be assumed that the applied magnetic field is equal to the total magnetic field $\mathbf{h}_0 = \mathbf{h}$ and hence from Eq. (3.2) $\mathbf{h}_1 = 0$. Consequently the condition for a local minimum of Π at $\mathbf{F},\mathbf{M},\mathbf{B}_1$ requires a positive definite second variational (Frechet) derivative of Π with respect to the independent variables $\mathbf{g} \equiv (\mathbf{u}, \mathbf{M}, \mathbf{A})$,

$$\Pi_{,\mathbf{gg}}(\delta\mathbf{g}, \delta\mathbf{g}) = \int_D \{\rho \widehat{\psi}_{,\mathbf{gg}}(\delta\mathbf{g}, \delta\mathbf{g}) + \mu_0 J \mathbf{h}_{1,g}(\delta\mathbf{g}) \bullet \mathbf{h}_{1,g}(\delta\mathbf{g})\}\, \mathrm{d}V \geqslant 0, \tag{4.6}$$

which satisfy the boundary conditions $\delta\mathbf{u} = \delta\mathbf{A}_1 = 0$ on ∂D. Observe that the variation of the magnetic perturbation energy—$\mu_0 J$ term—is always positive and consequently the quasiconvexity of $\widehat{\psi}$[9] will imply the quasiconvexity of Π. The quasiconvexity condition for $\widehat{\psi}$ implies (now $\mathbf{g} \equiv (\mathbf{u}, \mathbf{M})$)

$$\begin{aligned} &\int_D \{\rho \widehat{\psi}_{,\mathbf{gg}}(\delta\mathbf{g}, \delta\mathbf{g})\}\, \mathrm{d}V \\ &\quad = \int_D \left\{ \rho \left[(\boldsymbol{\nabla}\delta\mathbf{u}) \bullet\bullet \frac{\partial^2 \widehat{\psi}}{\partial\mathbf{F}\partial\mathbf{F}} \bullet\bullet (\boldsymbol{\nabla}\delta\mathbf{u}) + (\boldsymbol{\nabla}\delta\mathbf{u}) \bullet\bullet \frac{\partial^2 \widehat{\psi}}{\partial\mathbf{F}\partial\mathbf{M}} \bullet \delta\mathbf{M} \right.\right. \\ &\qquad \left.\left. + \delta\mathbf{M} \bullet \frac{\partial^2 \widehat{\psi}}{\partial\mathbf{M}\partial\mathbf{F}} \bullet\bullet (\boldsymbol{\nabla}\delta\mathbf{u}) + \delta\mathbf{M} \bullet \frac{\partial^2 \widehat{\psi}}{\partial\mathbf{M}\partial\mathbf{M}} \bullet \delta\mathbf{M} \right] \right\} \mathrm{d}V \geqslant 0, \end{aligned} \tag{4.7}$$

for all $\delta\mathbf{u} = 0$ on ∂D. Since all the second-order derivatives of the potential $\widehat{\psi}$ are constants over D, an application of the Fourier–Plancherel identity over the compact domain D gives

$$\begin{aligned} &\int_D \{\rho \widehat{\psi}_{,\mathbf{gg}}(\delta\mathbf{g}, \delta\mathbf{g})\}\, \mathrm{d}V \\ &\quad = \int_D \left\{ \rho \left[(\overline{\Xi \Delta\mathbf{u}}) \bullet\bullet \frac{\partial^2 \widehat{\psi}}{\partial\mathbf{F}\partial\mathbf{F}} \bullet\bullet (\Xi \Delta\mathbf{u}) + (\overline{\Xi \Delta\mathbf{u}}) \bullet\bullet \frac{\partial^2 \widehat{\psi}}{\partial\mathbf{F}\partial\mathbf{M}} \bullet (\Delta\mathbf{M}) \right.\right. \\ &\qquad \left.\left. + \overline{\Delta\mathbf{M}} \bullet \frac{\partial^2 \widehat{\psi}}{\partial\mathbf{M}\partial\mathbf{F}} \bullet\bullet (\Xi \Delta\mathbf{u}) + \overline{\Delta\mathbf{M}} \bullet \frac{\partial^2 \widehat{\psi}}{\partial\mathbf{M}\partial\mathbf{M}} \bullet \Delta\mathbf{M} \right] \right\} \mathrm{d}V \geqslant 0, \end{aligned} \tag{4.8}$$

where $\Delta\mathbf{u}$, $\Delta\mathbf{M}$ are the Fourier transforms of $\delta\mathbf{u}$ and $\delta\mathbf{M}$ and Ξ is the Fourier transform variable corresponding to $\mathbf{X}$ and where $\overline{(\,)}$ denotes complex conjugation. From the arbitrariness of the subdomain D one obtains from Eq. (4.8) (using a standard condensation argument for positive definite quadratic forms) the following sufficient pointwise conditions for the quasiconvexity of $\widehat{\psi}$ for all arbitrary real vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$

$$\begin{aligned} &(\mathbf{ab}) \bullet\bullet \left[\frac{\partial^2 \widehat{\psi}}{\partial\mathbf{F}\partial\mathbf{F}} - \frac{\partial^2 \widehat{\psi}}{\partial\mathbf{F}\partial\mathbf{M}} \bullet \left(\frac{\partial^2 \widehat{\psi}}{\partial\mathbf{M}\partial\mathbf{M}} \right)^{-1} \bullet \frac{\partial^2 \widehat{\psi}}{\partial\mathbf{M}\partial\mathbf{F}} \right] \bullet\bullet (\mathbf{ab}) \geqslant 0, \\ &\mathbf{c} \bullet \frac{\partial^2 \widehat{\psi}}{\partial\mathbf{M}\partial\mathbf{M}} \bullet \mathbf{c} \geqslant 0. \end{aligned} \tag{4.9}$$

[9] Defined in a similar fashion by $\int_D \{\rho \widehat{\psi}(\mathbf{F} + \delta\mathbf{u}\boldsymbol{\nabla}, \mathbf{M} + \delta\mathbf{M})\}\, \mathrm{d}V \geqslant \mathrm{Vol}(D) \rho \widehat{\psi}(\mathbf{F}, \mathbf{M})$.

For special case of elasticity (where $\widehat{\psi} \equiv \widehat{\psi}(\mathbf{F})$), the above conditions reduce to rank one convexity of $\widehat{\psi}$, while for a purely magnetic solid (where $\widehat{\psi} \equiv \widehat{\psi}(\mathbf{M})$), one obtains the convexity of $\widehat{\psi}$ with respect to $\mathbf{M}$.

4.1. Isotropic case

For the special case of an isotropic magnetoelastic solid, the material's symmetry group is $\mathscr{G} = \mathscr{O}(3)$, the orthogonal group in $\mathbb{R}^3$, in which case the rank two tensor $\mathbf{P}$ in Eq. (4.2) is any orthogonal tensor $\mathbf{Q}$. It is a known result of representation theory, e.g. Green and Adkins (1970), that an isotropic scalar function which depends on one symmetric rank two tensor $\mathbf{A}$ and one vector argument $\mathbf{a}$ is a function of the following six invariants:

$$\begin{aligned}
&\widehat{\psi}(\mathbf{A}, \mathbf{a}) = \widehat{\psi}(I_1, I_2, I_3, J_1, J_2, J_3),\\
&I_1 \equiv \operatorname{tr} \mathbf{A}, \quad I_2 \equiv \tfrac{1}{2}\,[(\operatorname{tr} \mathbf{A})^2 - \operatorname{tr}(\mathbf{A} \bullet \mathbf{A})], \quad I_3 \equiv \det \mathbf{A},\\
&J_1 \equiv \mathbf{a} \bullet \mathbf{A}^{-1} \bullet \mathbf{a}, \quad J_2 \equiv \mathbf{a} \bullet \mathbf{a}, \quad J_3 \equiv \mathbf{a} \bullet \mathbf{A} \bullet \mathbf{a},
\end{aligned} \tag{4.10}$$

where according to Eq. (4.1) $\mathbf{A}=\mathbf{F}^{\mathrm{T}} \bullet \mathbf{F} \equiv \mathbf{C}$ and $\mathbf{a}=\mathbf{F}^{\mathrm{T}} \bullet \mathbf{M}$. Consequently the invariants appearing in Eq. (4.10) for the isotropic material are

$$\begin{aligned}
&\widehat{\psi}(\mathbf{B}, \mathbf{M}) = \widehat{\psi}(I_1, I_2, I_3, J_1, J_2, J_3), \quad \mathbf{B} \equiv \mathbf{F} \bullet \mathbf{F}^{\mathrm{T}},\\
&I_1 \equiv \operatorname{tr} \mathbf{B}, \quad I_2 \equiv \tfrac{1}{2}\,[(\operatorname{tr} \mathbf{B})^2 - \operatorname{tr}(\mathbf{B} \bullet \mathbf{B})], \quad I_3 = \det \mathbf{B},\\
&J_1 \equiv \mathbf{M} \bullet \mathbf{M}, \quad J_2 \equiv \mathbf{M} \bullet \mathbf{B} \bullet \mathbf{M}, \quad J_3 \equiv \mathbf{M} \bullet \mathbf{B}^2 \bullet \mathbf{M},
\end{aligned} \tag{4.11}$$

where the rank two tensor $\mathbf{B}$ (not to be confused with the reference magnetic flux vector) is the left Cauchy–Green tensor of continuum mechanics. Notice that isotropy automatically satisfies the time reversibility requirement (4.3) since according to Eq. (4.11) the invariants J_1, J_2 and J_3 are quadratic in $\mathbf{M}$.

At this point it is of interest to record the constitutive equations (2.42) and (2.44) for the special case of an isotropic elastic solid. Using Eq. (4.11) into the constitutive equations (2.42) for the magnetic field

$$\mu_0 \mathbf{h} = 2 \left[\frac{\partial \widehat{\psi}}{\partial J_1} \mathbf{I} + \frac{\partial \widehat{\psi}}{\partial J_2} \mathbf{B} + \frac{\partial \widehat{\psi}}{\partial J_3} \mathbf{B}^2 \right] \bullet \mathbf{M}. \tag{4.12}$$

The calculation of the stress $\boldsymbol{\sigma}$ from Eq. (2.44) makes use of the objectivity of $\widehat{\psi}$ according to Eq. (4.1), of the symmetry of $\boldsymbol{\sigma}$ and of the constitutive relation for $\mathbf{h}$ in Eq. (2.42) to give

$$\boldsymbol{\sigma} = \rho \left[\frac{\partial \widehat{\psi}}{\partial \mathbf{B}} \bullet \mathbf{B} + \mathbf{B} \bullet \frac{\partial \widehat{\psi}}{\partial \mathbf{B}} + \mu_0 (\mathbf{M}\mathbf{h} + \mathbf{h}\mathbf{M}) \right] + \mu_0 [\mathbf{h}\mathbf{h} - \frac{1}{2} (\mathbf{h} \bullet \mathbf{h}) \mathbf{I}]. \tag{4.13}$$

Using the results in Eq. (4.13) in conjunction with the definitions of the invariants in Eq. (4.11), one obtains from Eq. (2.48) that the surface traction $\mathbf{t}$ is given by the following expression:

$$\mathbf{t} = \left\{ \rho \left[\frac{\partial\widehat{\psi}}{\partial\mathbf{B}} \bullet \mathbf{B} + \mathbf{B} \bullet \frac{\partial\widehat{\psi}}{\partial\mathbf{B}} + \mu_0\mathbf{M}\mathbf{h} \right] \right\} \bullet \mathbf{n} - \frac{\mu_0}{2} (\mathbf{m} \bullet \mathbf{n})^2 \mathbf{n}. \tag{4.14}$$

In both of the above equations, the common expression involving $\partial\widehat{\psi}/\partial\mathbf{B}$ can be expanded as follows:

$$\begin{aligned} \frac{\partial\widehat{\psi}}{\partial\mathbf{B}} \bullet \mathbf{B} + \mathbf{B} \bullet \frac{\partial\widehat{\psi}}{\partial\mathbf{B}} = 2 \left(\frac{\partial\widehat{\psi}}{\partial I_1} + I_1 \frac{\partial\widehat{\psi}}{\partial I_2} \right) \mathbf{B} - 2 \frac{\partial\widehat{\psi}}{\partial I_2} \mathbf{B}^2 + 2I_3 \frac{\partial\widehat{\psi}}{\partial I_3} \mathbf{I} \\ + \frac{\partial\widehat{\psi}}{\partial J_2} [\mathbf{M}(\mathbf{B} \bullet \mathbf{M}) + (\mathbf{M} \bullet \mathbf{B})\mathbf{M}] \\ + \frac{\partial\widehat{\psi}}{\partial J_3} [\mathbf{M}(\mathbf{B}^2 \bullet \mathbf{M}) + (\mathbf{M} \bullet \mathbf{B}^2)\mathbf{M} + 2(\mathbf{M} \bullet \mathbf{B})(\mathbf{B} \bullet \mathbf{M})]. \end{aligned} \tag{4.15}$$

4.2. Incompressible isotropic case

For the special case of incompressibility ($I_3 = 1$), the I_3 dependent term in Eq. (4.15) is replaced by $(p/\rho)\mathbf{I}$, where p is the hydrostatic pressure (needed to maintain this constraint and which is solved from application of the boundary condition), as follows:

$$\begin{aligned} \frac{\partial\widehat{\psi}}{\partial\mathbf{B}} \bullet \mathbf{B} + \mathbf{B} \bullet \frac{\partial\widehat{\psi}}{\partial\mathbf{B}} = 2 \left[\left(\frac{\partial\widehat{\psi}}{\partial I_1} + I_1 \frac{\partial\widehat{\psi}}{\partial I_2} \right) \mathbf{B} - \frac{\partial\widehat{\psi}}{\partial I_2} \mathbf{B}^2 \right] + \frac{p}{\rho} \mathbf{I} \\ + \frac{\partial\widehat{\psi}}{\partial J_2} [\mathbf{M}(\mathbf{B} \bullet \mathbf{M}) + (\mathbf{M} \bullet \mathbf{B})\mathbf{M}] + \frac{\partial\widehat{\psi}}{\partial J_3} [\mathbf{M}(\mathbf{B}^2 \bullet \mathbf{M}) \\ + (\mathbf{M} \bullet \mathbf{B}^2)\mathbf{M} + 2(\mathbf{M} \bullet \mathbf{B})(\mathbf{B} \bullet \mathbf{M})]. \end{aligned} \tag{4.16}$$

Also for the incompressible case a modification is required in the pointwise necessary conditions for the quasiconvexity of $\widehat{\psi}$. The vectors $\mathbf{a}$ and $\mathbf{b}$ appearing in Eq. (4.9) are no longer independent. Indeed the linearization of the incompressibility constraint $\det \mathbf{F} = 0$ results in $\nabla \bullet \delta\mathbf{u} = 0$ whose Fourier transform is $\Xi \bullet \Delta\mathbf{u} = 0$ and thus Eq. (4.9) must hold for all real vectors $\mathbf{a}$, $\mathbf{b}$ satisfying

$$\mathbf{a} \bullet \mathbf{b} = \mathbf{0}. \tag{4.17}$$

5. Illustrative example

The magneto-mechanical coupling effects of the solids considered are illustrated through the uniaxial stretching and torsional response of a MRE cylinder subjected to a uniform magnetic field along its axis. The constitutive characterization of this incompressible elastomer is approximated from magnetization, uniaxial stretching and small field magnetostriction experimental data of Ginder et al. (1999), Clark and Ginder (2002) and Ginder et al. (2002), respectively, as described in Kankanala et al.

Table 2
Chosen coefficients for the free energy of an MRE

G (MPa)	C_{10}	C_{20}	$\mu_0\rho M_s$ (T)	C_{11}	C_{21}	C_{01}	C_{02}	C^*_{01}
1.0	5/9	4/9	1/2	8/25	3/25	1/16	3/80	1/25

(2003).[10] These results have been fitted to the following free energy per unit volume $\rho\widehat{\psi}(I_1, I_2, J_1, J_2)$ which depends only on four of the six invariants ($I_3 = 1$ because of incompressibility while for simplicity $\partial\widehat{\psi}/\partial J_3 = 0$):

$$\rho\widehat{\psi} = \frac{G}{2}\left\{\left[C_{10} + C_{11}\frac{J_1}{(M_s)^2}\right](I_1 - 3) + \left[C_{20} + C_{21}\frac{J_1}{(M_s)^2}\right](I_2 - 3) + C_{01}\frac{J_1}{(M_s)^2} + C_{02}\frac{J_2}{(M_s)^2} + C^*_{01}\left[\cosh\left(\frac{J_1}{(M_s)^2}\right) - 1\right]\right\}. \tag{5.1}$$

The values of the material constants G (initial shear modulus in the absence of a magnetic field) and M_s (saturation magnetization) as well as the values of the dimensionless constants C_{ij} and C^*_{ij} ($0 \leqslant i, j \leqslant 2$) are given in Table 2. The notational convention employed in these constants is that their first index corresponds to the presence of invariants I_i and the second to the presence of invariants J_j with a zero when the corresponding invariant is absent. The constants C_{ij} are associated with the linearly dependent terms on the invariants I_i and J_j while C^*_{ij} indicate a nonlinear dependence of the same invariants.

Before proceeding with the applications, we must check if the proposed free energy satisfies the necessary pointwise conditions for quasiconvexity given in Eq. (4.9).[11] The first two inequalities are easy to verify that they hold for all possible deformation gradients $\mathbf{F}$ and magnetizations $\mathbf{M}$. Unfortunately no proof could be found so far for the third inequality in Eq. (4.9) which was verified numerically for arbitrary magnetizations and for large enough strains $|\ln\lambda_i| \leqslant 3$ where λ_i are the stretch ratios (principal values of $\mathbf{F}$). The maximum strains and magnetizations occurring in the two illustrative examples that follow, fall well inside the domains of $\mathbf{F}$ and $\mathbf{M}$ for which Eq. (4.9) is satisfied.

[10] We would like to mention at this point the recent work of Brigadnov and Dorfmann (2003) and Dorfmann and Ogden (2003) on (isotropic) modeling of similar MREs. Their (direct) formulation is, however, based on a free energy depending on magnetic flux $\mathbf{b}$ and linear magnetic constitutive equations.

[11] It should be pointed out that even in the purely elastic case ($\mathbf{M} = \mathbf{0}$), it is difficult to check general pointwise rank one convexity conditions (even for the isotropic case); for instance, as seen in Zee and Sternberg (1983) for incompressible or Rosakis (1990) for compressible hyperelastic solids. Ball's (1977a, b), Ball and Marsden (1984) polyconvexity concept is a mathematically attractive and physically meaningful way to guarantee quasiconvexity. Unfortunately we were not able to see an easy generalization for the magnetoelastic case.

5.1. Uniaxial stretching of a cylinder

The first boundary value problem to be solved pertains to the uniaxial stretching of a cylinder with initial aspect ratio R (radius)$/L$ (length) $= 1/2$ under the influence of constant end tractions $\pm t\mathbf{e}_z$ and subjected to a uniform axial magnetic field $h\mathbf{e}_z$. Denoting by λ the stretch ratio in the axial direction and considering the material's incompressibility and isotropy, the left Cauchy–Green tensor $\mathbf{B}$, the magnetic field $\mathbf{h}$ and the magnetization per mass $\mathbf{M}$ take the following form in cylindrical coordinates:

$$\mathbf{B} = \frac{1}{\lambda}(\mathbf{e}_r\mathbf{e}_r + \mathbf{e}_\theta\mathbf{e}_\theta) + \lambda^2\mathbf{e}_z\mathbf{e}_z, \quad \mathbf{h} = h\mathbf{e}_z, \quad \mathbf{M} = M\mathbf{e}_z, \tag{5.2}$$

while the invariants I_1, I_2, J_1, J_2 defined in Eq. (4.11) become in view of Eq. (5.2)

$$I_1 = \lambda^2 + \frac{2}{\lambda}, \quad I_2 = 2\lambda + \frac{1}{\lambda^2}, \quad J_1 = M^2, \quad J_2 = (\lambda M)^2. \tag{5.3}$$

The unknown scalar pressure p is found from the requirement that the lateral traction $\mathbf{t} = 0$ for $\mathbf{n} = \mathbf{e}_r$. When this result is combined with Eqs. (4.14) and (4.16), the axial traction at the end section $\mathbf{t} = t\mathbf{e}_z$ for $\mathbf{n} = \mathbf{e}_z$ is found to be, in view also of Eqs. (5.2) and (5.3):

$$t = 2\rho\left[\left(\frac{\partial\widehat{\psi}}{\partial I_1} + \frac{1}{\lambda}\frac{\partial\widehat{\psi}}{\partial I_2}\right)\left(\lambda^2 - \frac{1}{\lambda}\right) + M^2\left(\frac{\partial\widehat{\psi}}{\partial J_1} + 2\lambda^2\frac{\partial\widehat{\psi}}{\partial J_2}\right)\right] - \frac{\mu_0}{2}(\rho M)^2. \tag{5.4}$$

Similarly, by using Eqs. (5.1)–(5.3) into the $\mathbf{M}-\mathbf{h}$ relation in Eq. (4.12) one obtains the following relation between h and M:

$$\mu_0 h = 2M\left(\frac{\partial\widehat{\psi}}{\partial J_1} + \lambda^2\frac{\partial\widehat{\psi}}{\partial J_2}\right). \tag{5.5}$$

The results illustrating the magnetoelastic coupling in a uniaxial stretching experiment of the material described by Eq. (5.1) are depicted in Figs. 3–5 for a cylinder of aspect ratio $R/L = 1/2$. In Fig. 3 is depicted the dimensionless traction, t/G vs. the natural (logarithmic) strain $\ln(\lambda)$ (stress–strain) as h varies, Fig. 4 gives the dimensionless magnetization M/M_s vs. the dimensionless applied magnetic field $h/\rho M_s$ at different preloads t/G while Fig. 5 shows the magnetostriction, i.e. the strain change Δe (where e is the engineering strain defined as $\lambda - 1$) vs. the dimensionless applied magnetic field $h/\rho M_s$ at three different levels of prestress t/G.

Note in Fig. 3 that for $h/\rho M_s \neq 0$ the traction vs. natural strain uniaxial response shows important deviations from the unmagnetized case $h=0$, up to a 30% for $h/\rho M_s = 0.8$ and for tensile strains of about 20%. Notice also that to maintain a zero strain under an external magnetic field, a tensile stress is required to overcome the attractive interparticle forces. As the tensile strain increases, the average interparticle distance increases and the interparticle forces are weaker for the same imposed external field h,

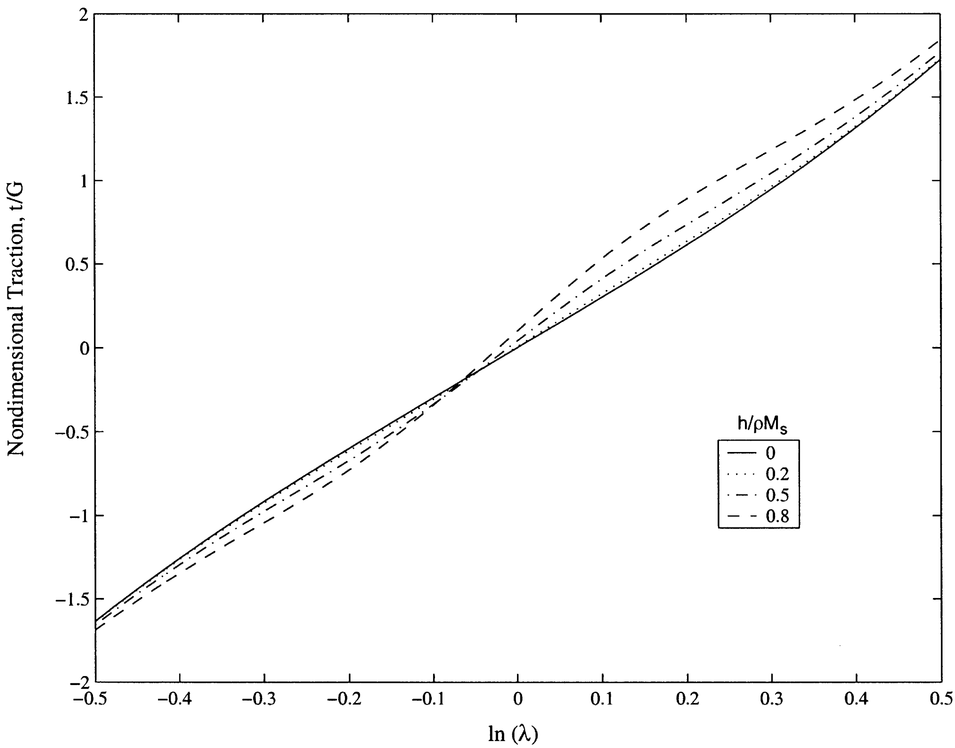

Fig. 3. Dimensionless traction t/G vs. engineering strain $e = \lambda - 1$ for a uniaxially loaded cylindrical MRE specimen under different dimensionless external magnetic fields $h/\rho M_s$. Note that for $h \neq 0$, a positive traction is needed to prevent magnetostriction at zero strain.

thus explaining the diminishing influence of h on the traction as the strains increase. Also observe that for high enough compressive strains ($< -5\%$) the magnitude of the compressive traction increases with an increasing applied magnetic field (indicating repulsive forces once the average particle distance is considerably diminished).

The magnetization response of the material to an external magnetic field, for different levels of prestress, is shown in Fig. 4. Again the shortening of the interparticle distance under compressive preloads effectively increases the magnetic susceptibility of the MRE, while the reverse is observed for tensile loads. Finally, Fig. 5 shows the magnetostriction behavior of the MRE under different preloads. Constriction of the material is expected under tensile (or zero—more obvious due to magnetostriction) preloads due to the attractive nature of interparticle forces for $h \neq 0$ under $\sigma \geqslant 0$. The elongation of the material observed for compressive preloads is consistent with observed increase in compressive loads for negative strains.

The results shown in Figs. 3–5 indicate that there is a strong magnetoelastic coupling effect (35% increase in stress) for relatively moderate strains ($\sim 20\%$ in tension) under important external fields ($\sim h/\rho M_s = 0.8$). The magnetoelastic coupling is found to be

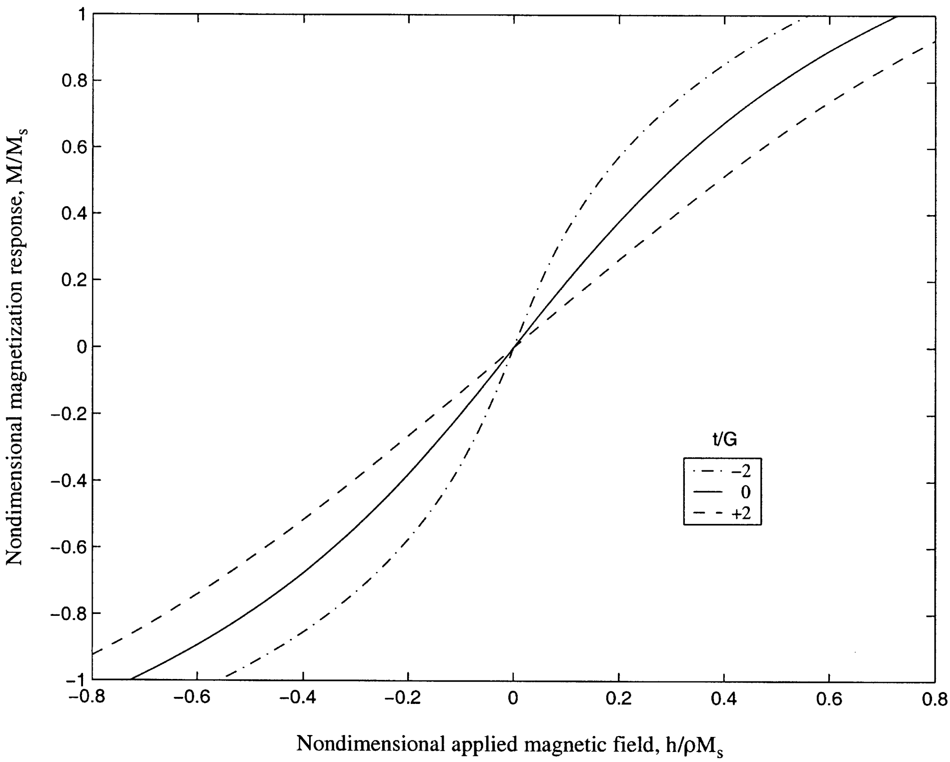

Fig. 4. Dimensionless magnetization M/M_s vs. dimensionless applied magnetic field $h/\rho M_s$ for a cylindrical MRE specimen under different dimensionless external tractions t/G. For tensile prestress ($t > 0$) iron particle distance increases thus lowering the specimen's magnetization compared to the stress-free case. The opposite is true for compression ($t < 0$) due to shorter distances among the iron particles.

the strongest when material is solicited in the direction of magnetization. This is why uniaxial results only are analyzed here. However, coupling effects also show up under shear in a way that alters the material's nonlinear response at high strains, and this is why the finite strain torsion problem in the presence of a magnetic field is examined next.

5.2. Torsion of a cylinder

The second boundary value problem solved is the finite torsion of the same incompressible circular cylinder of aspect ratio $R/L = 1/2$ subjected to a rate of twist χ (the rotation angle of a section at a distance z from the origin is $\Delta\theta = \chi z$). The kinematics of this deformation which lead to the left Cauchy–Green tensor $\mathbf{B}$ have been repeatedly presented in the literature (e.g. Green and Zerna, 1954) and need not be repeated here. Consequently, the expressions for $\mathbf{B}, \mathbf{M}$ and $\mathbf{h}$ are given at a point with cylindrical

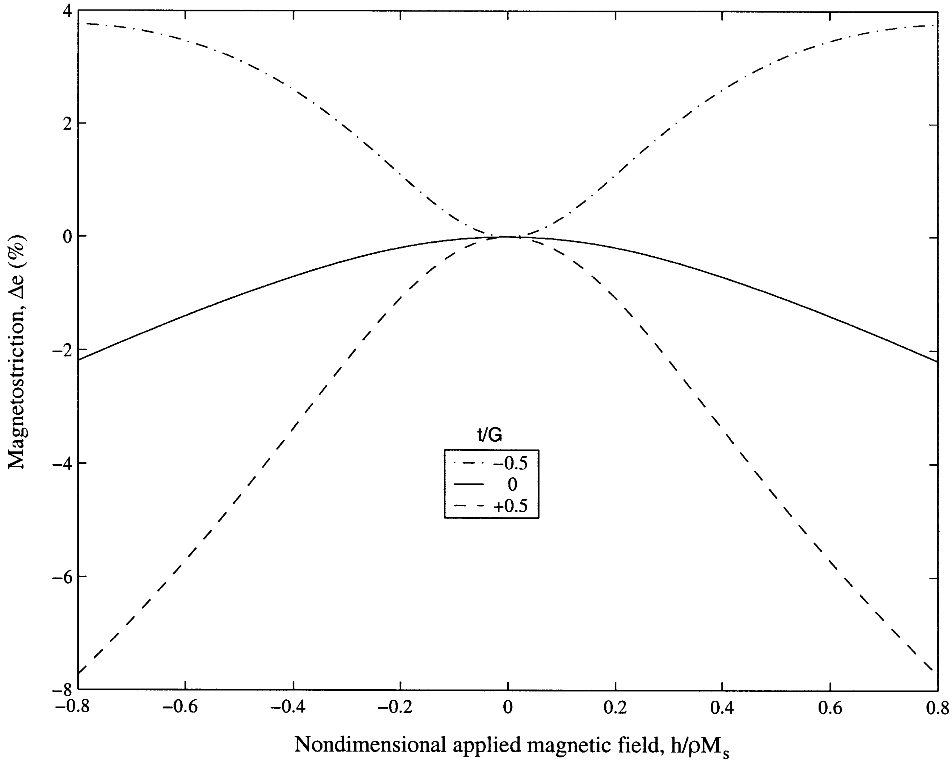

Fig. 5. Engineering strain change Δe due to magnetostriction of a cylindrical MRE specimen subjected to an applied dimensionless magnetic field $h/\rho M_s$ under different dimensionless external prestress t/G. For compressive stresses a positive strain has to be applied to prevent the stress from a further decrease as the iron particles are pulled together due to an increase in the magnetic field h. The opposite is true for tension ($t \geqslant 0$).

coordinates r, θ, z by

$$\mathbf{B} = \mathbf{e}_r\mathbf{e}_r + [1 + (r\chi)^2]\mathbf{e}_\theta\mathbf{e}_\theta + \mathbf{e}_z\mathbf{e}_z + r\chi(\mathbf{e}_\theta\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_\theta),$$
$$\mathbf{M} = M_\theta(r)\mathbf{e}_\theta + M_z(r)\mathbf{e}_z, \quad \mathbf{h} = h\mathbf{e}_z, \tag{5.6}$$

while the invariants I_1, I_2, J_1, J_2 defined in Eq. (4.11) are found with the help of Eq. (5.6) to be

$$I_1 = I_2 = 3 + (r\chi)^2, \quad J_1 = (M_\theta)^2 + (M_z)^2,$$
$$J_2 = (M_\theta)^2[1 + (r\chi)^2] + (M_z)^2 + 2M_\theta M_z r\chi. \tag{5.7}$$

The two nontrivial components (along $\mathbf{e}_\theta$ and $\mathbf{e}_z$ of the $\mathbf{M} - \mathbf{h}$ relationship in Eq. (4.12)), specialized for $\widehat{\psi}$ in Eq. (5.1) (i.e. considering that $\partial\widehat{\psi}/\partial J_3 = 0$)

result in

$$\left[\frac{\partial\widehat{\psi}}{\partial J_1}+\frac{\partial\widehat{\psi}}{\partial J_2}[1+(r\chi)^2]\right]M_\theta+\left[\frac{\partial\widehat{\psi}}{\partial J_2}r\chi\right]M_z=0,$$

$$2\left[\frac{\partial\widehat{\psi}}{\partial J_2}r\chi M_\theta+\left(\frac{\partial\widehat{\psi}}{\partial J_1}+\frac{\partial\widehat{\psi}}{\partial J_2}\right)M_z\right]=\mu_0 h. \tag{5.8}$$

The only unknown that we need to solve for in this problem, before determining all the field quantities, is the pressure $p(r)$. To this end we first use Eqs. (4.13) and (4.16) to obtain the radial and azimuthal stresses in terms of p, respectively, as

$$\sigma_{rr}=2\rho\left\{\frac{\partial\widehat{\psi}}{\partial I_1}+[2+(r\chi)^2]\frac{\partial\widehat{\psi}}{\partial I_2}\right\}+p-\frac{\mu_0}{2}h^2,$$

$$\sigma_{\theta\theta}=2\rho\left\{\frac{\partial\widehat{\psi}}{\partial I_1}[1+(r\chi)^2]+\frac{\partial\widehat{\psi}}{\partial I_2}[2+(r\chi)^2]\right.$$
$$\left.+\frac{\partial\widehat{\psi}}{\partial J_2}[(M_\theta)^2[1+(r\chi)^2]+M_\theta M_z r\chi]\right\}+p-\frac{\mu_0}{2}h^2. \tag{5.9}$$

Applying next the equilibrium equation (2.18) (in the absence of acceleration and body forces) results in the only nontrivial (radial component) equation

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0. \tag{5.10}$$

Upon integrating Eq. (5.10), applying the traction condition $t_r=0$ at the lateral surface $r=R$, and using Eqs. (4.14) and (5.7), one finds

$$2\rho\left[\frac{\partial\widehat{\psi}}{\partial I_1}+[2+(r\chi)^2]\frac{\partial\widehat{\psi}}{\partial I_2}\right]_{r=R}+p(R)=0, \tag{5.11}$$

resulting in the following final expression for $p(r)$

$$p(r)=2\rho\left\{\int_R^r\left[\frac{\partial\widehat{\psi}}{\partial I_1}(r\chi)^2+\frac{\partial\widehat{\psi}}{\partial J_2}[(M_\theta)^2(1+(r\chi)^2)+M_\theta M_z r\chi]\right]\frac{\mathrm{d}r}{r}\right.$$
$$\left.-\frac{\partial\widehat{\psi}}{\partial I_1}-\frac{\partial\widehat{\psi}}{\partial I_2}[2+(r\chi)^2]\right\}. \tag{5.12}$$

We now have all the ingredients at hand to calculate the axial force N and twisting moment T at each end of the cylinder, which are given by

$$N=2\pi\int_0^R t_z^{\mathrm{top}}r\,\mathrm{d}r,\qquad M=2\pi\int_0^R t_\theta^{\mathrm{top}}r^2\,\mathrm{d}r, \tag{5.13}$$

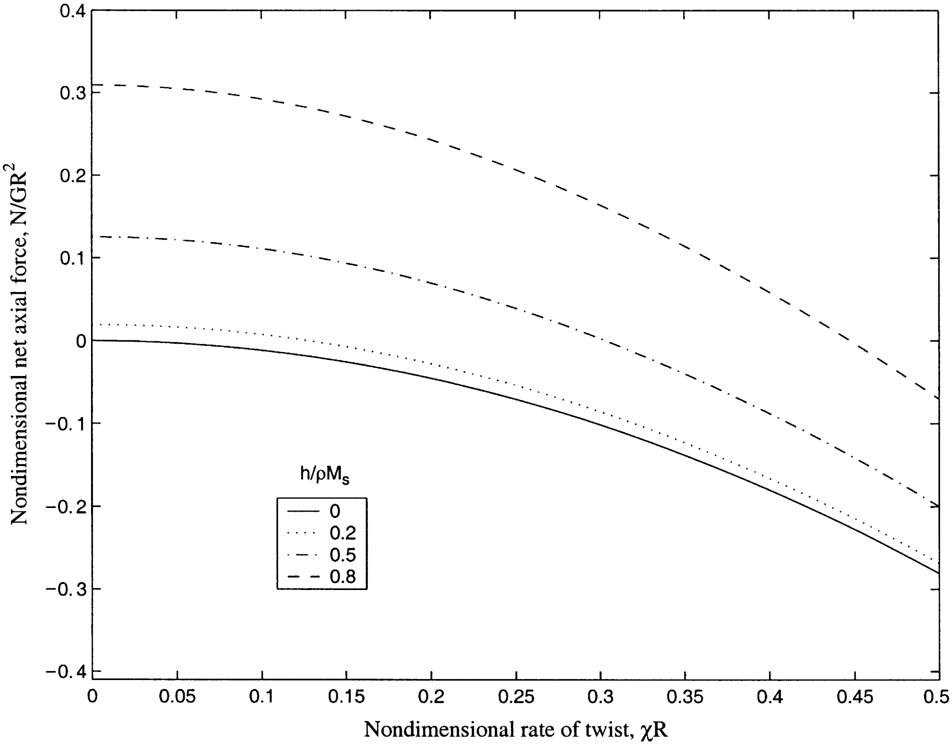

Fig. 6. Dimensionless axial force N/GR^2 vs. dimensionless rate of twist χR for the pure torsion of a cylindrical MRE specimen under various applied axial dimensionless magnetic fields $h/\rho M_s$. For the purely mechanical response the axial force is compressive and proportional to $(\chi R)^2$ as expected from finite elasticity in Mooney materials. However as h increases the curves are shifted by a positive amount since a tensile force is required to keep the particles from getting closer.

where $\mathbf{t}^{\text{top}} = t_\theta^{\text{top}}\mathbf{e}_\theta + t_z^{\text{top}}\mathbf{e}_z$ is the traction vector at the top surface ($z = L$, $\mathbf{n} = \mathbf{e}_z$) of the cylinder. The final expression for N and M are found with the help of the traction definition for $\mathbf{t}$ in Eqs. (4.14) and (4.16) using $\mathbf{n} = \mathbf{e}_z$ and employing the equation for pressure in Eq. (5.11) as well as Eq. (5.8) to arrive at the following expressions:

$$N = 2\pi \int_0^R \left\{ 2\rho \left[-(r\chi)^2 \left[\frac{1}{2}\frac{\partial\widehat{\psi}}{\partial I_1} + \frac{\partial\widehat{\psi}}{\partial I_2} \right] + \frac{\partial\widehat{\psi}}{\partial J_1}(M_z)^2 + \frac{\partial\widehat{\psi}}{\partial J_2} \left[2(M_z)^2 \right.\right.\right.$$

$$\left.\left.\left. + \frac{3}{2} M_\theta M_z r\chi - \frac{1}{2}(M_\theta)^2(1 + (r\chi)^2) \right] \right] - \frac{\mu_0}{2}(\rho M_z)^2 \right\} r\,\mathrm{d}r,$$

$$M = 2\pi \int_0^R 2\rho \left\{ \left(\frac{\partial\widehat{\psi}}{\partial I_1} + \frac{\partial\widehat{\psi}}{\partial I_2} \right) r\chi + \frac{\partial\widehat{\psi}}{\partial J_2} \left[\frac{3}{2}(M_\theta)^2 r\chi + M_\theta M_z \left[1 - \frac{1}{2}(r\chi)^2 \right] \right.\right.$$

$$\left.\left. - \frac{1}{2}(M_z)^2 r\chi \right] \right\} r^2\,\mathrm{d}r. \tag{5.14}$$

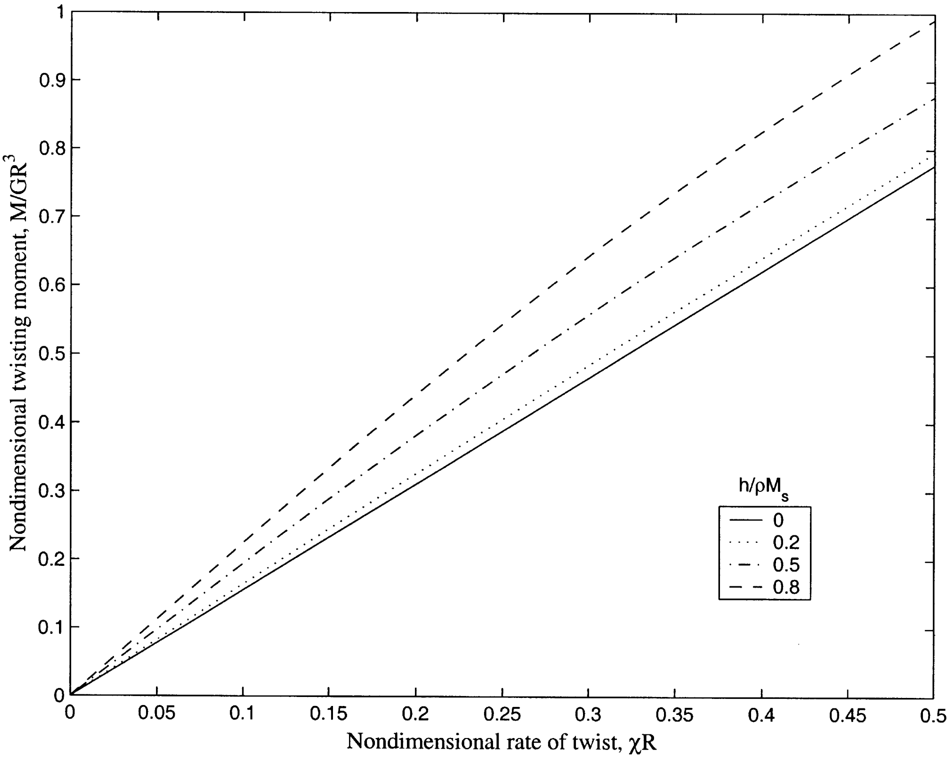

Fig. 7. Dimensionless twisting moment M/GR^3 vs. dimensionless rate of twist χR for the pure torsion of a cylindrical MRE specimen under various applied axial dimensionless magnetic fields $h/\rho M_S$. In the absence of a magnetic field, the twisting moment is linear (proportional to χR). As h increases, the interparticle forces serve to "stiffen" the MRE for up to moderate rates of twist. As rates of twist are further increased, the magnetic effect begins to diminish leading to a nonlinear moment-twist behavior.

The influence of an external magnetic field on the dimensionless axial force N/GR^2 and twisting moment M/GR^3, respectively, as a function of the dimensionless rate of twist χR, is depicted in Figs. 6 and 7. A known phenomenon in nonlinear elasticity is the development of a compressive axial force in the pure twist experiment of a cylindrical specimen (e.g. Green and Zerna, 1954). This compressive force, corresponding to no external magnetic field, is for a Mooney–Rivlin material (to which Eq. (5.1) reduces to for $\mathbf{M}=\mathbf{0}$) proportional to the square of the applied rate of twist as shown in solid line in Fig. 6. In the presence of an externally applied field, the force-rate of twist curves look still parabolic but are shifted upwards (towards the tensile range) with an increasing applied magnetic field. Again the explanation for this phenomenon can be found in the tendency of the particles to decrease their average distance under an external magnetic field, which then requires the application of a tensile force to prevent shortening of the specimen.

The dimensionless moment M/GR^3 vs. dimensionless twist χR response under various external magnetic fields is plotted in Fig. 7 and shows, over a wide range of twists, a linear increase of the torsional stiffness (slope of the curve) in terms of the applied magnetic field (the max 30% increase corresponds to $h/\rho M_s = 0.8$).

The results for the torsion case show the importance of magnetoelastic coupling under torsion as well as the consequences of this coupling on the nonlinear aspects of the material's response (influence on the restraining force, changing from compressive to tensile). The exploration of coupling effects in more complicated geometries requires numerical techniques based on the previously presented general formulation and is the object of current studies.

6. Conclusions

In this work we present the continuum formulation for finitely strained magnetorheological elastomers, i.e. rubbers filled with micron-sized ferromagnetic particles. Two different approaches are presented for MREs with negligible dissipative and hysteretic behaviors: a direct one based on balance laws formulated in the current configuration and a novel energy approach based on the unconstrained minimization of a potential energy functional formulated in the reference configuration. The main advantage of the direct approach presented is that no a priori assumptions are made for the expressions of electromagnetic body forces and electromagnetic part of the stress. Unlike in earlier works, an energy minimizer is used in the latter energetic approach to derive Ampère's law, equilibrium equation and traction condition. It is shown that both approaches result in the same governing equations and boundary conditions. In discussing the properties of the solid's free energy we pay particular attention to the quasiconvexity of the potential energy and derive sufficient, for the quasiconvexity, pointwise conditions on the free energy. To illustrate the magnetoelastic coupling in MREs we propose a free energy function inferred from existing experimental data and proceed to show the influence of magnetic field on uniaxial traction and pure torsion of cylinders as well as the influence of prestress on magnetization curves.

The proposed theory can be easily finetuned to account for anisotropic MREs, as is the case of MREs that are cured in the presence of strong magnetic fields. Moreover, the proposed new variational formulation intended to be used in numerical calculations of application devices made of MREs, where its Lagrangian formulation and energy minimization features plus the absence of constraints in the variables used, make for an efficient numerical algorithm. In addition, the proposed theory, because of its energy minimization feature, is ideally suited for the study of stability problems in MREs, currently under investigation.

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