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Asymptotic Stability Analysis for Sheet Metal Forming—Part I: Theory

In this paper is presented a general methodology for predicting puckering instabilities in sheet metal forming applications. A novel approach is introduced which does not use shell theory approximations. The starting point is Hill's stability functional for a three-dimensional rate-independent stressed solid which is modified for contact. By using a multiple scale asymptotic technique with respect to the small dimensionless thickness parameter ε*, one can derive the two-dimensional version of the stability functional which is accurate up to* $O(\varepsilon^4)$*, thus taking into account bending effects. Loss of positive definiteness of this functional indicates possibility of a puckering instability in a sheet metal forming problem with a known stress and deformation state. An advantage of the proposed method is that the puckering investigation is independent of the algorithm used for calculating the deformed state of the sheet.* [S0021-8936(00)00804-7]

1 Introduction and Motivation

Stamping of sheet metal is one of the most widely used industrial manufacturing processes. There are three major problems that limit the formability of a stamped part which have to be accounted for in its design: springback, tearing, and puckering/wrinkling. Springback is the change in shape of the part that occurs after a part is removed from the blankholder/die assembly and is due to the elastic unloading of the part. Tearing is the splitting of the part in areas of high strain concentrations and is due to the localized necking of the sheet. Puckering, as defined by Devons [1] is a waviness of the sheet that is not in contact with the tooling surfaces and is a bifurcation buckling phenomenon due to the presence of compressive in-plane stresses in the sheet. When the same phenomenon occurs in areas that come into contact with the tooling, usually the flat part of the binder, the surface waviness phenomena is known as wrinkling.

In modeling a tearing problem, the difficulty is in the determination of the proper constitutive law since the phenomenon is local in nature. The difficulty in modeling springback is due to the geometry of the part and in the determination of its prestress state from which an elastic unloading takes place. Modeling of puckering requires both an accurate description of the constitutive behavior of the material and the solution of a boundary value problem. Moreover, experimental investigation of puckering faces the difficulty of the determination of the onset of the phenomenon, since imperfections in the form of minute amounts of surface waviness are always present in stamped parts.

Of interest here is the modeling of puckering instabilities for general stamping geometries. The standard approach thus far uses a shell-type analysis (usually in conjunction with a finite element method code) and follows the deformation of the part all the way to the formation of finite amplitude wrinkles (e.g., Taylor et al. [2]). More refined analyses use a linearized stability method to check for bifurcation in a part with a known prestress state obtained by using a shell-type analysis (e.g., Neal and Tugcu [3]). The obvious shortcoming of this approach is the stability results' dependence on the shell theory employed.

To overcome the inconsistencies associated with the use of a particular shell theory to calculate the onset of bifurcation in shell buckling problems, Triantafyllidis and Kwon [4] proposed reversing the order of the limiting process in the analysis, by first formulating the stability problem of the three-dimensional solid and then finding its critical load and buckling mode as the dimensionless thickness parameter, ε, goes to zero. This asymptotic methodology has recently been applied by Scherzinger and Triantafyllidis [5] to another similar problem, namely the buckling of slender beams with arbitrary cross sections (there the ε parameter is the beam's slenderness defined as the square root of its sectional area over its length) where the interested reader can find another comprehensive application of the proposed methodology.

The departing point for our analysis is Hill's [6,7] stability functional for a three-dimensional elastoplastic solid, properly modified to account for contact with tooling surfaces. Since stability against puckering depends on the sign of the functional's minimum eigenvalue, the present work consists of a multiple scale asymptotic analysis to obtain the minimum eigenvalue and the corresponding eigenmode in terms of ε. The multiple scale analysis is a finite strain adaptation of the methodology proposed by Destuynder [8] for the consistent derivation of linear elastic shell theories from the corresponding three-dimensional equations of elasticity. The present method results in the calculation of the stability functional of a prestressed stamped sheet that is accurate to $O(\varepsilon^4)$. The functional includes bending stiffness effects and only requires a two-dimensional stress state and eigenmode (the degrees-of-freedom are the displacements of the midsurface). No shell theory assumptions are required and normality of plane sections and the plane stress assumption arise naturally as a part of the analysis.

The outline of this work is as follows: The presentation begins with a description of the kinematics for a shell of arbitrary shape in Section 2.1. The treatment of contact, essential for the stability of sheet metal forming problems, is presented in Section 2.2 followed by the statement of the variational problem to find the minimum eigenvalue in Section 2.3. The asymptotic analysis of the problem is presented in Section 3. Expansion of field quantities are given and substituted into the stability functional. The stability functional is evaluated in Section 3.2 and its two-dimensional form, suitable for sheet metal forming, is found accurate to $O(\varepsilon^4)$.

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2 Formulation

To begin the stability analysis for a sheet metal forming problem, one starts with Hill's [6] stability functional evaluated at a given equilibrium state[1] of a solid of volume V and boundary ∂V

$$\mathcal{F}(\lambda,\mathbf{v}) = \frac{1}{2}\int_V L^{ijkl}(\lambda) v_{k,l} v_{i,j} dV + \frac{1}{2}\int_{\partial V} K^{ik} v_k v_i dS. \quad (2.1)$$

Here λ is a load parameter, which determines the current stress state and the internal variables of the solid, v_i are the covariant components of a kinematically admissible perturbation away from the equilibrium state, $v_{i,j}$ denotes the covariant derivative with respect to the three-dimensional basis in the current configuration, L^{ijkl} are the contravariant components of the incremental moduli that relate the rate of the first Piola-Kirchoff stress to the rate of the deformation gradient ($\dot{\mathbf{\Pi}}^T = \mathbf{L}:\dot{\mathbf{F}}$). The second integral in (2.1) is a penalty term that is introduced to account for the contact between the tooling surfaces and the sheet. The symmetry of the incremental moduli, L^{ijkl}, and the contact terms, K^{ik}, implies real eigenvalues for $\mathcal{F}$. Positive definiteness of (2.1) ensures stability of the structure at the given equilibrium state, in the sense that positive energy has to be externally supplied into the structure for any admissible perturbation. The idea proposed in this work is to take advantage of the slenderness of the structure to develop a two-dimensional form of the stability functional which is accurate to any desired order of the slenderness parameter.

2.1 Geometric Preliminaries. Consider a shell-like structure in its current configuration as shown in Fig. 1. The solid occupies a volume V and has boundary ∂V consisting of the top, bottom and lateral surfaces ($\partial V = \partial V_+ \cup \partial V_- \cup \partial V_n$). Material points in the shell are identified by their convected coordinates, $\theta^i = (\theta^\alpha, \theta^3)$; a material point on the midsurface of the shell has position vector $\mathbf{r}(\theta^\alpha)$, and the domain of the midsurface of the sheet is denoted by A. The position vector, $\mathbf{p}$, for an arbitrary material point in the shell can be written as

$$\mathbf{p}(\theta^i) = \mathbf{r}(\theta^\alpha) + \theta^3 \mathbf{n}(\theta^\alpha), \quad -\frac{h(\theta^\alpha)}{2} \leqslant \theta^3 \leqslant \frac{h(\theta^\alpha)}{2}, \quad (2.2)$$

where $\mathbf{n}(\theta^\alpha)$ is the unit outward normal to the midsurface of the shell. At the material point on the midsurface, given by θ^α, the thickness of the shell is $h(\theta^\alpha)$.

The basis vectors for the midsurface, $\mathbf{a}_i$, and the three-dimensional solid, $\mathbf{g}_i$, are defined as follows:

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha}, \quad \mathbf{a}_3 = \mathbf{n}; \quad \mathbf{g}_i = \frac{\partial \mathbf{p}}{\partial \theta^i}. \quad (2.3)$$

The midsurface and three-dimensional metrics are given by $a_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j$ and $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ with inverses a^{ij} and g^{ij}. From (2.3) the relation between the $\mathbf{a}_i$ basis and the $\mathbf{g}_i$ basis is established with the help of μ_i^j,

$$\mathbf{g}_i = \mu_i^j \mathbf{a}_j; \quad \mu_\beta^\alpha = \delta_\beta^\alpha - \theta^3 b_\beta^\alpha, \quad \mu_i^3 = \mu_3^i = \delta_i^3,$$
$$\mathbf{a}_i = q_i^j \mathbf{g}_j; \quad q_i^j = (\mu_j^i)^{-1}, \quad (2.4)$$

where b_β^α are the mixed components of the curvature tensor for the midsurface. The evaluation of the gradient of the perturbation, introduced in (2.1), requires the derivatives of the midsurface basis vectors with respect to the convected coordinates

$$\frac{\partial \mathbf{a}_i}{\partial \theta^j} = t_{ij}^k \mathbf{a}_k. \quad (2.5)$$

[1]Here and subsequently Greek indices range from 1 to 2 and Latin indices range from 1 to 3.

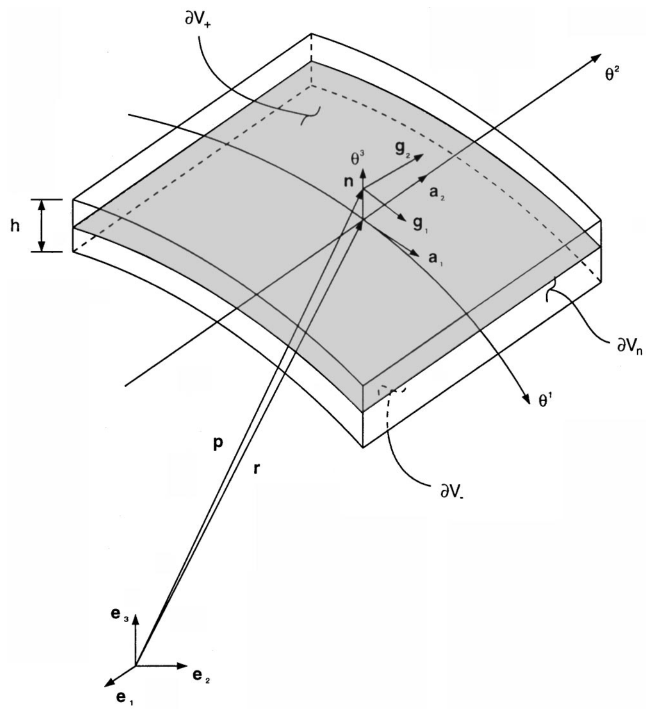

Fig. 1 Three-dimensional kinematics for a shell-like structure. The midsurface is defined by the vector r while a point off the midsurface is defined by position vector p. The covariant midsurface basis is $\mathbf{a}_i$ and the covariant three-dimensional basis is $\mathbf{g}_i$.

Since $\mathbf{a}_\alpha$ are the basis vectors for the midsurface, the coefficients $t_{\alpha\beta}^\gamma = \gamma_{\alpha\beta}^\gamma$ are the Christoffel symbols for the midsurface. From differential geometry (e.g., Goetz [9]), the only other nonzero components of t_{ij}^k are

$$t_{\alpha\beta}^3 = b_{\alpha\beta} = \alpha_{\alpha\gamma} b_\beta^\gamma, \quad t_{3\beta}^\gamma = -b_\beta^\gamma. \quad (2.6)$$

The components of any tensor in the subsequent analysis can be defined with respect to the three-dimensional basis, $\mathbf{g}_i$, or the midsurface basis, $\mathbf{a}_i$ (e.g., $\mathbf{v} = v^i \mathbf{g}_i = \overline{v}^i \mathbf{a}_i$).[2] The tensor components in (2.1) are referred to the three-dimensional basis; however, the following analysis naturally employs the midsurface basis. Therefore, relationships between the components of tensors referred to the two bases are needed. In particular, relationships are sought, with the help of (2.4) and (2.5), for the components of the incremental moduli, $\mathbf{L}$, and the gradient of the perturbation, $\mathbf{v}\nabla$,

$$L^{ijkl} = \overline{L}^{mnrs} q_m^i q_n^j q_r^k q_s^l; \quad v_{i,j} = \left(\frac{\partial \overline{v}_k}{\partial \theta^j} - t_{kj}^l \overline{v}_l\right)\mu_i^k = \overline{v}_{k,j}\mu_i^k. \quad (2.7)$$

2.2 Treatment of Contact. In metal forming applications, the sheet is in contact with the rigid surfaces of the punch, die, and blank holder, and this contact will constrain any kinematically admissible perturbation, $\mathbf{v}$. Therefore the effect of contact must be accounted for in the statement of the stability problem; this is done using a penalty-type formulation.

It is assumed, without loss of generality, that contact occurs between the tooling and the bottom surface of the sheet (∂V_-). Denoting the energy due to contact as

[2]Here and subsequently components of tensors with respect to the midsurface basis will be denoted with a bar surmounting the symbol, $(\overline{\cdot})$.

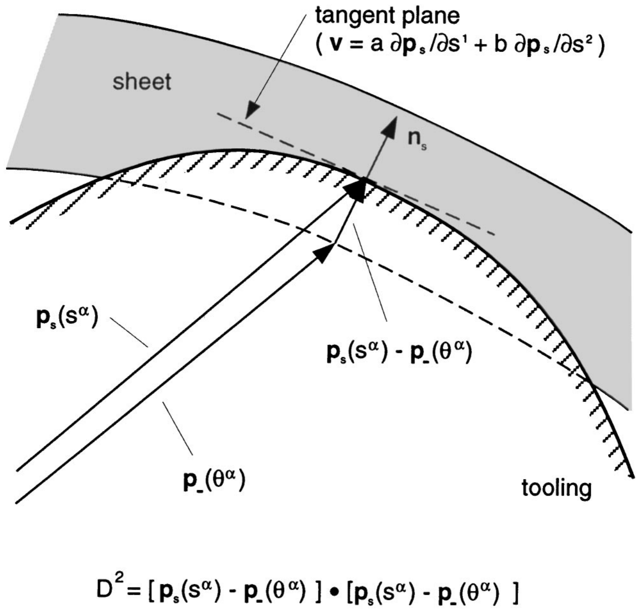

Fig. 2 Contact between tooling surface and sheet. A point on the sheet's lower surface with position vector $\mathbf{p}_-$ is at a distance D from the rigid surface. The penetration distance is positive when $\mathbf{p}_- - \mathbf{p}_s$ has the same orientation as $\mathbf{n}_s$, the outward unit normal to the rigid surface.

$$\mathcal{E}=\frac{1}{2}\int_{\partial V_-} kH(D)D^2 dA, \tag{2.8}$$

where k is a foundation stiffness for the tooling, $D(\mathbf{u})$, a function of the displacement $\mathbf{u}$, is the interpenetration distance between the bottom surface of the sheet and the tooling surface, and $H(D)$ is the Heaviside step function ($H(D)=1$ if $D\geqslant 0$ and $H(D)=0$ if $D<0$), the contact term for the stability functional is augmented by

$$\frac{1}{2}(\mathcal{E}_{,uu}\delta_u)\delta u=\frac{1}{2}\int_{\partial V_-} kH(D)(\delta D\,\delta D+D\,\delta(\delta D))dA, \tag{2.9}$$

where $\mathcal{E}_{,uu}$ denotes the second functional derivative of $\mathcal{E}$ with respect to the displacement $\mathbf{u}$. Given a parameterization of the rigid surface in terms of the coordinates s^α, the position vector for a point on this surface is given by $\mathbf{p}_s=\mathbf{p}_s(s^\alpha)$. The interpenetration distance between the sheet and the surface is defined as follows (see Fig. 2):

$$D^2=[\mathbf{p}_s(s^\alpha)-\mathbf{p}_-(\theta^\alpha)]\cdot[\mathbf{p}_s(s^\alpha)-\mathbf{p}_-(\theta^\alpha)], \tag{2.10}$$

where $\mathbf{p}_-(\theta^\alpha)=\mathbf{p}(\theta^\alpha,-h(\theta^\alpha)/2)$ is the position vector of a material point on the bottom surface of the sheet that is in contact with the rigid surface. The distance D is taken positive if there is a penetration of the sheet on the rigid surface, i.e., when $\mathbf{p}_s-\mathbf{p}_-$ is on the same direction as the outward normal to the rigid surface $\mathbf{n}_s$.

Given a material point $\mathbf{p}_-(\theta^\alpha)$, for a given equilibrium state, the point on the tooling surface, $\mathbf{p}_s(s^\alpha)$, is sought such that it minimizes D. This point is found to satisfy the following condition:

$$[\mathbf{p}_s(s^\alpha)-\mathbf{p}_-(\theta^\alpha)]\cdot\frac{\partial\mathbf{p}_s}{\partial s^\beta}=0, \tag{2.11}$$

which states that the vector $\mathbf{p}_s-\mathbf{p}_-$ is orthogonal to the tangent plane of the surface at the point $\mathbf{p}_s$. With the above relationship, the penalty term used for modeling contact in the stability functional becomes

$$\frac{1}{2}\int_{\partial V_-}\mathbf{v}\cdot\mathbf{K}\cdot\mathbf{v}dA$$
$$=\frac{1}{2}\int_{\partial V_-}kH(D)\left[\mathbf{v}\cdot\mathbf{v}-\left(\mathbf{v}\cdot\frac{\partial\mathbf{p}_s}{\partial s^\alpha}\right)\left(\mathbf{v}\cdot\frac{\partial\mathbf{p}_s}{\partial s^\beta}\right)\psi^{\alpha\beta}\right]dA, \tag{2.12}$$

where $\psi^{\alpha\beta}$ is

$$\psi^{\alpha\beta}=\left[\frac{\partial\mathbf{p}_s}{\partial s^\alpha}\cdot\frac{\partial\mathbf{p}_s}{\partial s^\beta}+(\mathbf{p}_s-\mathbf{p}_-)\cdot\frac{\partial^2\mathbf{p}_s}{\partial s^\alpha\partial s^\beta}\right]^{-1}. \tag{2.13}$$

Finally, it is noted that the effect of friction is seen only through the principal solution of the sheet metal forming process; it is assumed that friction does not affect the stability calculations.

2.3 Problem Statement. Stability of the prestressed solid is guaranteed (see Hill [6]) if the functional (2.1) is positive definite, i.e., if its minimum eigenvalue, defined below, is positive

$$\beta=\min_{\mathbf{v}\in\mathcal{D}}\frac{2\mathcal{F}}{\langle\mathbf{v},\mathbf{v}\rangle}. \tag{2.14}$$

Here $\mathcal{D}$ is the space of kinematically admissible perturbations and $\langle\mathbf{v},\mathbf{v}\rangle$ is an appropriately chosen inner product. For the problem examined here, the following choice for the inner product is adopted:

$$\langle\mathbf{v},\mathbf{v}\rangle=\int_A\int_{-h/2}^{h/2}a^{ik}\overline{v_k}\overline{v_i}d\theta^3 dA. \tag{2.15}$$

The minimization problem in (2.14) can be reformulated as a variational problem

$$\beta\int_A\int_{-h/2}^{h/2}a^{ik}\overline{v_k}\delta\overline{v_i}d\theta^3 dA=\int_A\int_{-h/2}^{h/2}\overline{L^{ijkl}}\overline{v_{k,l}}\delta\overline{v_{i,j}}\mu d\theta^3 dA$$
$$+\int_A[\overline{K^{ik}}\overline{v_k}\delta\overline{v_i}\mu]_{\theta^3=-h/2}dA. \tag{2.16}$$

In addition to the minimum eigenvalue, a mode uniqueness condition is also introduced:

$$\langle\mathbf{v},\mathbf{u}\rangle=\int_A\int_{-h/2}^{h/2}a^{ik}\overline{v_k}\overline{u_i}d\theta^3 dA=C \tag{2.17}$$

where $\mathbf{u}$ and C are an appropriately chosen vector field and constant, respectively.

3 Asymptotics

The solution of the stability problem for a sheet metal forming application is found using the following approach: Given an equilibrium state for the solid, an asymptotic analysis is employed to deduce a stability criterion for the shell-like structure based on the minimum eigenvalue of Hill's functional for the three-dimensional solid. In this approach, the current geometry, stress state, and as a result, the incremental moduli are known a priori. Since the thickness is assumed to be small relative to the dimensions of the sheet and the minimum radius of curvature of the midsurface, the asymptotic analysis is performed using the ratio of the sheet thickness, h, relative to a characteristic length of the sheet, l, as the small parameter in the problem. Denoting this small parameter by $\varepsilon\equiv h/l$, asymptotic expansions are carried out in terms of ε. The scaled thickness coordinate, $\xi=\theta^3/\varepsilon$, is also used in the subsequent analysis and varies between $-\zeta\leqslant\xi\leqslant\zeta$, where $\zeta\equiv h/2\varepsilon$.

The minimum eigenvalue and all the field quantities are expanded in a regular perturbation series in terms of ε^n, $n=0,1,2,\ldots$. By taking the expansions for these quantities, substituting them into (2.1), and evaluating the minimum eigenvalue of the stability functional, the stability of the sheet metal part is examined.

3.1 Expansions. For the analysis of the stability functional, the minimum eigenvalue, corresponding eigenmode, contact terms, and incremental moduli are expanded in a regular perturbation series in ε,

$$\beta(\varepsilon)=\overset{0}{\beta}+\varepsilon\overset{1}{\beta}+\varepsilon\overset{2}{\beta}+\cdots, \tag{3.1}$$

$$\overline{v}_i(\theta^j;\varepsilon)=\overset{0}{\overline{v}}_i(\theta^\alpha,\xi)+\varepsilon\overset{1}{\overline{v}}_i(\theta^\alpha,\xi)+\varepsilon^2\overset{2}{\overline{v}}_i(\theta^\alpha,\xi)+\cdots, \tag{3.2}$$

$$\overline{K}^{ik}(\theta^\alpha;\varepsilon)=\overset{0}{\overline{K}}{}^{ik}(\theta^\alpha)+\varepsilon\overset{1}{\overline{K}}{}^{ik}(\theta^\alpha)+\varepsilon^2\overset{2}{\overline{K}}{}^{ik}(\theta^\alpha)+\cdots, \tag{3.3}$$

$$\overline{L}^{ijkl}(\theta^\alpha;\varepsilon)=\overset{0}{\overline{L}}{}^{ijkl}(\theta^\alpha)+\varepsilon\overset{1}{\overline{L}}{}^{ijkl}(\theta^\alpha,\xi)+\varepsilon^2\overset{2}{\overline{L}}{}^{ijkl}(\theta^\alpha,\xi)+\cdots. \tag{3.4}$$

Similar expansions can be obtained for the geometric quantities μ^i_j and its inverse q^i_j which relate the midsurface and three-dimensional covariant basis vectors (see (2.4)). Finally, it is noted that the foundation stiffness for the contact term in the problem must also be scaled with ε, $k\to\varepsilon k$. In order to physically motivate this rescaling, we recall from the theory of shells that the normal and shear stresses are on the order of ε^2 and ε, respectively. The rescaling is necessary to enforce this condition as $\varepsilon\to 0$. It should also be noted at this point that the adopted asymptotic expansions for the various field quantities are expected to be valid several thicknesses away from the midsurface boundary. Moreover, it is tacitly assumed that the corresponding boundary layer effects are inconsequential for the overall stability analysis of the structure.

3.2 Evaluation of Stability Functional. Using the above introduced rescaled expressions for θ^3 and k, the variational equation for the minimum eigenvalue of the stability functional in (2.16) is rewritten as

$$\begin{aligned}\varepsilon\beta\int_A\int_{-\zeta}^{\zeta}a^{ik}\overline{v}_k\delta\overline{v}_i d\xi dA &= \varepsilon\int_A\int_{-\zeta}^{\zeta}\Bigg\{\frac{1}{\varepsilon^2}\overline{L}^{i3k3}\frac{\partial\overline{v}_k}{\partial\xi}\frac{\partial\delta\overline{v}_i}{\partial\xi}+\frac{1}{\varepsilon}(\overline{L}^{i3k\gamma}q^\delta_\gamma)\overline{v}_{k,\delta}\frac{\partial\delta\overline{v}_i}{\partial\xi}\\ &\quad+\frac{1}{\varepsilon}(\overline{L}^{i\alpha k3}q^\beta_\alpha)\frac{\partial\overline{v}_k}{\partial\xi}\delta\overline{v}_{i,\beta}+(\overline{L}^{i\alpha k\gamma}q^\beta_\alpha q^\delta_\gamma)\overline{v}_{k,\delta}\delta\overline{v}_{i,\beta}\Bigg\}\mu d\xi dA\\ &\quad+\varepsilon\int_A[\overline{K}^{ik}\overline{v}_k\delta\overline{v}_i\mu]_{\xi=-\zeta}dA.\end{aligned} \tag{3.5}$$

Substituting the expansions (3.1)–(3.4) into (3.5), and collecting terms of like order, the following governing equations are found that must be satisfied for the various orders of ε.

Equations of O(ε^{-1}). The lowest order governing equations are those of O(ε^{-1}), namely

$$\int_A\int_{-\zeta}^{\zeta}\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{0}{\overline{v}}_k}{\partial\xi}\frac{\partial\delta\overline{v}_i}{\partial\xi}d\xi dA=0. \tag{3.6}$$

Integrating by parts, and in view of the arbitrariness of $\delta\overline{v}_i$, the governing differential equations and boundary conditions for $\overset{0}{\overline{v}}_k$ are found

$$\begin{aligned}&\frac{\partial}{\partial\xi}\left[\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{0}{\overline{v}}_k}{\partial\xi}\right]=0 \quad\text{in}\quad -\zeta\leqslant\xi\leqslant\zeta\\ &\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{0}{\overline{v}}_k}{\partial\xi}=0 \quad\text{at}\quad \xi=\pm\zeta.\end{aligned} \tag{3.7}$$

The solution of (3.7) is

$$\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{0}{\overline{v}}_k}{\partial\xi}=0,\quad \Rightarrow\frac{\partial\overset{0}{\overline{v}}_k}{\partial\xi}=0. \tag{3.8}$$

Since $\overset{0}{\overline{L}}{}^{i3k3}$ is nonsingular for all applications (the material is assumed to be in the elliptic range of its response), the solution of (3.8) implies $\overset{0}{\overline{v}}_k=\overset{0}{\overline{v}}_k(\theta^\alpha)$, which states that the lowest order term in the expansion of the mode is only dependent on the midsurface coordinates.

Equations of O(ε^0). Terms of O(ε^0) are collected next and, making use of (3.8), the following governing equations are found:

$$\int_A\int_{-\zeta}^{\zeta}\left[\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{1}{\overline{v}}_k}{\partial\xi}+\overset{0}{\overline{L}}{}^{i3k\delta}\overset{0}{\overline{v}}_{k,\delta}\right]\frac{\partial\delta\overline{v}_i}{\partial\xi}d\xi dA=0. \tag{3.9}$$

Following the same steps as in (3.6), we obtain the governing differential equations and boundary conditions for $\overset{1}{\overline{v}}_k$:

$$\begin{aligned}&\frac{\partial}{\partial\xi}\left[\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{1}{\overline{v}}_k}{\partial\xi}+\overset{0}{\overline{L}}{}^{i3k\delta}\overset{0}{\overline{v}}_{k,\delta}\right]=0 \quad\text{in}\quad -\zeta\leqslant\xi\leqslant\zeta\\ &\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{1}{\overline{v}}_k}{\partial\xi}+\overset{0}{\overline{L}}{}^{i3k\delta}\overset{0}{\overline{v}}_{k,\delta}=0 \quad\text{at}\quad \xi=\pm\zeta.\end{aligned} \tag{3.10}$$

Solving (3.10) gives the following result for $\partial\overset{1}{\overline{v}}_k/\partial\xi$:

$$\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{1}{\overline{v}}_k}{\partial\xi}+\overset{0}{\overline{L}}{}^{i3k\delta}\overset{0}{\overline{v}}_{k,\delta}=0. \tag{3.11}$$

Recalling the plane stress assumption for the principal solution, $\tau^{i3}=0$ and $\dot{\tau}^{i3}=0$, and the resulting orthotropy of the incremental moduli, the Kirchoff-Love hypothesis for the deformation of thin shells is recovered for the eigenmode

$$\frac{\partial\overset{1}{\overline{v}}_\alpha}{\partial\xi}=-\overset{0}{\overline{v}}_{3,\alpha}. \tag{3.12}$$

Equations of O(ε^1). The next lowest order equations are those of O(ε^1). Making use of (3.8) and (3.11), the governing equations of O(ε^1) are derived:

$$\begin{aligned}&\int_A\int_{-\zeta}^{\zeta}\Bigg\{\Bigg[\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{2}{\overline{v}}_k}{\partial\xi}+\overset{1}{\overline{L}}{}^{i3k3}\frac{\partial\overset{1}{\overline{v}}_k}{\partial\xi}+\overset{0}{\overline{L}}{}^{i3k\delta}\overset{1}{\overline{v}}_{k,\delta}\\ &\quad+\left(\overset{1}{\overline{L}}{}^{i3k\delta}+\overset{0}{\overline{L}}{}^{i3k\gamma}\overset{1}{q}{}^\delta_\gamma\right)\overset{0}{\overline{v}}_{k,\delta}\Bigg]\frac{\partial\delta\overline{v}_i}{\partial\xi}\\ &\quad+\left(\overset{0}{\overline{L}}{}^{i\beta k3}\frac{\partial\overset{1}{\overline{v}}_k}{\partial\xi}+\overset{0}{\overline{L}}{}^{i\beta k\delta}\overset{0}{\overline{v}}_{k,\delta}\right)\delta\overline{v}_{i,\beta}\Bigg\}d\xi dA\\ &\quad+\int_A[\overset{0}{\overline{K}}{}^{ik}\overset{0}{\overline{v}}_k\delta\overline{v}_i]_{\xi=-\zeta}dA\\ &=\overset{0}{\beta}\int_A\int_{-\zeta}^{\zeta}a^{ik}\overset{0}{\overline{v}}_k\delta\overline{v}_i d\xi dA.\end{aligned} \tag{3.13}$$

From the results in (3.11), it becomes convenient to introduce the plane stress incremental moduli, $\overline{\mathcal{P}}^{i\beta k\delta}$,

$$\overline{\mathcal{P}}^{i\beta k\delta}\equiv[\overline{L}^{i\alpha k\gamma}-\overline{L}^{i\alpha m3}(\overline{L}^{n3m3})^{-1}\overline{L}^{n3k\gamma}]q^{\beta}_{\alpha}q^{\delta}_{\gamma}. \tag{3.14}$$

Using (3.14) and integrating (3.13) by parts, and assuming the thickness varies slowly throughout the sheet (i.e., $\partial\zeta/\partial\theta^{\alpha}=\mathrm{O}(\varepsilon^{n}),\ n\geqslant 2$), the following governing partial differential equations and boundary conditions are found:

$$\begin{aligned}
&\frac{\partial}{\partial\xi}\left[\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{2}{\overline{v}}_k}{\partial\xi}+\overset{1}{\overline{L}}{}^{i3k3}\frac{\partial\overset{1}{\overline{v}}_k}{\partial\xi}+\overset{0}{\overline{L}}{}^{i3k\delta}\overset{1}{\overline{v}}_{k,\delta}+(\overset{1}{\overline{L}}{}^{i3k\delta}+\overset{0}{\overline{L}}{}^{i3k\gamma}\overset{1}{q}{}^{\delta}_{\gamma})\overset{0}{\overline{v}}_{k,\delta}\right]\\
&\quad+\frac{\partial}{\partial\theta^{\beta}}\left[\overset{0}{\overline{\mathcal{P}}}{}^{i\beta k\delta}\overset{0}{\overline{v}}_{k,\delta}\right]+(\overset{0}{\overline{\mathcal{P}}}{}^{i\beta k\delta}t^{\gamma}_{\beta\gamma}+\overset{0}{\overline{\mathcal{P}}}{}^{m\beta k\delta}t^{i}_{m\beta})\overset{0}{\overline{v}}_{k,\delta}+\overset{0}{\beta}a^{ik}\overset{0}{\overline{v}}_k\\
&\quad=0\quad\text{in}\quad-\zeta\leqslant\xi\leqslant\zeta\\
&\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{2}{\overline{v}}_k}{\partial\xi}+\overset{1}{\overline{L}}{}^{i3k3}\frac{\partial\overset{1}{\overline{v}}_k}{\partial\xi}+\overset{0}{\overline{L}}{}^{i3k\delta}\overset{1}{\overline{v}}_{k,\delta}+(\overset{1}{\overline{L}}{}^{i3k\delta}+\overset{0}{\overline{L}}{}^{i3k\gamma}\overset{1}{q}{}^{\delta}_{\gamma})\overset{0}{\overline{v}}_{k,\delta}\\
&\quad=\overset{0}{\overline{K}}{}^{ik}\overset{0}{\overline{v}}_k\quad\text{at}\quad\xi=-\zeta\\
&\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{2}{\overline{v}}_k}{\partial\xi}+\overset{1}{\overline{L}}{}^{i3k3}\frac{\partial\overset{1}{\overline{v}}_k}{\partial\xi}+\overset{0}{\overline{L}}{}^{i3k\delta}\overset{1}{\overline{v}}_{k,\delta}+(\overset{1}{\overline{L}}{}^{i3k\delta}+\overset{0}{\overline{L}}{}^{i3k\gamma}\overset{1}{q}{}^{\delta}_{\gamma})\overset{0}{\overline{v}}_{k,\delta}\\
&\quad=0\quad\text{at}\quad\xi=\zeta.
\end{aligned}\tag{3.15}$$

Considering (3.13) with $\delta\overline{v}_i=\delta\overline{v}_i(\theta^{\alpha})$, one finds the following governing equations for $\overset{0}{\overline{v}}_i(\theta^{\alpha})$ in variational form

$$\int_A\left\{\overset{0}{\overline{\mathcal{P}}}{}^{i\beta k\delta}\overset{0}{\overline{v}}_{k,\delta}\delta\overline{v}_{i,\beta}+\frac{1}{2\zeta}\overset{0}{\overline{K}}{}^{ik}\overset{0}{\overline{v}}_k\delta\overline{v}_i\right\}dA=\overset{0}{\beta}\int_A a^{ik}\overset{0}{\overline{v}}_k\delta\overline{v}_i dA, \tag{3.16}$$

while the equations for $\partial\overset{2}{\overline{v}}_k/\partial\xi$ are found from (3.15) and (3.16):

$$\begin{aligned}
&\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{2}{\overline{v}}_k}{\partial\xi}+\overset{1}{\overline{L}}{}^{i3k3}\frac{\partial\overset{1}{\overline{v}}_k}{\partial\xi}+\overset{0}{\overline{L}}{}^{i3k\delta}\overset{1}{\overline{v}}_{k,\delta}+(\overset{1}{\overline{L}}{}^{i3k\delta}+\overset{0}{\overline{L}}{}^{i3k\gamma}\overset{1}{q}{}^{\delta}_{\gamma})\overset{0}{\overline{v}}_{k,\delta}\\
&\quad=f(\xi)\overset{0}{\overline{K}}{}^{ik}\overset{0}{\overline{v}}_k,
\end{aligned}\tag{3.17}$$

where $f(\xi)\equiv(\zeta-\xi)/2\zeta$.

Equations of O(ε^2). The next lowest order equations are those of $\mathrm{O}(\varepsilon^{2})$. Making use of (3.8), (3.11), (3.14), and (3.17), the governing equations of $\mathrm{O}(\varepsilon^{2})$ are found:

$$\begin{aligned}
&\int_A\int_{-\zeta}^{\zeta}\left\{\left[\overset{0}{\overline{L}}{}^{i3k3}\frac{\partial\overset{3}{\overline{v}}_k}{\partial\xi}+\overset{1}{\overline{L}}{}^{i3k3}\frac{\partial\overset{2}{\overline{v}}_k}{\partial\xi}+\overset{2}{\overline{L}}{}^{i3k3}\frac{\partial\overset{1}{\overline{v}}_k}{\partial\xi}+\overset{0}{\overline{L}}{}^{i3k\delta}\overset{2}{\overline{v}}_{k,\delta}\right.\right.\\
&\quad+(\overset{1}{\overline{L}}{}^{i3k\delta}+\overset{0}{\overline{L}}{}^{i3k\gamma}\overset{1}{q}{}^{\delta}_{\gamma})\overset{1}{\overline{v}}_{k,\delta}+(\overset{2}{\overline{L}}{}^{i3k\delta}+\overset{1}{\overline{L}}{}^{i3k\gamma}\overset{1}{q}{}^{\delta}_{\gamma}+\overset{0}{\overline{L}}{}^{i3k\gamma}\overset{2}{q}{}^{\delta}_{\gamma})\overset{0}{\overline{v}}_{k,\delta}\\
&\quad\left.+\overset{1}{\mu}f(\xi)\overset{0}{\overline{K}}{}^{ik}\overset{0}{\overline{v}}_k\right]\frac{\partial\delta\overline{v}_i}{\partial\xi}+\left[\overset{0}{\overline{\mathcal{P}}}{}^{i\beta k\delta}\overset{1}{\overline{v}}_{k,\delta}+\overset{1}{\overline{\mathcal{P}}}{}^{i\beta k\delta}\overset{0}{\overline{v}}_{k,\delta}+\overset{1}{\mu}\overset{0}{\overline{\mathcal{P}}}{}^{i\beta k\delta}\overset{0}{\overline{v}}_{k,\delta}\right.\\
&\quad\left.+f(\xi)\overset{0}{\overline{L}}{}^{i\beta m3}(\overset{0}{\overline{L}}{}^{n3m3})^{-1}\overset{0}{\overline{K}}{}^{nk}\overset{0}{\overline{v}}_k\right]\delta\overline{v}_{i,\beta}d\xi dA\\
&\quad+\int_A\left[(\overset{1}{\mu}\overset{0}{\overline{K}}{}^{ik}\overset{0}{\overline{v}}_k+\overset{1}{\overline{K}}{}^{ik}\overset{0}{\overline{v}}_k+\overset{0}{\overline{K}}{}^{ik}\overset{1}{\overline{v}}_k)\delta\overline{v}_i\right]_{\xi=-\zeta}dA\\
&=\int_A\int_{-\zeta}^{\zeta}a^{ik}(\overset{0}{\beta}\overset{1}{\overline{v}}_k+\overset{1}{\beta}\overset{0}{\overline{v}}_k)\delta\overline{v}_i d\xi dA.
\end{aligned}\tag{3.18}$$

Choosing $\delta\overline{v}_i=\delta\overline{v}_i(\theta^{\alpha})$, and integrating through the thickness, gives the following equations for the ξ independent part of $\overset{1}{\overline{v}}_i$:

$$\begin{aligned}
&\int_A\left\{\left(\overset{0}{\overline{\mathcal{P}}}{}^{i\beta k\delta}\langle\overset{1}{\overline{v}}_{k,\delta}\rangle+\langle\overset{1}{\overline{\mathcal{P}}}{}^{i\beta k\delta}\rangle\overset{0}{\overline{v}}_{k,\delta}+\frac{1}{2}\overset{0}{\overline{L}}{}^{i\beta m3}(\overset{0}{\overline{L}}{}^{n3m3})^{-1}\overset{0}{\overline{K}}{}^{nk}\overset{0}{\overline{v}}_k\right)\delta\overline{v}_{i,\beta}\right.\\
&\quad\left.+\frac{1}{2\zeta}\left[(\overset{1}{\mu}\overset{0}{\overline{K}}{}^{ik}+\overset{1}{\overline{K}}{}^{ik})\overset{0}{\overline{v}}_k+\overset{0}{\overline{K}}{}^{in}\overset{1}{\overline{v}}_k\right]_{\xi=-\zeta}\delta\overline{v}_i\right\}dA\\
&\quad=\int_A a^{ik}(\overset{0}{\beta}\langle\overset{1}{\overline{v}}_k\rangle+\overset{1}{\beta}\overset{0}{\overline{v}}_k)\delta\overline{v}_i dA,
\end{aligned}\tag{3.19}$$

where the thickness average of a function f is defined as

$$\langle f\rangle\equiv\frac{1}{2\zeta}\int_{-\zeta}^{\zeta}f d\xi. \tag{3.20}$$

With the solution of (3.19), the mode, including first-order bending terms, is determined up to order ε:

$$\overline{v}_i=\overset{0}{\overline{v}}_i+\varepsilon(\langle\overset{1}{\overline{v}}_i\rangle-\xi(\overset{0}{\overline{L}}{}^{n3i3})^{-1}\overset{0}{\overline{L}}{}^{n3k\delta}\overset{0}{\overline{v}}_{k,\delta})+\mathrm{O}(\varepsilon^{2}). \tag{3.21}$$

An expression for the stability functional, accurate to $\mathrm{O}(\varepsilon^{3})$, can be found using this form of the mode that includes bending terms.

Stability Functional up to O(ε^3). Using the previous results, the stability functional is assembled with accuracy up to $\mathrm{O}(\varepsilon^{3})$. Starting with the following expression for $\mathcal{F}$,

$$\begin{aligned}
\mathcal{F}=&\frac{\varepsilon}{2}\int_A\int_{-\zeta}^{\zeta}\left\{\overline{L}^{i3k3}\frac{\partial\overline{v}_k}{\partial\xi}\frac{\partial\overline{v}_i}{\partial\xi}+\overline{L}^{i3k\gamma}q^{\delta}_{\gamma}\overline{v}_{k,\delta}\frac{\partial\overline{v}_i}{\partial\xi}\right.\\
&\left.+\overline{L}^{i\alpha k3}q^{\beta}_{\alpha}\frac{\partial\overline{v}_k}{\partial\xi}\overline{v}_{i,\beta}+\overline{L}^{i\alpha k\gamma}q^{\delta}_{\gamma}q^{\beta}_{\alpha}\overline{v}_{k,\delta}\overline{v}_{i,\beta}\right\}\mu d\xi dA\\
&+\frac{\varepsilon}{2}\int_A[\overline{K}^{ik}\overline{v}_k\overline{v}_i\mu]_{\xi=-\zeta}dA,
\end{aligned}\tag{3.22}$$

making the substitutions of the expansions (3.1)–(3.4) into (3.22), using (3.8), (3.11) and choosing $\delta\overline{v}_i=\langle\overset{2}{\overline{v}}_i\rangle$ in (3.16) and invoking the mode uniqueness condition, we find the following simplified expression for $\mathcal{F}$:

$$\begin{aligned}
\mathcal{F}=&\frac{\varepsilon}{2}\int_A\left\{\int_{-\zeta}^{\zeta}[(\overset{0}{\overline{\mathcal{P}}}{}^{i\beta k\delta}+\varepsilon\overset{1}{\overline{\mathcal{P}}}{}^{i\beta k\delta}+\varepsilon^{2}\overset{2}{\overline{\mathcal{P}}}{}^{i\beta k\delta})(\overset{0}{\overline{v}}_{k,\delta}+\varepsilon\overset{1}{\overline{v}}_{k,\delta})\right.\\
&\times(\overset{0}{\overline{v}}_{i,\beta}+\varepsilon\overset{1}{\overline{v}}_{i,\beta})(\overset{0}{\mu}+\varepsilon\overset{1}{\mu}+\varepsilon^{2}\overset{2}{\mu})]d\xi+[(\overset{0}{\overline{K}}{}^{ik}+\varepsilon\overset{1}{\overline{K}}{}^{ik}+\varepsilon^{2}\overset{2}{\overline{K}}{}^{ik})\\
&\times(\overset{0}{\overline{v}}_k+\varepsilon\overset{1}{\overline{v}}_k)(\overset{0}{\overline{v}}_i+\varepsilon\overset{1}{\overline{v}}_i)(\overset{0}{\mu}+\varepsilon\overset{1}{\mu}+\varepsilon^{2}\overset{2}{\mu})]_{\xi=-\zeta}\\
&+\varepsilon^{2}\int_{-\zeta}^{\zeta}[2f(\xi)\overset{0}{\overline{K}}{}^{im}(\overset{0}{\overline{L}}{}^{n3m3})^{-1}[(\overset{1}{\overline{L}}{}^{n3k\delta}+\overset{0}{\overline{L}}{}^{n3k\gamma}\overset{1}{q}{}^{\delta}_{\gamma}\\
&-\overset{1}{\overline{L}}{}^{n3p3}(\overset{0}{\overline{L}}{}^{q3p3})^{-1}\overset{0}{\overline{L}}{}^{q3k\delta})\overset{0}{\overline{v}}_{k,\delta}+\overset{0}{\overline{L}}{}^{n3k\delta}\overset{1}{\overline{v}}_{k,\delta}]\overset{0}{\overline{v}}_i\\
&\left.+(f(\xi))^{2}\overset{0}{\overline{K}}{}^{im}(\overset{0}{\overline{L}}{}^{n3m3})^{-1}\overset{0}{\overline{K}}{}^{nk}\overset{0}{\overline{v}}_k\overset{0}{\overline{v}}_i]d\xi\right\}dA+\mathrm{O}(\varepsilon^{4}).
\end{aligned}\tag{3.23}$$

Examining the ε^{3} contact terms in (3.23) it is recalled that for adequately large values of the foundation stiffness, the mode $\overset{0}{\overline{v}}_i=\mathrm{O}(\varepsilon)$ in the areas of contact. This makes the second integral though the thickness $\mathrm{O}(\varepsilon^{4})$. In metal forming applications, the tooling is assumed to be rigid, and thus the foundation stiffness will be very large, and an expression for the stability functional accurate to $\mathrm{O}(\varepsilon^{4})$ is found:

$$\mathcal{F}=\frac{1}{2}\int_A\left\{\int_{-h/2}^{h/2}[\overline{\mathcal{P}}^{i\beta k\delta}\hat{v}_{k,\delta}\hat{v}_{i,\beta}\mu]d\theta^{3}+[\overline{K}^{ik}\hat{v}_k\hat{v}_i]_{\xi=-\zeta}\right\}dA \tag{3.24}$$

where the mode $\hat{v}_i$ is given by

$$\hat{v}_i(\theta^i) = \overset{0}{v}_i(\theta^\alpha) + \langle \overset{1}{v}_i \rangle(\theta^\alpha) - \theta^3(\overline{L}^{m3i3})^{-1}\overline{L}^{m3k\delta}\left[\frac{\partial \overset{0}{v}_k(\theta^\alpha)}{\partial \theta^\delta} - t^r_{k\delta}\overset{0}{v}_r(\theta^\alpha)\right], \tag{3.25}$$

and the plane stress incremental moduli, $\overline{\mathcal{P}}^{i\beta k\delta}$, and the contact terms, $\overline{K}^{ik}$, can be found to the accuracy of the prestressed solution.

For puckering problems in metal forming operations, the mode of the instability is a bending mode. When $\beta = 0$, and assuming $\langle \overset{1}{\mathcal{P}}{}^{i\beta k\delta} \rangle \approx 0$, a reasonable assumption in metal forming problems, it can be shown that $\langle \overset{1}{v}_i \rangle \approx 0$, i.e., the mode is a bending mode. When this is the case, the $\hat{v}_i$ only depends on $\overset{0}{v}_i$ and the expression for the stability functional simplifies further.

In practice, the positive definiteness of (3.24) is determined numerically. It is assumed that the stress state, and therefore the incremental moduli for the material, are known at any point of the loading path. For the sheet metal forming applications of interest, the load parameter λ is taken to be the punch displacement H. A stamping process is stable against puckering at a height H, if for all punch displacements between 0 and H the functional (3.24) is positive definite. An application of the general theory for the case of the hemispherical cup test is discussed in detail in Part II.

4 Conclusions

The goal of the present paper is to present a general and consistent methodology to model puckering in sheet metal forming processes. The starting point is Hill's three-dimensional stability functional for rate-independent solids, appropriately modified for contact. Positive definiteness of this functional, i.e., a positive minimum eigenvalue β, ensures the stability of the corresponding prestressed elastoplastic solid, while the onset of buckling corresponds to a vanishing β. The slenderness of the solid, i.e., the dimensionless thickness parameter $\varepsilon \ll 1$, permits using a multiple scale asymptotic method, the construction of a two-dimensional stability functional which is accurate up to $O(\varepsilon^4)$ and which takes into account bending effects. The advantage here lies in the avoidance of any shell theory type approximation. The result is a consistently derived stability functional which is defined on the middle surface of the sheet.

The method is meant to be employed with the finite element discretization of the sheet forming problem of interest and has several advantages. It can use any equilibrium prestress to check for the stability of the corresponding deformed state. It is particularly useful when a membrane solution is available, as in the case of tearing calculations, where the present methodology also allows a check for puckering. The stability functional is defined independently of the algorithm used for the principal solution, and hence one can selectively refine the mesh in those areas prone to puckering. It also provides β which is a measure of the stability of the sheet against puckering. Since the stability functional is symmetric, the criterion for β is equivalent to the minimum diagonal entry of **D** in an **LDU** decomposition of the stiffness matrix **K** which results from the finite element discretization of the stability functional. The application to puckering experiments for the hemispherical cup test are presented in Part II.

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