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A sufficient condition for the linear instability of strain-rate-dependent solids

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Abstract. The linear stability criterion proposed generalizes to continua earlier results obtained for the viscoplastic Shanley column. A sufficient condition for instability is marked by the change of sign of the real part of the dominant eigenvalue for the system of equations constructed from the linearized version of the first and second derivatives with respect to time of the principle of virtual work. In particular, it is shown that the conditions for first instability are those for the first bifurcation of a strain-rate-independent solid obtained in the limit of inviscid plastic flow.

Condition suffisante d'instabilité linéaire pour les matériaux viscoplastiques

Résumé. *Le critère de stabilité proposé généralise aux milieux continus les résultats obtenus précédemment pour la colonne de Shanley viscoplastique. Une condition suffisante d'instabilité est liée à l'existence d'une valeur propre à partie réelle positive dans le système d'équations provenant du principe des travaux virtuels et de ses dérivées première et seconde par rapport au temps. Il est montré que les conditions de première instabilité sont celles de la première bifurcation du système élasto-plastique obtenu pour une viscosité tendant vers zéro.*

Version française abrégée

Une condition suffisante d'instabilité est proposée pour des milieux continus élasto-viscoplastiques. Les deux échelles de temps présentes sont t_l, liée à la vitesse de chargement, et t_r, le temps de relaxation du solide. Le rapport t_r/t_l est le nombre adimensionel T.

Note présentée par Pierre SUQUET.

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Les conditions de stabilité linéaire pour les solides visqueux sous chargement indépendant du temps sont identiques à celles obtenues en l'absence de viscosité (Nguyen et Radenkovic, 1975). Les résultats de stabilité diffèrent de ces dernières pour des valeurs de T petites devant l'unité mais différentes de zéro. Ils sont obtenus en perturbant les équations d'équilibre quasistatique du système dont les coefficients sont figés à l'instant de la perturbation. Une approche plus rigoureuse de la stabilité, qui peut s'appliquer à des valeurs quelconques de T, nécessite de ne pas figer les coefficients des équations au moment de la perturbation. Le solide, dont le comportement élasto-viscoplastique est donné par les équations (1) et (2) est soumis à une déformation principale qui dépend des deux temps caractéristiques du problème. La solution principale est perturbée en modifiant le chargement pendant un intervalle de temps infiniment petit comparé aux deux temps caractéristiques du problème (*fig. 1*). Il résulte de ce choix de perturbation, déjà validé lors de l'étude de la colonne de Shanley (Massin *et al.*, 1996), que les variables internes sont inchangées à la fin de la séquence de perturbation. De fait, la variation de contrainte lors de la perturbation est approximée par la réponse élastique du système (4) bien qu'une décharge élastique soit exclue. Le critère prédit la stabilité si la derivée par rapport au temps de la norme de la vitesse de toutes les perturbations admissibles décroît avec le temps à la fin de la séquence de perturbation (*fig. 1*).

L'évolution initiale de la perturbation est donnée par la résolution du système des trois équations en (5) et (6) qui proviennent de la linéarisation du principe des travaux virtuels (3) et de ses dérivées première et seconde par rapport au temps. Ces deux dérivées sont nécessaires afin d'obtenir une relation entre la perturbation de vitesse $\Delta\dot{\mathbf{u}}$ et sa variation dans le temps $\Delta\ddot{\mathbf{u}}$ car une dépendance temporelle de type exponentiel n'est pas postulée. On montre qu'une condition suffisante d'instabilité, avec le critère énoncé précédemment, est équivalente à l'existence d'une valeur propre à partie réelle positive pour le système d'équations en (10).

La dernière partie de notre analyse justifie le critère de stabilité en prouvant que l'instabilité est détectée à la première bifurcation du système élastoplastique obtenu dans la limite d'une viscosité nulle. Ce résultat est en accord avec les analyses de Massin *et al.* (1996) sur la colonne de Shanley à supports visco-plastiques pour laquelle on détecte l'instabilité à la charge tangente de la structure lorsque la viscosité tend vers zéro. Pour les milieux continus, le problème aux valeurs propres (10) est réécrit pour un matériau dont la viscosité tend vers zéro (14). À la stabilité neutre, cette équation (14) est identique à l'équation qui marque la ruine du critère d'exclusion de Hill (1958) présentée en (12). Cette limite est singulière car c'est le module réduit, et non le module tangent, qui contrôle la stabilité à $T=0$ (Nguyen et Radenkovic, 1975). La bifurcation s'accompagne alors instantanément d'une décharge élastique sur une partie de la structure. L'obtention du module tangent lors du passage à la limite ($T \to 0$) est possible grâce au choix d'une taille de perturbation ε suffisamment petite pour éviter toute décharge élastique.

1. Introduction

A sufficient condition for linear instability is proposed for boundary value problems involving viscoplastic solids with an elastic instantaneous response. The only two timescales which enter this class of problems are t_l, related to the rate of applied loading, and t_r, the relaxation time of the viscous solid. The ratio t_r/t_l defines the dimensionless number T which takes a zero value if either the viscosity is suppressed or the applied loading is independent of time. The proposed stability condition generalizes to continua an equivalent condition derived and validated by the same authors for Shanley's column with elastic-viscoplastic supports (Massin *et al.*, 1996).

The stability of rate-independent solids in the elastic-plastic range of deformation is usually based on the dissipation induced by a perturbation of the equilibrium ($T = 0$). The introduction of a viscosity does not change the equilibrium stability predictions (Nguyen and Radenkovic, 1975). Stability results differ if the dimensionless time T is different from zero but small compared to one. In that instance, stability is studied by perturbing the quasi-static equilibrium and by freezing the time-dependence of the coefficients entering the governing equations. This simple procedure results in first-instability predictions which are those that lead to the failure of Hill's bifurcation exclusion criterion (Hill, 1958) for a rate-independent solid obtained in the limit of inviscid plastic flow (Leroy, 1991).

This Note proposes a stability criterion valid for arbitrary values of T. The stability transition is said to be marked by the change in sign of the maximum time derivative of the L_2 norm of all admissible velocity perturbations. The relation between the velocity perturbation and its first rate of change with time is obtained from the linearized version of the first and second derivatives with respect to time of the principle of virtual work.

2. Constitutive relations and principle of virtual work

With the hypothesis of small deformation and finite rotation, the adopted elastic-viscoplastic constitutive model with a single internal variable (the equivalent plastic strain γ) reads:

$$\boldsymbol{\sigma} = \mathbf{E} : (\epsilon - \epsilon^{p}) \quad \text{with} \quad \epsilon = \frac{1}{2} (\mathbf{u}\nabla + \nabla\mathbf{u} + \nabla\mathbf{u} \cdot \mathbf{u}\nabla) \quad \text{and} \quad \dot{\epsilon}^{p} = \dot{\gamma} \frac{\partial \psi}{\partial \boldsymbol{\sigma}} (\boldsymbol{\sigma}) \tag{1}$$

in which $\mathbf{E}$, $\mathbf{u}$, ϵ^{p} and $\boldsymbol{\sigma}$ denote the linear elasticity tensor, the displacement vector, the plastic part of the Lagrangian deformation ϵ and the second Piola–Kirchhoff stress tensor, respectively. The flow potential is denoted by ψ. The evolution in time of the internal variable γ is governed by:

$$\dot{\gamma} = \frac{1}{t_r} \mathscr{F} (\phi(\boldsymbol{\sigma}), g(\gamma), t_r) \tag{2}$$

where $\mathscr{F}$ is a positive function of ϕ and g, which denote the equivalent stress and the hardening function of γ, respectively. Note that the limit of inviscid plastic flow is marked by t_r equal to zero. In that instance, $\mathscr{F}(\phi, g, 0)$ must also be equal to zero for $\dot{\gamma}$ to be bounded and serves the purpose of a yield criterion.

The viscous solid occupies the reference volume V delimited by the surface ∂V, on part of which a traction $\mathbf{T}$ is applied. The conditions of equilibrium are expressed by the principle of virtual work:

$$\int_V \boldsymbol{\sigma}:\delta\epsilon \quad \mathrm{d}V = \int_{\partial V} \mathbf{T} \cdot \delta\mathbf{u} \quad \mathrm{d}S \tag{3}$$

in which $\delta\mathbf{u}$ is an admissible displacement field and $\delta\epsilon$ the corresponding variation of ϵ. The fundamental solution of that equation is denoted by a subscript 0 and is a function of the two timescales of the problem.

3. Linear stability analysis

The time interval, during which the boundary conditions are perturbed, is first analysed to provide the initial conditions for the evolution problem studied next. This evolution of the perturbation is described by the linearized version of the first and second derivatives with respect to time of the principle of virtual work. Those two equations form the basis of the eigenvalue analysis resulting from the stability criterion proposed. Time t is now normalized with respect to the loading time t_l and henceforth a dot denotes the derivative with respect to the dimensionless time: $\tau = t/t_l$.

3.1. *Initial perturbation sequence*

The field variables of the problem are perturbed during a physical time interval which is small compared to the two timescales of the problem ($\Delta\tau << 1$ and T). The perturbation results from a modification of the applied loading which lasts for only $\Delta\tau$ (*fig. 1*). The difference between any quantity $A(\tau)$ measured on the perturbed and principal trajectories is denoted by $\varepsilon\, \Delta A(\tau)$ in which ε is small compared to one and also sufficiently small to avoid any unloading during perturbation for any non-zero value of T. Assuming sufficient continuity of all fields during the perturbation sequence, one finds from the results of an asymptotic expansion for the small parameter $\Delta\tau/T$ that the stress and plastic strain perturbation after $\Delta\tau$ are

$$\Delta\boldsymbol{\sigma} = \mathbf{E} : \Delta\boldsymbol{\epsilon} + \mathcal{O}\left(\frac{\Delta\tau}{T}\right) \quad \text{and} \quad \Delta\boldsymbol{\epsilon}^p = \frac{\Delta\tau}{T}\left(\mathscr{F}_{,\phi}\frac{\partial\psi}{\partial\boldsymbol{\sigma}}\frac{\partial\phi}{\partial\boldsymbol{\sigma}} + \mathscr{F}\frac{\partial^2\psi}{\partial\boldsymbol{\sigma}\,\partial\boldsymbol{\sigma}}\right) : \mathbf{E} : \Delta\boldsymbol{\epsilon} + \mathcal{O}\left[\left(\frac{\Delta\tau}{T}\right)^2\right] \quad (4)$$

The perturbation in stress is zero-order in $\Delta\tau/T$ and generated by the elastic response of the solid. The plastic strain perturbation is first-order in $\Delta\tau/T$ and thus disregarded compared to the perturbation of the total strain. Consequently, at the end of the time interval $\Delta\tau$, the perturbation in any field variable is independent of the details of the perturbation sequence as well as of the size of this time interval.

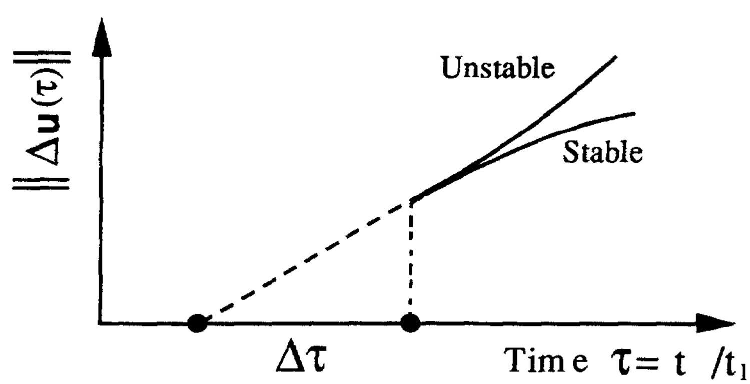

Fig. 1. – Schematic representation of the perturbation sequence and illustration of the stability criterion.
Fig. 1. – Présentation schématique du temps de perturbation et illustration du critère de stabilité.

3.2. *Evolution of the initial perturbation*

All relations concerning the evolution problem are evaluated at the end of the initial perturbation sequence, referred to as the onset. The principle of virtual work (3), once linearized and with the help of (4), relates the perturbations in loading and displacement:

$$\int_V \delta\mathbf{u}\nabla{:}\mathbf{L}^e{:}\Delta\mathbf{u}\nabla \,\mathrm{d}V = \int_{\partial V} \Delta\mathbf{T}\cdot\delta\mathbf{u}\,\mathrm{d}S \quad \text{with} \quad \mathbf{L}^e = \mathbf{E}+\mathbf{S} \quad \text{and} \quad S_{ijkl} = \boldsymbol{\sigma}^o_{jl}\,\delta_{ik} \tag{5}$$

where the fundamental solution $\mathbf{u}_0$ (and thus its gradient $\nabla\mathbf{u}_0$) is set to zero since the principal configuration at the onset is the reference configuration. Complementary equations relating perturbations in the displacement, the velocity and its rate of change are now obtained by linearizing the first and second derivatives with respect to time of the principle of virtual work (3). They read:

$$\begin{cases} \displaystyle\int_V \delta\mathbf{u}\nabla : (\,\mathbf{L}^e : \Delta\dot{\mathbf{u}}\nabla + \mathbf{A} : \nabla\Delta\mathbf{u}\,)\;\mathrm{d}V = 0 \\ \displaystyle\int_V \delta\mathbf{u}\nabla : (\,\mathbf{L}^e : \Delta\ddot{\mathbf{u}}\nabla + \mathbf{B} : \nabla\Delta\dot{\mathbf{u}} + \mathbf{C} : \nabla\Delta\mathbf{u}\,)\;\mathrm{d}V = 0 \end{cases} \tag{6}$$

in which the fourth-order tensors **A**, **B** and **C** are defined by:

$$\begin{cases} \mathbf{A} = -\,\mathbf{E} : \mathbf{M} : \mathbf{E}\dfrac{1}{T} + \dot{\mathbf{R}}_0 + \mathbf{E}\cdot\nabla\dot{\mathbf{u}}_0 + \dot{\mathbf{u}}_0\nabla\cdot\mathbf{E} \\ \mathbf{B} = -\,\mathbf{E} : \mathbf{M} : \mathbf{E}\dfrac{1}{T} + 2\,\dot{\mathbf{R}}_0 + 2\,\mathbf{E}\cdot\nabla\dot{\mathbf{u}}_0 + 2\,\dot{\mathbf{u}}_0\nabla\cdot\mathbf{E} \\ \mathbf{C} = \mathbf{E} : \left(\mathbf{M} : \mathbf{E} : \mathbf{M}\dfrac{1}{T} - \mathbf{N}\right) : \mathbf{E}\dfrac{1}{T} + \ddot{\mathbf{R}}_0 + \ddot{\mathbf{u}}_0\nabla\cdot\mathbf{E} + \mathbf{E}\cdot\nabla\ddot{\mathbf{u}}_0 \\ \quad + 2\,\dot{\mathbf{u}}_0\nabla\cdot\mathbf{E}\cdot\nabla\dot{\mathbf{u}}_0 - \dfrac{1}{T}\mathbf{E} : \mathbf{M} : \mathbf{E}\cdot\nabla\dot{\mathbf{u}}_0 - 2\,\dot{\mathbf{u}}_0\nabla : \mathbf{E} : \mathbf{M} : \mathbf{E}\dfrac{1}{T} \end{cases} \tag{7}$$

with $R_{ijkl} = S_{ijlk}$. In (7), the two tensors **M** and **N** are defined by:

$$\begin{cases} \mathbf{M} = \mathscr{F}\,\dfrac{\partial^2\psi}{\partial\boldsymbol{\sigma}\,\partial\boldsymbol{\sigma}} + \mathscr{F}_{,\phi}\,\dfrac{\partial\psi}{\partial\boldsymbol{\sigma}}\,\dfrac{\partial\phi}{\partial\boldsymbol{\sigma}} \\ \mathbf{N} = \mathscr{F}_{,\phi\phi}\,\dfrac{\partial\phi}{\partial\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}_0\,\dfrac{\partial\psi}{\partial\boldsymbol{\sigma}}\,\dfrac{\partial\phi}{\partial\boldsymbol{\sigma}} + \dot{\gamma}_0\,g_{,\gamma}\left[\mathscr{F}_{,g}\,\dfrac{\partial^2\psi}{\partial\boldsymbol{\sigma}\,\partial\boldsymbol{\sigma}} + \mathscr{F}_{,g\phi}\,\dfrac{\partial\psi}{\partial\boldsymbol{\sigma}}\,\dfrac{\partial\phi}{\partial\boldsymbol{\sigma}}\right] + \mathscr{F}\dfrac{\partial^3\psi}{\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}_0 \\ \quad + \mathscr{F}_{,\phi}\left[\dfrac{\partial\phi}{\partial\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}_0\,\dfrac{\partial^2\psi}{\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}} + \dfrac{\partial\psi}{\partial\boldsymbol{\sigma}}\,\dfrac{\partial^2\phi}{\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}}{:}\dot{\boldsymbol{\sigma}}_0 + \dfrac{\partial^2\psi}{\partial\boldsymbol{\sigma}\,\partial\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}_0\,\dfrac{\partial\phi}{\partial\boldsymbol{\sigma}} + \dfrac{1}{T}\mathscr{F}_{,g}\,g_{,\gamma}\,\dfrac{\partial\psi}{\partial\boldsymbol{\sigma}}\,\dfrac{\partial\phi}{\partial\boldsymbol{\sigma}}\right] \end{cases} \tag{8}$$

Note that **M** in (8) is invertible as long as $\mathscr{F}$ differs from zero. This condition permits the inversion of the operator **A** in (6) which is necessary to express $\Delta\mathbf{u}$ in terms of $\Delta\dot{\mathbf{u}}$ in what follows.

3.3. *Eigenvalue analysis and stability criterion*

For loads smaller than the Euler load, the operator $\mathbf{L}^e$ in (5) is positive definite and hence invertible. Since **A** is also invertible, the system of equations (6) can be rewritten as:

$$\Delta\ddot{\mathbf{u}} = \mathscr{L}[\Delta\dot{\mathbf{u}}] \tag{9}$$

in terms of the linear operator $\mathscr{L}$. This operator is in general not self-adjoint even for associated plasticity models (*i.e.*, $\psi = \phi$).

The stability criterion proposed requires all perturbations in the velocity field to have the time derivative of their L_2 norm decreasing at the onset (*fig. 1*). Hence, stability is preserved if the maximum of the time derivative of that norm is negative. Equivalently, it requires the largest eigenvalue of the self-adjoint part $\mathscr{L}^s$ of the operator $\mathscr{L}$ to be negative. An analytical expression for $\mathscr{L}^s$ being often difficult to obtain, we choose to restrict our attention to the study of the sign of the eigenvalues of the operator $\mathscr{L}$ in (9). The corresponding eigenvalue problem is then:

$$\begin{cases} \int_V \delta\mathbf{u}\,\nabla : (\,\mathbf{L}^e : \Delta\dot{\mathbf{u}}\,\nabla + \mathbf{A} : \nabla\Delta\mathbf{u}\,)\ \mathrm{d}V = 0 \\ \int_V \delta\mathbf{u}\,\nabla : (\,\Lambda\mathbf{L}^e : \Delta\dot{\mathbf{u}}\,\nabla + \mathbf{B} : \nabla\Delta\dot{\mathbf{u}} + \mathbf{C} : \nabla\Delta\mathbf{u}\,)\ \mathrm{d}V = 0 \end{cases} \tag{10}$$

The existence of an eigenvalue Λ with positive real part in (10) implies that $\mathscr{L}^s$ also has an eigenvalue with positive real part and hence signal instability. Consequently, the study of (10) instead of (9) provides a sufficient condition for instability.

4. The limit of inviscid plastic flow

As a preliminary step to the validation of the stability criterion by non-linear analyses (Massin, 1995), our proposition is justified by comparing the stability predictions for vanishing values of T with the bifurcation conditions obtained for a rate-independent solid. That limit is for vanishing values of the relaxation time and a fixed rate of loading.

The constitutive model of the rate-independent solid in the limit of inviscid flow were discussed after eq. (2). The function $\mathscr{F}$ must equal zero for the equivalent plastic strain rate to remain bounded. That condition, checked during the whole loading process, provides the consistency condition which relates the equivalent plastic strain rate to the stress rate. Inserting this result in the rate form of (1), we obtain the following incremental moduli:

$$\dot{\boldsymbol{\sigma}} = \left(\mathbf{E} - \mathbf{Q}\frac{1}{H}\mathbf{P}\right) : \dot{\boldsymbol{\epsilon}} \quad \text{with} \quad \mathbf{Q} = \mathbf{E} : \frac{\partial\psi}{\partial\boldsymbol{\sigma}},\quad \mathbf{P} = \frac{\partial\phi}{\partial\boldsymbol{\sigma}} : \mathbf{E} \quad \text{and} \quad H = \frac{\partial\phi}{\partial\boldsymbol{\sigma}} : \mathbf{E} : \frac{\partial\psi}{\partial\boldsymbol{\sigma}} - \frac{F_{,g}}{F_{,\phi}}\, g_{,\gamma} \tag{11}$$

which characterizes the rate-independent plastic model for an inviscid plastic flow. Loss of uniqueness for that solid is marked by the failure of Hill's exclusion criterion (Hill, 1958):

$$\int_V \delta\mathbf{u}\,\nabla : \left(\mathbf{L}^e - \mathbf{Q}\frac{1}{H}\mathbf{P}\right) : \delta\mathbf{u}\,\nabla \quad \mathrm{d}V > 0 \tag{12}$$

when a single mode $\delta\mathbf{u} = \Delta\dot{\mathbf{u}}$ makes the integrand in the left-hand side of (12) vanish, if an associated flow rule is considered ($\mathbf{P} = \mathbf{Q}$).

The rest of this section demonstrates that for vanishing values of t_r and at neutral stability, the system (10) admits for solution the one that first fails Hill's exclusion criterion (12). To proceed, limiting values of the tensors **M**, **A**, **B** and **N** are necessary. These are obtained by observing that the stress rate $\dot{\boldsymbol{\sigma}}$ and the velocity gradient $\dot{\mathbf{u}}_0\nabla$ can be disregarded compared to the stress and the velocity gradient divided by T, respectively. These limiting values inserted in (10) lead to the following system of equations:

$$\begin{cases} \displaystyle\int_V \delta\mathbf{u}\,\nabla : \left(\mathbf{L}^e : \Delta\dot{\mathbf{u}}\,\nabla - \frac{\mathcal{F}_{,\phi}}{T}\,\mathbf{QP} : \Delta\mathbf{u}\,\nabla\right)\,\mathrm{d}V = 0 \\ \displaystyle\int_V \delta\mathbf{u}\,\nabla : \left[\left(A\mathbf{L}^e - \frac{\mathcal{F}_{,\phi}}{T}\,\mathbf{QP}\right) : \Delta\dot{\mathbf{u}}\,\nabla + \left(\frac{\mathcal{F}_{,\phi}}{T}\right)^2 \mathbf{Q}H\mathbf{P} : \Delta\mathbf{u}\,\nabla\right]\,\mathrm{d}V = 0 \end{cases} \tag{13}$$

which are combined to obtain

$$\int_V \delta\mathbf{u}\,\nabla : \left[\mathbf{L}^e - \frac{\mathbf{QP}}{AT/\mathcal{F}_{,\phi} + H}\right] : \Delta\dot{\mathbf{u}}\,\nabla \quad \mathrm{d}V = 0 \tag{14}$$

This equation at neutral stability is identical to (12) at its first failure, proving the proposition made above. It should be stressed that the limit obtained is singular: it is the reduced modulus load, and not the tangent modulus load, which pinpoints the limit of stability (Nguyen and Radenkovic, 1975) for T equal to zero. At this reduced modulus load, bifurcation is marked by an instantaneous elastic unloading over part of the structure. This unloading is absent from the results of our stability analysis in view of the choice made above of a sufficiently small perturbation size ε to avoid elastic unloading for any small but non-zero value of T.

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References

Hill R., 1958. A general theory of uniqueness and stability in elastic-plastic solids, *J. Mech. Phys. Solids*, 6, p. 236-249.

Leroy Y. M., 1991. Linear stability analysis of rate-dependent discrete systems, *Int. J. Solids Struct.*, 27, p. 783-808.

Massin P., 1995. On the stability of strain-rate dependent solids and structures, Ph.D. Thesis, University of Michigan, Ann Arbor, MI, USA.

Massin P., Triantafyllidis N. and Leroy Y. M., 1996. On the stability of strain-rate dependent solids. I. Structural examples. Submitted for publication.

Nguyen Q. S. and Radenkovic D., 1975. Stability of equilibrium in elastic plastic solids. *IUTAM Symposium*, Marseille, France.