On the Comparison Between Microscopic and Macroscopic Instability Mechanisms in a Class of Fiber-Reinforced Composites

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To investigate the relation between the macroscopic and microscopic instability predictions for certain microstructured media, we study the stability of an axially stretched fiber-reinforced composite under plane strain conditions. The microstructural instability is attributed to a bifurcation buckling of the fibers while the corresponding macroscopic one occurs at the loss of ellipticity in the homogenized incremental equilibrium equations of the material. The effects of geometry and material properties on those phenomena are analyzed. The macroscopic approach consistently and sometimes considerably overestimates the stable region of the composite. The attractive feature in this work is that all the investigations can be done by using analytical solutions instead of the numerical ones employed in similar investigations so far.

1 Introduction and Motivation

One of the most interesting problems in the study of composite media is the prediction of their failure mechanisms due to adequately large loads. Customarily a composite medium is idealized by a homogeneous continuum whose properties have been obtained by using an appropriate averaging technique (e.g., self-consistent method, homogenization method) which takes into account the solid's microstructure. The question of the composite's stability is usually addressed by investigating the (homogeneous) macroscopic properties of the solid and not by an analysis at the microstructural level. Of the numerous examples of this approach we can state here the Hill-Tsai-Wu (see [1]) failure criterion for fiber-reinforced composites and the work of Rudnicki and Rice [2] for the effects of porosity on the localization strain for microporous metallic aggregates. For a more extensive account on the flow of metals and geomaterials under large strains using the aforementioned approach, the interested reader is referred to S. Nemat-Nasser [3] and references quoted therein. So far relatively few attempts have been made to correlate the stability predictions of the continuum model with the corresponding exact microstructural failure mechanism for a given composite.

One of the first studies in this direction for the case of finitely strained solids was presented by V. Tvergaard [4]. In investigating the effects of voids on the shear band formation of a finitely strained periodic porous material he found the microscopical critical loads and buckling modes and compared them with the predictions of shear band instabilities based on an approximate continuum model of the void-matrix aggregate. More recently Abeyaratne and Triantafyllidis [5] using a consistent homogenization theory approach (needed for proper modeling of the strong interactions between voids) predicted the loss of ellipticity in a finitely strained highly porous periodic elastic material. The overall loss of ellipticity (i.e., the emergence of shear bands) in the homogenized composite is again due to the buckling of the solid at the microstructural level. In view of the composite's geometry though, an exact solution to the corresponding microbuckling problem is very complicated (the method of solution presented in [4] gives only an upper bound for the microbuckling loads) and thus no quantitative comparison between and micro and macrofailure mechanisms was attempted.

The purpose of the present work (which is, so to speak, a continuation of [5]) is a consistent study of the relation between micro and corresponding macroinstability phenomena in finitely strained two-phase composite solids. For more clarity attention will be focused on a model where both the micro and macroinstability problems have an exact analytical solution. As such a case we have decided to investigate a fiber-reinforced composite (made of alternating layers of two different materials) under plane strain conditions and stretched (stretch ratio λ) along x_1, the direction of the reinforcement. The microstructural instability is attributed to a bifurcation (i.e., the loss of uniqueness) of the uniform (i.e., independent of x_1) solution in the form of a wavy pattern. Assuming no layer separation, the bifurcation results for the problem can be obtained in an attractive

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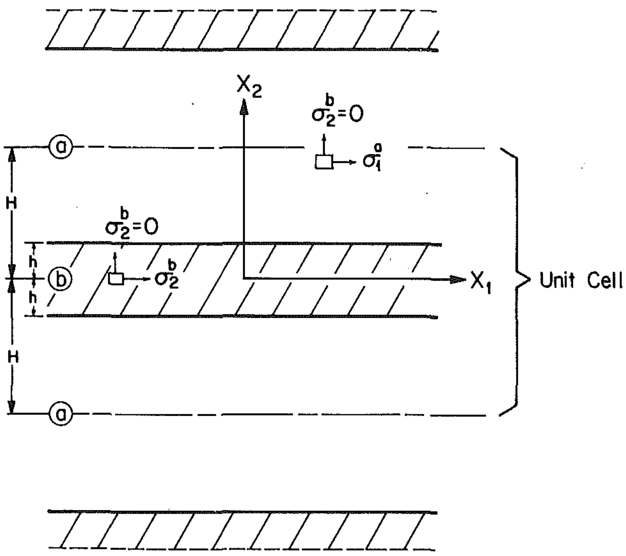

Fig. 1 Schematic representation of the composite

analytical solution (a special case of this problem for layers of equal thickness has been investigated by Biot [6] using a different approach). After calculating the homogenized incremental moduli of the composite we find the macroscopical instability mode which corresponds to the loss of ellipticity in the aforementioned homogenized solid. For given geometry and material properties of the layers we compare the macro and microcritical values for the stretch ratio λ as well as the corresponding instability patterns.

The investigation is carried out for two particular composites. The first application involves a constitutive equation developed by Stören and Rice [7] for modeling the behavior of metals at finite strain while the second one deals with a Mooney-Rivlin solid. The motivation for this choice lies in the different stability properties of the two materials in the finite strain regime. The biggest discrepancies between the micro and macrocritical stretches were found in the case of tension ($\lambda > 1$). For compression the two different mechanisms give the same results for certain range of geometry and material properties. Some very interesting asymptotic (with respect to the wavelength of the bifurcation eigenmode) cases are also discussed. The results prove the dramatic effect of softer inclusions on the stability properties of a two-phase composite and provide a method for quantitative comparison between a macroscopic instability mode and the corresponding microstructural failure mechanism.

2 Microscopic Instability Analysis

2.1 General Problem Formulation. Consider the composite medium, shown in Fig. 1, made of alternating identical layers of two different materials: material (b) with thickness $2h$ and material (a) with thickness $2(H\text{-}h)$. The composite deforms under plane strain conditions and both materials are taken to be incompressible, initially orthotropic (with axes of orthotropy x_1 and x_2) and perfectly bonded between them. At the current state of deformation materials (a) and (b) are under principal stresses $\overset{a}{\sigma}_1$, $\overset{a}{\sigma}_2$ and $\overset{b}{\sigma}_1$, $\overset{b}{\sigma}_2$ along the x_1, x_2 directions respectively. The corresponding principal stretch ratios $\overset{a}{\lambda}_1$, $\overset{a}{\lambda}_2$ and $\overset{b}{\lambda}_1$, $\overset{b}{\lambda}_2$ satisfy the following kinematical relations in view of the incompressibility and the perfect bonding between layers.

$$\overset{a}{\lambda}_1 = \overset{b}{\lambda}_1 \equiv \lambda, \qquad \overset{a}{\lambda}_2 = \overset{b}{\lambda}_2 \equiv 1/\lambda \tag{2.1}$$

Equilibrium across the (a)–(b) interface requires

$$\overset{a}{\sigma}_2 = \overset{b}{\sigma}_2 \equiv \sigma_2 \tag{2.2}$$

The question to be addressed in this section is the following: assuming a displacement (or traction) increase at the composite's infinitely remote boundary corresponding to a uniform strain increment $\mathbf{D}$ = constant ($D_{11} = \dot{\lambda}/\lambda$, $D_{22} = -\dot{\lambda}/\lambda$, $D_{12} = 0$) when does the associated incremental solution for the solid cease to be unique, i.e., when a bifurcated solution for the composite occurs?

The prebifurcation state of the solid can be completely specified as a function of the applied stretch ratio λ. The fundamental solution is obviously the one corresponding to a (piecewise) constant stress state. At some stage of deformation suppose that bifurcation is possible so that for a given increment $\dot{\lambda}$ there exist two different solutions $\dot{\mathbf{u}}^\alpha$ and $\dot{\mathbf{u}}^\beta$ of the incremental equilibrium equations. Introduce the following notation for the difference between two possible solutions in any field quantity

$$\Delta(\cdot) \equiv (\cdot)^\beta - (\cdot)^\alpha \tag{2.3}$$

Taking for simplicity the reference configuration to coincide with the current one at the onset of bifurcation (so that $\mathbf{u} = 0$ but $\dot{\mathbf{u}} \neq 0$) the incremental equilibrium equations, in terms of the first Piola-Kirchhoff (or nominal) stress $T_{\alpha\beta}$, are

$$[\Delta \dot{T}_{\alpha\beta}],_\alpha = 0 \tag{2.4}$$

while the interface traction continuity condition requires

$$[[\Delta \dot{T}_{2\beta}]] = 0 \tag{2.5}$$

with $[[\cdot]]$ denoting the jump of a field quantity across a layer interface.

Here and subsequently the summation convention for repeated indices is tacitly assumed for all Greek subscripts which always range from 1 to 2. Moreover a comma followed by a subscript denotes differentiation with respect to the corresponding coordinate.

For the case of an incompressible, pressure insensitive and initially orthotropic rate-independent material, the most general structure of its constitutive law is, as first noted by Biot [6]

$$\begin{aligned} \overset{\nabla}{\sigma}_{11} &= 2\mu^* \dot{u}_{1,1} - \dot{p}, \\ \overset{\nabla}{\sigma}_{22} &= 2\mu^* \dot{u}_{2,2} - \dot{p}, \\ \overset{\nabla}{\sigma}_{12} &= \overset{\nabla}{\sigma}_{21} = \mu(\dot{u}_{1,2} + \dot{u}_{2,1}) \end{aligned} \tag{2.6}$$

Where $\overset{\nabla}{\sigma}$ denotes the Jaumann (or corrotational) rate of the Cauchy stress and $\dot{p}$ denotes the hydrostatic pressure increment. The quantities μ and μ^* are called the instantaneous shear moduli and depend in general on the deformation history. A more convenient, for the subsequent analysis, form of (2.6) in terms of the nominal stress rate $\dot{\mathbf{T}}$ is given by (see for example Hill and Hutchinson [8])

$$\begin{aligned} &\dot{T}_{11} = (2\mu^* - \sigma_1)\dot{u}_{1,1} - \dot{p}, \quad \dot{T}_{22} = (2\mu^* - \sigma_2)\dot{u}_{2,2} - \dot{p} \\ &\dot{T}_{12} = \left(\mu + \frac{\sigma_1 - \sigma_2}{2}\right)\dot{u}_{2,1} + \left(\mu - \frac{\sigma_1 + \sigma_2}{2}\right)\dot{u}_{1,2}, \\ &\dot{T}_{21} = \left(\mu + \frac{\sigma_2 - \sigma_1}{2}\right)\dot{u}_{1,2} + \left(\mu - \frac{\sigma_1 + \sigma_2}{2}\right)\dot{u}_{2,1} \end{aligned} \tag{2.7}$$

In view of the assumed incompressibility $\Delta \dot{u}_{\alpha,\alpha} = 0$, the following potential function ψ can be introduced for convenience

$$\Delta \dot{u}_1 = \frac{\partial \psi}{\partial x_2}, \quad \Delta \dot{u}_2 = -\frac{\partial \psi}{\partial x_1} \tag{2.8}$$

which upon substitution in the constitutive equations (2.7) gives for the equilibrium equation (2.4)

$$\left(\mu+\frac{\sigma_1-\sigma_2}{2}\right)\psi,_{1111}+2(2\mu^*-\mu)\psi,_{1122}+\left(\mu+\frac{\sigma_2-\sigma_1}{2}\right)\psi,_{2222}=0 \tag{2.9}$$

while the corresponding interface traction continuity conditions (2.5) yield

$$\left[\left[\left(\mu-\frac{\sigma_1-\sigma_2}{2}\right)\psi,_{22}-\left(\mu-\frac{\sigma_1+\sigma_2}{2}\right)\psi,_{11}\right]\right]=0$$
$$\left[\left[\left(\mu-4\mu^*+\frac{\sigma_1+\sigma_2}{2}\right)\psi,_{112}-\left(\mu-\frac{\sigma_1-\sigma_2}{2}\right)\psi,_{222}\right]\right]=0 \tag{2.10}$$

Note that in (2.9) and (2.10) the constants μ, μ^*, σ_1, σ_2 are all functions of the stretch ratio λ. The pertaining explicit functional relations are to be specified subsequently according to the particular constitutive equation choices. The bifurcation problem considered here consists of finding the closest to unity values of λ (one higher for tension, one lower for compression) for which a nontrivial, bounded in $\mathbb{R}^2$, solution to (2.9), (2.10) becomes possible. At this point one can also assume without loss of generality (in view of the material's incompressibility and pressure insensitivity) that the lateral stress $\sigma_2 = 0$. For additional convenience we denote $\sigma_1 \equiv \sigma$.

Observe that in (2.9) the functions μ, μ^*, σ depend solely on x_2 (piecewise constants). Consequently by taking the Fourier transform of (2.9) and (2.10) with respect to x_1 ($x_1 \rightarrow \omega$) the initial two-dimensional problem in ψ is reduced to the following one-dimensional problem in terms of its Fourier transform $\tilde{\psi}$: find conditions for the existence of a nontrivial, bounded in x_2, solution $\tilde{\psi}(\omega, x_2)$ to the fourth-order ordinary differential equation

$$\left(\mu+\frac{\sigma}{2}\right)(i\omega)^4+2(2\mu^*-\mu)(i\omega)^2\tilde{\psi},_{22}+\left(\mu-\frac{\sigma}{2}\right)\tilde{\psi},_{2222}=0 \tag{2.11}$$

with (piecewise constant) periodic coefficients of period $2H$. In addition at the points of discontinuity of these coefficients i.e., at $x_2 = \pm h + 2nH$; $n \epsilon Z$, the following jump conditions are satisfied

$$[[\tilde{\psi}]]=0, \quad [[\tilde{\psi},_2]]=0$$
$$\left[\left[\left(\mu-\frac{\sigma}{2}\right)(\tilde{\psi},_{22}-(i\omega)^2\tilde{\psi})\right]\right]=0 \tag{2.12}$$
$$\left[\left[\left(\mu-4\mu^*+\frac{\sigma}{2}\right)(i\omega)^2\tilde{\psi},_2-\left(\mu-\frac{\sigma}{2}\right)\tilde{\psi},_{222}\right]\right]=0$$

The first two interface conditions are the transformed displacement rate continuity requirements while the last two are the transformed traction rate continuity conditions (2.10). It should be noted at this point that the Fourier transform $\tilde{\psi}(\omega, x_2)$ of ψ (x_1 x_2) is not an ordinary function of ω but a generalized one and thus all subsequent equations in this subsection are taken in the sense of distributions.

From (2.11), in the interval $[-H, H]$, $\tilde{\psi}$ must be of the form

$$\tilde{\psi}(\omega,x_2) = \sum_{j=1}^{4} A_j^+(\omega)\exp(i\omega\overset{a}{z}_j x_2) \qquad h \le x_2 \le H$$
$$\tilde{\psi}(\omega,x_2) = \sum_{j=1}^{4} B_j(\omega)\exp(i\omega\overset{b}{z}_j x_2) \qquad -h \le x_2 \le h \tag{2.13}$$
$$\tilde{\psi}(\omega,x_2) = \sum_{j=1}^{4} A_j^-(\omega)\exp(i\omega\overset{a}{z}_j x_2) \qquad -H \le x_2 \le -h$$

where z_j ($z_3 = -z_1$, $z_4 = -z_2$) are the four distinct roots of the biquadratic

$$\left(\mu+\frac{\sigma}{2}\right)+2(2\mu^*-\mu)z^2+\left(\mu-\frac{\sigma}{2}\right)z^4=0 \tag{2.14}$$

The a or b superscripts in (2.13) indicate which layer's stresses and moduli have to be used in (2.14). All the roots z_j are assumed to be complex with nonzero imaginary part, thus excluding a loss of ellipticity (shear band) type of instability in either constituent of the composite prior to bifurcation. For a more detailed description of the different regimes in a finitely strained solid and the corresponding physical interpretations, the interested reader is referred to Hill and Hutchinson [8].

The determination of the unknown constants A_j^+, B_j, A_j^- proceeds as follows. From Floquet's theory for ordinary differential equations (see, for example, Ince [9]) the solution of (2.11) and (2.12) has to satisfy the quasiperiodicity conditions

$$\tilde{\psi}(H)=\Lambda\tilde{\psi}(-H),\ \ \tilde{\psi},_2(H)=\Lambda\tilde{\psi},_2(-H),\ \ \tilde{\psi},_{22}(H)=\Lambda\tilde{\psi},_{22}(H),$$
$$\tilde{\psi},_{222}(H)=\Lambda\tilde{\psi},_{222}(-H);\ \ \Lambda\equiv\exp(iq2H) \tag{2.15}$$

where q is a real number to be subsequently specified. Substituting (2.13) into (2.12) and (2.15) it is found that a nonzero solution to our problem exists (i.e., nonzero coefficients A_j^+, B_j, A_j^-) when

$$\mathrm{Det}[\mathbf{F}-\exp(2iqH)\mathbf{I}]=0;$$
$$\mathbf{F}\equiv\overset{a}{\mathbf{V}}\exp[2i\omega(H-h)\overset{a}{\mathbf{Z}}]\overset{a}{\mathbf{V}}{}^{-1}\overset{b}{\mathbf{V}}\exp[2i\omega h\overset{b}{\mathbf{Z}}]\overset{b}{\mathbf{V}}{}^{-1} \tag{2.16}$$

with the four by four matrices $\mathbf{V}$, $\mathbf{Z}$, and $\mathbf{I}$ given by

$$V_{1i}=1, V_{2i}=z_i, V_{3i}=\left(\mu-\frac{\sigma}{2}\right)(z_i^2-1),$$
$$V_{4i}=\left(\mu-\frac{\sigma}{2}\right)z_i{}^3+\left(4\mu^*-\mu-\frac{\sigma}{2}\right)z_i \tag{2.17}$$
$$Z_{ij}=z_i\delta_{ij}(\text{no sum}),\quad \mathbf{I}=\delta_{ij}\quad 1\le i,j\le 4$$

Observing that Det $\mathbf{F} = 1$ and that tr $(\mathbf{F})$ = tr $(\mathbf{F}^{-1})$ the characteristic polynomial of $\mathbf{F}$ is

$$\Lambda^4-(\mathrm{tr}\mathbf{F})\Lambda^3+\frac{1}{2}[(\mathrm{tr}\mathbf{F})^2-\mathrm{tr}\mathbf{F}^2]\Lambda^2-(\mathrm{tr}\mathbf{F})\Lambda+1=0 \tag{2.18}$$

with all real coefficients, as it can be easily shown, and with Λ given by (2.15). It is a matter of straightforward algebra to show that $\mathbf{F}$ admits a unimodular (since $q \epsilon \mathbb{R}$, $|\Lambda| = 1$) eigenvalue when the following conditions are met

$$(|\mathrm{tr}\mathbf{F}|+2)^2 \ \ge\ \mathrm{tr}\mathbf{F}^2\ge(|\mathrm{tr}\mathbf{F}|-2)^2 \qquad \text{if } |\mathrm{tr}\mathbf{F}|\ge 4$$
$$(|\mathrm{tr}\mathbf{F}|+2)^2 \ \ge\ \mathrm{tr}\mathbf{F}^2\ge\frac{1}{2}(\mathrm{tr}\mathbf{F})^2-4 \qquad \text{if } \mathrm{tr}\mathbf{F}\ge 4 \tag{2.19}$$

Observing that $\mathbf{F} = \mathbf{F}(\lambda)$, the determination of the composite's critical stretch ratio is then achieved as follows: for fixed ω – and for given geometry and material properties – and for $\lambda > 1$ in tension ($\lambda < 1$ in compression) one seeks the lowest (highest) value of λ say $\hat{\lambda}$ for which (2.19) is satisfied. Obviously at $\hat{\lambda}$ one of these three inequalities will be satisfied as an equality. The wanted critical value of the stretch ratio is given by $\lambda_{cr} = \min \hat{\lambda}(\omega)$ ($\lambda_{cr} = \max\hat{\lambda}(\omega)$) over all $\omega \ge 0$. As one can see from (2.16) and (2.18) only positive values of ω need be considered. Results from calculations using specific constitutive models (i.e., using particular functions μ^* (λ), $\mu(\lambda)$ and $\sigma(\lambda)$) will be presented in a subsequent section.

2.2 Critical Stretch Ratios for the Long and Short Wavelength Limits. In determining $\hat{\lambda}(\omega)$ two cases are of particular interest: the long wavelength limit $\omega \rightarrow 0$ and the short wavelength limit $\omega \rightarrow \infty$. The employed nomenclature stems from the fact that ω is the frequency of the corresponding bifurcation eigenmode $\exp(i\omega x_1)f(\omega,x_2)$.

2.2.1 Long Wavelength Limit ($\omega \to 0$). A (matrix) series expansion of the exponential terms in **F** (see (2.16)) in powers of ω and groupment of the same order terms, yields after some lengthy but straightforward calculations

$$\begin{aligned} \mathrm{tr}\mathbf{F} &= 4+(\omega H)^2\chi+(\omega H)^4\zeta+\ldots; \\ \mathrm{tr}\mathbf{F}^2 &= 4+(\omega H)^2 4\chi+(\omega H)^4(16\zeta+\eta)+\ldots \end{aligned} \qquad (2.20)$$

where χ, ζ, and η are given by

$$\chi = \frac{\xi_a^2}{2}\mathrm{tr}(\overset{a}{\mathbf{Z}}{}^2)+\xi_a\xi_b\mathrm{tr}(\overset{a}{\mathbf{V}}\overset{a}{\mathbf{Z}}\overset{a}{\mathbf{V}}{}^{-1}\overset{b}{\mathbf{V}}\overset{b}{\mathbf{Z}}\overset{b}{\mathbf{V}}{}^{-1})+\frac{\xi_b^2}{2}\mathrm{tr}(\overset{b}{\mathbf{Z}}{}^2)$$

$$\begin{aligned} \zeta = {} & \frac{\xi_a^4}{24}\mathrm{tr}(\overset{a}{\mathbf{Z}}{}^4)+\frac{\xi_a^3\xi_b}{6}\mathrm{tr}(\overset{a}{\mathbf{V}}\overset{a}{\mathbf{Z}}{}^3\overset{a}{\mathbf{V}}{}^{-1}\overset{b}{\mathbf{V}}\overset{b}{\mathbf{Z}}\overset{b}{\mathbf{V}}{}^{-1}) \\ & +\frac{\xi_a^2\xi_b^2}{4}\mathrm{tr}(\overset{a}{\mathbf{V}}\overset{a}{\mathbf{Z}}{}^2\overset{a}{\mathbf{V}}{}^{-1}\overset{b}{\mathbf{V}}\overset{b}{\mathbf{Z}}{}^2\overset{b}{\mathbf{V}}{}^{-1}) \\ & +\frac{\xi_a\xi_b^3}{6}\mathrm{tr}(\overset{a}{\mathbf{V}}\overset{a}{\mathbf{Z}}\overset{a}{\mathbf{V}}{}^{-1}\overset{b}{\mathbf{V}}\overset{b}{\mathbf{Z}}{}^3\overset{b}{\mathbf{V}}{}^{-1})+\frac{\xi_b^4}{24}\mathrm{tr}(\overset{b}{\mathbf{Z}}{}^4) \end{aligned}$$

$$\eta = \xi_a^2\xi_b^2[\mathrm{tr}(\overset{a}{\mathbf{V}}\overset{a}{\mathbf{Z}}\overset{a}{\mathbf{V}}{}^{-1}\overset{b}{\mathbf{V}}\overset{b}{\mathbf{Z}}\overset{b}{\mathbf{V}}{}^{-1})^2-\mathrm{tr}(\overset{a}{\mathbf{V}}\overset{a}{\mathbf{Z}}{}^2\overset{a}{\mathbf{V}}{}^{-1}\overset{b}{\mathbf{V}}\overset{b}{\mathbf{Z}}{}^2\overset{b}{\mathbf{V}}{}^{-1})]$$

with

$$\xi_a \equiv 2i(1-\epsilon), \quad \xi_b \equiv 2i\epsilon, \quad \epsilon \equiv h/H \qquad (2.21)$$

From (2.21) one can deduce the following useful identities

$$\begin{aligned} 2\chi &= \mathrm{tr}\mathbf{g}^2, \quad 24\zeta+2\eta=\mathrm{tr}\mathbf{g}^4 \\ \mathbf{g} &= \xi_a\overset{a}{\mathbf{V}}\overset{a}{\mathbf{Z}}\overset{a}{\mathbf{V}}{}^{-1}+\xi_b\overset{b}{\mathbf{V}}\overset{b}{\mathbf{Z}}\overset{b}{\mathbf{V}}{}^{-1} \end{aligned} \qquad (2.22)$$

A further simplification of the foregoing result is possible; upon substitution of (2.17) into (2.22) and using also (2.12) the matrix **g** is found to be

$$\begin{aligned} g_{12} &= g_{21}=g_{34}=g_{43}=2i, \quad g_{41}=-2i[(1-\epsilon)\overset{a}{\sigma}+\epsilon\overset{b}{\sigma}]\equiv 2iv \\ g_{23} &= 2i[(1-\epsilon)/(\overset{a}{\mu}-\overset{a}{\sigma}/2)+\epsilon/(\overset{b}{\mu}-\overset{b}{\sigma}/2)]\equiv 2i\pi \qquad (2.23) \\ g_{32} &= 2i[(1-\epsilon)(4\overset{a}{\mu}{}^*-\overset{a}{\sigma})+\epsilon(4\overset{b}{\mu}{}^*-\overset{b}{\sigma})]\equiv -2i\rho \end{aligned}$$

while all the other entries of **g** are zero.

Introducing the asymptotic expansions for tr **F** and tr $\mathbf{F}^2$ (2.20) into the characteristic equation (2.18) and recalling that $\Lambda = \exp(2iqH)$ one obtains, after grouping the terms of the like order in ωH, that in the limit $\omega H \to 0$

$$\hat{q}^4+\hat{q}^2\chi/4+(\chi^2-\eta-12\zeta)/32=0; \qquad \hat{q}\equiv q/\omega \qquad (2.24)$$

which with the help of (2.21)-(2.23) can be rewritten as

$$\hat{q}^4+\hat{q}^2(\pi\rho-2)+(1-\pi v)=0 \qquad (2.25)$$

The existence of a real in ($\hat{q}$) solution of (2.25) signals the presence of a long wavelength bifurcation mode. The parameter $\hat{q}$ admits an interesting physical interpretation: it is the ratio of frequencies of the buckling pattern with respect to the x_2 and x_1 directions, respectively. Consequently $\hat{q} = \tan\phi$ where ϕ is the inclination of the normal (with respect to x_1) to the diagonal of the fundamental rectangle of the quasi periodic buckling eigenmode.

2.2.2 Short Wavelength Limit ($\omega \to \infty$). Unlike the previous case, the determination of the critical stretch $\hat{\lambda}(\infty)$ for the short wavelength case is easier achieved by analyzing the initial equations (2.11), and (2.12), (2.15).

From (2.13) $\tilde{\psi}(\omega, x_2)$ can be recasted in the more convenient for this case form:

$$\begin{aligned} \tilde{\psi}(\omega,x_2) = {} & \hat{A}_1^+(\omega)\exp[i\omega\overset{a}{z}_1(x_2-h)] \\ & +\hat{A}_2^+(\omega)\exp[i\omega\overset{a}{z}_2(x_2-h)] \\ & +\hat{A}_3^+(\omega)\exp[-i\omega\overset{a}{z}_1(x_2-H)] \\ & +\hat{A}_4^+(\omega)\exp[-i\omega\overset{a}{z}_2(x_2-H)] \quad \text{for} \quad h\le x_2\le H \\ \tilde{\psi}(\omega,x_2) = {} & \hat{B}_1(\omega)\exp[-i\omega\overset{b}{z}_1(x_2-h)] \\ & +\hat{B}_2(\omega)\exp[-i\omega\overset{b}{z}_2(x_2-h)] \\ & +\hat{B}_3(\omega)\exp[i\omega\overset{b}{z}_1(x_2+h)] \\ & +\hat{B}_4(\omega)\exp[i\omega\overset{b}{z}_2(x_2+h)] \quad \text{for} \quad -h\le x_2\le h \\ \tilde{\psi}(\omega,x_2) = {} & \hat{A}_1^-(\omega)\exp[i\omega\overset{a}{z}_1(x_2+H)] \\ & +\hat{A}_2^-(\omega)\exp[i\omega\overset{a}{z}_2(x_2+H)] \\ & +\hat{A}_3^-(\omega)\exp[-i\omega\overset{a}{z}_1(x_2+h)] \\ & +\hat{A}_4^-(\omega)\exp[-i\omega\overset{a}{z}_2(x_2+h)] \quad \text{for} \ -H\le x_2\le -h \end{aligned} \qquad (2.26)$$

where z_1, z_2 are the two roots of (2.14) with positive imaginary parts.

For $\omega \to \infty$ the boundary conditions at the interfaces $x_2 = \pm h$[1] are satisfied (for nontrivial $\tilde{\psi}$ in (2.26)) when

$$\mathrm{Det}\mathbf{W}=0;$$

$$[W_{ij}=\overset{a}{V}_{ij}\ 1\le j\le 2,\ W_{ij}=(-1)^i\overset{b}{V}_{ij-2}\ 3\le j\le 4],\ 1\le i\le 4 \qquad (2.27)$$

The preceding condition after considerable manipulations using (2.14) and (2.17) can be equivalently written as

$$\begin{aligned} & (\overset{a}{\mu}-\overset{a}{\sigma}/2)^2[-1+(\overset{a}{z}_1\overset{a}{z}_2)^2-(\overset{a}{z}_1\overset{a}{z}_2)(2-(\overset{a}{z}_1)^2-(\overset{a}{z}_2)^2)] \\ & +(\overset{b}{\mu}-\overset{b}{\sigma}/2)^2[-1+(\overset{b}{z}_1\overset{b}{z}_2)^2-(\overset{b}{z}_1\overset{b}{z}_2)(2-(\overset{b}{z}_1)^2-(\overset{b}{z}_2)^2)] \\ & +(\overset{a}{\mu}-\overset{b}{\sigma}/2)({}^b\mu-{}^b\sigma/2)[2(1+\overset{a}{z}_1\overset{a}{z}_2)(1+\overset{b}{z}_1\overset{b}{z}_2) \\ & +(\overset{a}{z}_1+\overset{a}{z}_2)(\overset{b}{z}_1+\overset{b}{z}_2)(\overset{a}{z}_1\overset{a}{z}_2+\overset{b}{z}_1\overset{b}{z}_2)]=0 \end{aligned} \qquad (2.28)$$

which is exactly the condition for the onset of on surface instability at the interface of two finitely strained and perfectly bonded incompressible solids (see, for example, Biot [6]) as one can easily observe from the exponentially decaying eigenmode in (2.26) away from the interfaces $x_2 = \pm h$ (as $\omega \to \infty$).

3 Macroscopic Instability Analysis

The equilibrium (2.4) compatibility (2.8) and constitutive (2.7) equations for the composite can alternatively be put in the form

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left[L_{\alpha\beta\gamma\delta}\frac{\partial^2\psi}{\partial x_\gamma \partial x_\delta}\right]=0 \qquad (3.1)$$

with the periodic in x_2 of period $2H$ functions $L_{\alpha\beta\gamma\delta}(x_2)$ given by

$$\begin{aligned} & L_{1111}=\mu+\sigma/2,\ L_{1122}=L_{2211}=-\mu+\sigma/2,\ L_{2222}=\mu-\sigma/2 \\ & L_{1212}=\mu+\mu^*,\ L_{1221}=L_{2112}=\mu^*-\mu,\ L_{2121}=\mu+\mu^*-\sigma \end{aligned} \qquad (3.2)$$

Of interest here is the behavior of the incremental displacement potential ψ^H (where the superscript (H) denotes the dependence of ψ on the period $2H$) as the periodic structure of the composite becomes finer, i.e., as $H \to 0$. The

[1] Note: The Floquet conditions (2.15) in the limit $\omega \to \infty$ turn out to be superfluous.

basic result from homogenization theory[2] (see, for example, Bensoussan, Lions, and Papanicolaou [10]) establishes that in the limit $H \to 0$, $\psi^H \to \psi^0$ which satisfies

$$\mathcal{L}_{\alpha\beta\gamma\delta} \frac{\partial^2 \psi^0}{\partial x_\alpha \partial x_\beta \partial x_\gamma \partial x_\delta} = 0 \tag{3.3}$$

where the constant (i.e., independent on $\mathbf{x}$) homogenized incremental moduli $\mathcal{L}$ are given by

$$\mathcal{L}_{\alpha\beta\gamma\delta} = \frac{1}{2} \int_{-1}^{1} \left[L_{\alpha\beta\gamma\delta}(y_2) + L_{\alpha\beta\epsilon\zeta}(y_2) \frac{d^2 \phi^{\gamma\delta}(y_2)}{dy_\epsilon dy_\zeta} \right] dy_2,$$

$$y_2 \equiv x_2/H \tag{3.4}$$

in which the functions $\phi^{\gamma\delta}(y_2)$ are periodic in $[-1, 1]$ i.e., $\phi^{\epsilon\zeta}(-1) = \phi^{\epsilon\zeta}(1)$ and satisfy

$$\frac{d^2}{dy_2^2} \left[L_{2222}(y_2) \frac{d^2 \phi^{\epsilon\zeta}}{dy_2^2} \right] = -\frac{d^2}{dy_2^2} L_{22\epsilon\zeta}(y_2)$$
$$\epsilon = \zeta = 1 \text{ or } \epsilon = \zeta = 2 \tag{3.5}$$

The foregoing result can be straightforwardly obtained using S. Palencia's [11] method of multiple scale expansions i.e., putting $\psi^H(x_1, x_2) = \psi^0(x_1, x_2; y_2) + H\psi^1(x_1, x_2; y_2) + \ldots + H^n \psi^n(x_1, x_2; y_2)$ substituting into (3.1) and subsequently collecting terms of the like order in H. Employing (3.4) and (3.5) the homogenized incremental moduli are found to be

$$\mathcal{L}_{1111} = \langle L_{1111} - L_{1122} L_{2211}/L_{2222} \rangle + \langle L_{2211}/L_{2222} \rangle^2 / \langle L_{2222}^{-1} \rangle$$

$$\mathcal{L}_{1122} = \mathcal{L}_{2211} = \langle L_{1122}/L_{2222} \rangle / \langle L_{2222}^{-1} \rangle, \mathcal{L}_{2222} = 1/\langle L_{2222}^{-1} \rangle$$

$$\text{all others } \mathcal{L}_{\alpha\beta\gamma\delta} = \langle L_{\alpha\beta\gamma\delta} \rangle \text{ with } \langle f \rangle \equiv 0.5 \int_{-1}^{1} f(y_2) dy_2 \tag{3.6}$$

Thus the investigation of the macroscopic stability properties of the composite is reduced to the examination of the loss of ellipticity in the incremental equilibrium equations of the homogenized material (3.3). The corresponding condition is the existence of real roots n_2/n_1 in the equation

$$\mathcal{L}_{1111} n_1^4 + (2\mathcal{L}_{1122} + 2\mathcal{L}_{1221} + \mathcal{L}_{1212} + \mathcal{L}_{2121}) n_1^2 n_2^2 + \mathcal{L}_{2222} n_2^4 = 0 \tag{3.7}$$

where n_1, n_2 are the direction cosines of the normal $\mathbf{n}$ to the characteristic direction. Denoting by ϕ the angle formed between $\mathbf{n}$ and x_1 (i.e., $\tan\phi = n_2/n_1$) and employing (3.6) and (3.2), (3.7) can be rewritten in the form

$$(\tan\phi)^4 + (\pi\rho - 2)(\tan\phi)^2 + (1 - \pi\upsilon) = 0 \tag{3.8}$$

where π, ρ, υ, are the material constants (functions of λ) defined in (2.3).

Observe that the loss of ellipticity condition for the homogenized material (3.8) is identical to equation (2.25) corresponding to the onset of the long wavelength bifurcation instability. This coincidence has to be expected since in both cases (i.e., in (2.25) as well in (3.8)) attention is focused on an instability mode with characteristic wavelength much higher than the unit cell size, i.e., $\omega H \to 0$. It is very interesting however to see how one achieves the same answer for the aforementioned long wavelength instability mode following two completely different analytical procedures.

4 Results From the Study of Two Particular Composites

In the present section the micro and macroinstability behavior of two particular composites is investigated. The first application involves a constitutive equation developed by

[2] Note: A rigorous mathematical proof of this result requires some restrictive assumptions on $\mathbf{L}$.

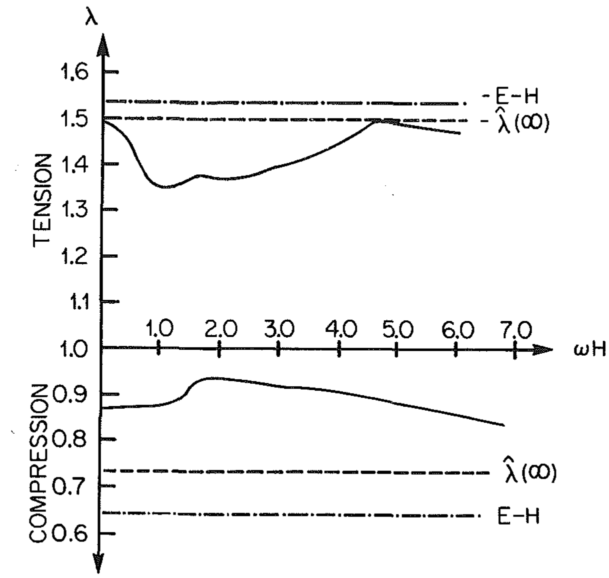

Fig. 2 Dependence of the critical stretch ratio λ on the nondimensionalized eigenmode frequency ωH for the hypoelastic composite with $E_a/E_b = 0.1, m_a = m_b = 4.0, \epsilon = 0.2$

Stören and Rice [7] for modeling the behavior of metals at finite strain. The second application deals with a Mooney-Rivlin solid. The motivation for this choice lies in the different stability properties of the two materials in the large strain regime, as it will be explained in the sequel.

4.1 Stören-Rice Material. The first composite to be considered consists of two alternating series of layers obeying the Stören-Rice [7] deformation theory of plasticity. This hypoelastic constitutive model has been successfully employed to predict the failure strains of finitely deformed metallic sheets. The reason for this model's use in the present work is that it loses ellipticity under plane strain conditions for rather realistic strain levels $0.5 < \lambda < 2$. The incremental moduli μ, μ^* as well as the stress σ under plane strain uniaxial stretching conditions are found to be (see for example Abeyaratne and Triantafyllidis [12] for the corresponding derivations)

$$\mu = \frac{E}{3} \left| \frac{2}{\sqrt{3}} \frac{ln\lambda}{\epsilon_y} \right|^{\frac{1}{m} - 1}, \quad \mu^* = \mu/m,$$

$$\sigma = E\epsilon_y \frac{2}{\sqrt{3}} \left| \frac{2 ln\lambda}{\sqrt{3}\epsilon_y} \right|^{\frac{1}{m}} \text{sgn}(ln\lambda) \tag{4.1}$$

when the uniaxial stress-strain curve is a power law with hardening exponent m, initial yield strain ϵ_y, and a Young's modulus E (i.e. $(\sigma/\sigma_y)^m = E(\epsilon/\epsilon_y)$; $\sigma_y = E\epsilon_y$). For additional information on this model and its physical relevance in the case of finitely strained metals the interested reader is referred to the original work by Stören and Rice [7].

The dependence of the critical stress $\hat{\lambda}$ on ωH the nondimensional frequency of the buckling eigenmode for a composite with $E_a/E_b = 0.1$, $m_a = m_b = 4.0$ and $\epsilon = h/H = 0.2$ is depicted in Fig. 2. Observe that the lowest (highest) stretch ratio corresponding to the onset of the first microstructural instability in the composite $\lambda_{cr} = \max\hat{\lambda}(\omega H)$ ($\lambda_{cr} = \min\hat{\lambda}(\omega H)$) is $\lambda_{cr} \cong 1.35$ ($\lambda_{cr} \cong 0.95$) while the macroscopic–loss of ellipticity–instability of the homogenized composite occurs at $\lambda = \hat{\lambda}(0) \cong 1.49$ ($\lambda = \hat{\lambda}(0) \cong 0.87$). The dashed lines in both figures indicate the onset of the surface buckling modes at the interface of the composite while the dash-dot lines correspond to the loss of ellipticity of the material which is the same for both layers since $m_a = m_b$

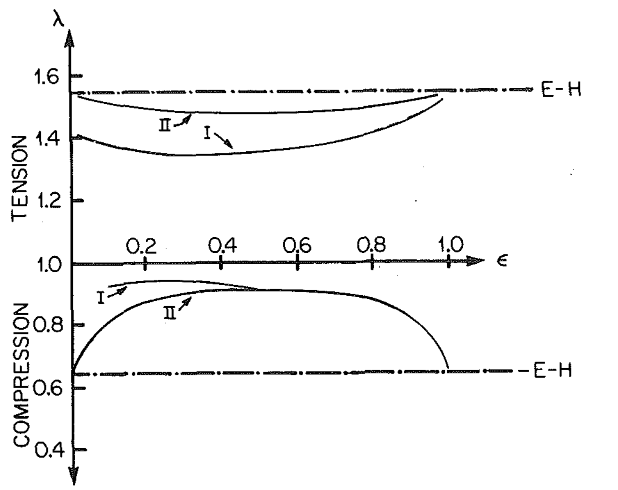

Fig. 3 Dependence of the microscopic (*I*) and macroscopic (*II*) critical stretch ratio λ on the layer thickness parameter ε for the hypoelastic composite with $E_a/E_b = 0.1, m_a = m_b = 4.0$

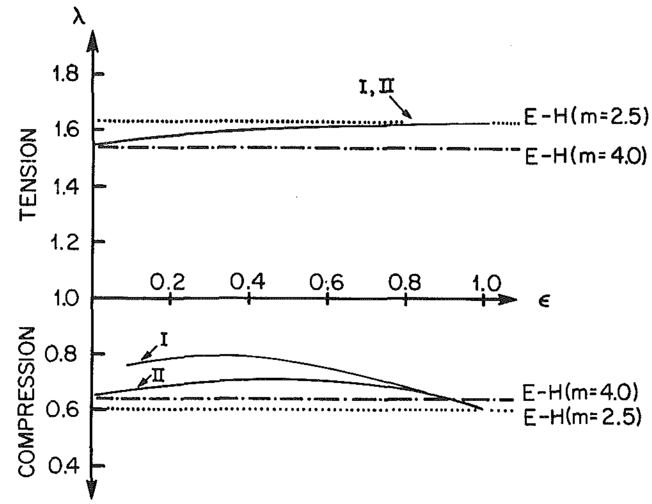

Fig. 4 Dependence of the microscopic (*I*) and macroscopic (*II*) critical stretch ratio λ on the layer thickness parameter ε for the hypoelastic composite with $E_a/E_b = 1, m_a = 4, m_b = 2.5$

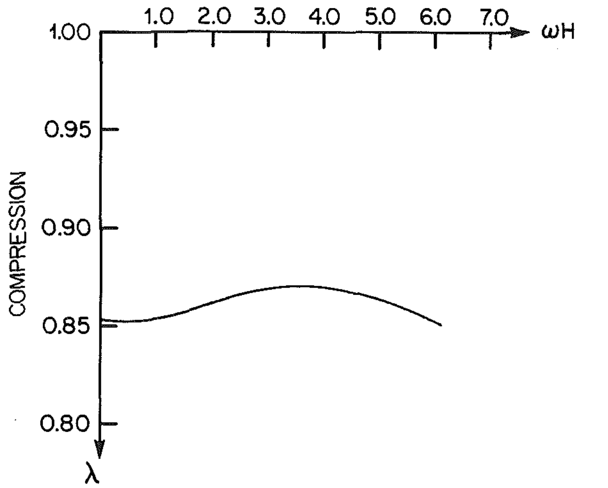

Fig. 5 Dependence of the critical stretch ratio λ on the nondimensionalized eigenmode frequency ωH for the hyperelastic composite with $E_a/E_b = 0.07, \epsilon = 0.1$

(see [12]). The results presented in Fig. 2 are very typical of all such $\hat{\lambda}(\omega H)$ curves obtained for the composites examined in this work.

The comparison between the critical stretch ratios for the onset of the microbuckling (curves *I* correspond to λ_{cr}) and those for the onset of the macroinstability (curves *II* correspond to $\hat{\lambda}(0)$) in a composite with $E_a/E_b = 0.1$, $m_a = m_b = 4$ for both tensile and compressive loading is depicted in Fig. 3 as a function of the composite's geometry, characterized by the thickness ratio $\epsilon = h/H$. Note that in tension the microbuckling instability always precedes the loss of ellipticity in the homogenized material with the biggest difference occurring around $\epsilon = 0.3$ (differences of the order of 30 percent). In compression the differences between the critical stretches corresponding to the onset of macro and microinstability are even greater. First observe that although each layer of the composite loses ellipticity at $\lambda_{cr} \approx 0.65$ the homogenized material loses ellipticity at much lower strains (e.g., $\hat{\lambda}(0) \cong 0.9$ for $\epsilon = 0.5$) Also note that for $\epsilon < 0.5$ (approximately) the microstructural instability occurs at lower strains than the macroscopic one with the difference between the corresponding stretch ratios increasing monotonically as the size of the stiffer inner fiber decreases i.e., as $\epsilon \to 0$. No microscopic instability occurs for $\epsilon > 0.5$.

The stability problem of a composite with two layers of identical initial stiffness, i.e., $E_a/E_b = 1$ but of different hardening exponents $m_a = 4$, $m_b = 2.5$ is examined in Fig. 4. Observe that in tension the lowest critical stretch ratio corresponds to the onset of the long wavelength mode i.e., $\lambda_{cr} = \hat{\lambda}(0)$ and thus the only instability occurring in the composite under tensional loading is the macroscopic one. It is also worth mentioning that the critical stretch in this case, which corresponds to the loss of ellipticity of the homogenized composite lies between the critical stretch ratios for the loss of ellipticity in each layer. Quite a different situation is encountered in the case of compression. There is considerable difference between the onset of the micro and the macroinstabilities which, in a way quite analogous to the compressive behavior of the previous composite (see Fig. 3) increases with decreasing layer thickness ratio ε. Note that the two critical stretches coincide for $\epsilon \geq 0.8$ approximately. In addition the behavior of the homogenized material in compression is also of interest in this case since its loss of ellipticity, for a wide range of ε, occurs at a strain level significantly inferior to those corresponding to the loss of ellipticity of each layer.

In conclusion one can deduce that the greater the difference in the layer stiffnesses, the more important the discrepancies between the macro and microinstability predictions are. Moreover, in the case of compressive loading, the thinner the stiffest layer is, the lower the critical strain for the onset of microbuckling is, and the bigger the difference between the micro and macroinstability critical stretches becomes.

4.2 Mooney-Rivlin Material. The second composite to be investigated is made up of alternating layers of rubber that obey the Mooney-Rivlin constitutive law. The choice in this case was made on the basis that this model remains always elliptic at any level of strain (see for example Knowles and Sternberg [13]).

For an incompressible, plane strain, uniaxial stress are given by

$$\mu = E(\lambda^2 + \lambda)^{-2}/6, \;\; \mu = \mu^*, \;\; \sigma = E(\lambda^2 - \lambda^{-2})/3 \qquad (4.2)$$

where E is the material's Young's modulus.

The long and short wavelength instabilities present a special interest in this case and will be studied further. For the macroinstability ($\omega H \to 0$) from (4.2) and (2.23) the characteristic equation for the loss of ellipticity in the homogenized composite (3.8) (or (2.25)) assumes the form

$$\hat{q}^4 + \hat{q}^2[(\lambda^4 + 3)s - 2] + 1 + (\lambda^4 - 1)s = 0; \qquad (4.3)$$
$$s \equiv [(1-\epsilon)/E_a + \epsilon/E_b][(1-\epsilon)E_a + \epsilon E_b]$$

where s satisfies $s \geq 1$. One can easily verify that for tension ($\lambda > 1$) the homogenized composite remains always elliptic

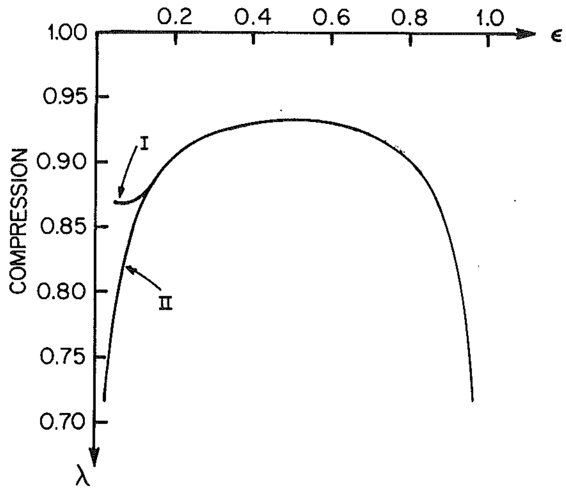

Fig. 6 Dependence of the microscopic (*I*) amd macroscopic (*II*) critical stretch ratio λ on the layer thickness parameter ε for the hyperelastic composite with $E_a/E_b = 0.07$

while in compression ($\lambda < 1$) the homogenized composite loses ellipticity at

$$\hat{\lambda}(0) = [(s-1)/s]^{1/4} \tag{4.4}$$

Similarly for the short wavelength limit ($\omega H \rightarrow \infty$) from (2.28) surface buckling at the layers' interface is found with the help of (4.2) and (2.14) to occur at (observing that in this case $z_1 = i\lambda^2$, $z_2 = i$)

$$x^6 + x^4(1+3t)/(1+t) + x^2(3-t)/(1+t) - (1-t)/(1+t) = 0$$

$$x \equiv \hat{\lambda}(\infty), \quad t \equiv 2E_aE_b/(E_a^2 + E_b^2) \tag{4.5}$$

where $0 \leq t \leq 1$. One can easily show that in this case (4.5) admits no interlayer surface instability for tensile loading while for compressive loading such an instability always occurs in the interval $0 \leq \hat{\lambda}(\infty) \leq 0.54$.

For the case of tensile loading the Mooney-Rivlin composite does not seem to admit a microbuckling instability either. Indeed no solution $\hat{\lambda}(\omega H)$ for (2.19) was found in the interval $1 \leq \lambda \leq 20$, $0 \leq \omega H \leq 16$ for a wide range of material properties $0.01 \leq E_a/E_b \leq 1$ and for all possible geometries $0 \leq \epsilon \leq 1$. Hence only the stability properties of the composite under compressive loading will be investigated.

The typical critical stretch ratio $\hat{\lambda}$ versus nondimensionalized frequency ωH of the buckling eigenmode is shown in Fig. 5 for a composite with $E_a/E_b = 0.07$ and $\epsilon = 0.1$. For this particular case the stretch corresponding to the onset of the first microstructural instability is $\lambda_{cr} = 0.87$.

Finally, Fig. 6 shows the comparison between the critical stretch ratios for the onset of the microbuckling (curve *I* corresponds to $\hat{\lambda}_{cr}$) and those for the onset of the macroinstability (curve *II* corresponds to $\hat{\lambda}(0)$ for a composite with $E_a/E_b = 0.07$ as a function of the layer thickness ratio ϵ. Note that $\lambda_{cr} < \hat{\lambda}(0)$ only for values of $\epsilon < 0.2$ approximately. Also note that the significant reduction for the macroscopic loss of ellipticity strain near $\epsilon = 0.5$ (as it can easily be shown from (4.4) curve *II* passes from $\hat{\lambda} = 0$ at $\epsilon = 0$, $\epsilon = 1$).

5 Conclusions

The purpose of this work has been the consistent quantitative comparison between the macroscopic and the corresponding microscopic instability mechanisms for composite media under finite strain by using a specific example and not the investigation of the stability properties of a particular composite.

Assuming a microbuckling mechanism responsible for the loss of ellipticity in the macroscopical level we have carried out an analytical investigation of the micro and macrostability problem for a fiber-reinforced composite using two different types of constitutive equations. Of the results obtained here the most dramatic evidence of the effect of the microstructure on the overall stability of a medium is the shear band type of instability appearing in the Mooney-Rivlin composite under compression in spite of the fact that the aforementioned material remains pointwise elliptic at any level of strain. This work can also be considered as a first step toward an improved quantitative understanding of the effects of geometry constraint in the stability properties of certain media with microstructure.

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References

1 Tsai, S., and Wu, E., "A General Theory of Strength for Anisotropic Materials," *J. Composite Materials*, Vol. 5, 1971, pp. 58–81.

2 Rudnicki, J., and Rice, J., "Conditions for the Localization of Deformation in Pressure Sensitive Dilatant Materials," *J. Mech. Physics Solids*, Vol. 23, 1975, pp. 371–394.

3 Nemat-Nasser, S., "On Finite Plastic Flow of Crystalline Solids and Geomaterials," ASME Journal of Applied Mechanics, Vol. 50, 1983, pp. 1114–1126.

4 Tvergaard, V., "Influence of Voids on Shear Band Instabilities Under Plane Strain Conditions," *Int. J. of Fracture*, Vol. 17, 1981, pp. 389–407.

5 Abeyaratne, R., and Triantafyllidis, N., "An Investigation of Localization in a Porous Plastic Material Using Homogenization Theory," ASME Journal of Applied Mechanics, Vol. 51, 1984, pp. 481=486.

6 Biot, M., *Mechanics of Incremental Deformation*, Wiley, New York, 1965.

7 Stören, S., and Rice, J., "Localized Necking in Thin Sheets," *J. Mech. Physics Solids*, Vol. 23, 1975, pp. 421=441.

8 Hill, R., and Hutchinson, J., "Bifurcation Phenomena in the Plane Strain Tension Test," *J. Mech. Physics Solids*, Vol. 23, 1975, pp. 239–264.

9 Ince, E., *Ordinary Differential Equations*, Dover, 1956.

10 Bensoussan, A., Lions, J., and Papanicolaou, G., *Asymptotic Analysis for Periodic Structures*, North Holland, 1978.

11 Sanchez-Palencia, E., "Comportement Local et Macroscopique d'un type de Milieux Physiques Heterogenes," *Int. J. Eng. Sci.*, Vol. 12, 1974, pp. 331-351.

12 Abeyaratne, R., and Triantafyllidis, N., "On the Emergence of Shear Bands in Plane Strains," *Int. J. Solids Structures*, Vol. 17, 1981, pp. 1113–1134.

13 Knowles, J., and Sternberg, E., "On the Failure of Ellipticity and the Emergence of Discontinuous Deformation Gradients in Plane Finite Elastostatics," *J. of Elasticity*, Vol. 8, 1978, pp. 329–379.