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An Investigation of Localization in a Porous Elastic Material Using Homogenization Theory

In this work we use a consistent homogenization theory approach to study the overall behavior of a highly porous elastic material at finite levels of strain. Although the matrix material in the problem considered here remains elliptic at all deformations, the resulting "homogenized" material is found to admit a loss of ellipticity at sufficiently high strains, i.e., can fail in a localized shear mode.

1 Introduction

To study the elastic behavior of a compressible material at finite strain, Blatz and Ko [1] carried out a series of experiments on a foamed polyurethane rubber. The material tested was obtained by chemically treating an (incompressible) vulcanized rubber, thereby generating a fine distribution of pores in it (with porosity ≈ 50 percent). The resulting inhomogeneous material is assumed, in an overall sense, to behave in a homogeneous and compressible manner. Based on their experiments, these investigators proposed a particular isotropic strain-energy density function characterizing the overall mechanical behavior of the foam rubber. This idealized model is commonly referred to as the "Blatz-Ko material."

A striking feature of the Blatz-Ko material is that the displacement equations of equilibrium associated with it lose ellipticity at sufficiently severe levels of strain (Knowles and Sternberg [2]) which implies the possibility of failure by shear localization (shear band formation). This is somewhat unexpected since existing constitutive models for vulcanized rubber (e.g. Mooney-Rivlin material) do not admit such a possibility. It should however be noted that this phenomenon is predicted to occur in a Blatz-Ko material at strain levels that are more severe than those tested and thus could possibly be a result of improper extrapolation. In addition, certain deficiencies in the fit of the aforementioned model with Blatz and Ko's own experiments have also been noted [2]. On the other hand, it is conceivable that the porous vulcanized rubber does indeed exhibit shear localization in its "macroscopic" behavior, which could, for example, be an averaged manifestation of a loss of stability at the "microstructural" level (e.g. void buckling).

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In the present study the plane strain problem for a rectangular block of a Mooney-Rivlin-like material having a very fine periodic distribution of identical cylindrical voids is considered. We determine the overall constitutive behavior of this porous material theoretically, and then examine the possibility of a loss of ellipticity in the "averaged" material. Despite the obvious differences between the orthotropic two-dimensional model analyzed here and the isotropic three-dimensional material tested by Blatz and Ko certain interesting qualitative similarities have been found.

A number of different techniques for determining the effective constitutive relations for inhomogeneous linearly elastic materials have been developed. (The interested reader is referred to the review articles by Watt et al. [3], Willis [4], and Walpole [5]). Both bounds on the overall material properties as well as direct estimates of them have been obtained. The so-called self-consistent estimates, based in part on Eshelby's strain transformation problem, appear to be the most widely used and are not restricted to problems in which the inhomogeneity is distributed periodically. It has been observed however that these estimates are inappropriate in the limiting case in which one of the constituents consists of voids and its concentration is large (Budiansky [6]) – precisely the case of interest in the present study.

A second method, applicable only in the particular case when the composite has a periodic structure, is the "homogenization theory" which was initiated by Sanchez-Palencia [7] and is described in detail in a monograph by Bensoussan, Lions, and Papanicolaou [8]. The method utilizes a singular perturbation analysis based on the multiple scales involved in the problem, viz. that of the entire body and that of the microstructure, and calculates the limiting material properties as the "size" of the microstructure tends to zero.

The method of homogenization is only applicable to situations governed by linear boundary-value problems. While the phenomenon of interest here is one that occurs at finite strain, it should be kept in mind that to examine the question of overall ellipticity, it is only the averaged incremental moduli of the body that one needs to know. These moduli in turn are associated with the equations of incremental equilibrium, which are in fact linear in the incremental displacements.

Many investigators, mainly in the areas of metal plasticity and geomechanics, have examined the role of voids in shear localization, e.g., [9–15]. It has been observed that the strain level at which most classical plastically incompressible elastic-plastic solids with a smooth yield surface admit shear bands is unrealistically high. Physically motivated modifications to the aforementioned constitutive descriptions have been proposed (e.g., Rudnicki and Rice [9], Gurson [10]) which allow for plastic dilatancy, arising for example as a macroscopic consequence of microvoids. These phenomenological constitutive relations lead to signficantly lower values for the strain at localization and have been utilized to study the development of shear bands, e.g., Yamamoto [11], Iwakuma and Nemat-Nasser [12], Saje, Pan, and Needleman [13], Tvergaard [14], and Ohno and Hutchinson [15].

It is important to emphasize the difference between the present study and the work on elastic-plastic solids referred to in the foregoing. As mentioned previously, a homogeneous block of a classical elastic-plastic solid loses ellipticity at an unrealistically high load level. If one were to consider a block of such material with a void in it, ellipticity would clearly be lost at smaller values of the applied load – the stress concentration effect of the void would generate the necessary conditions locally, near the cavity. Consequently, one can argue that it is merely the stress concentration at the voids that leads to the smaller critical load level in these studies of porous elastic-plastic solids. If one wishes to determine, conclusively, whether the voids play a role that is more fundamental than this, it is essential to consider a porous material whose matrix material itself does *not* lose ellipticity, no matter how large the loads. This is the case considered in the present paper.

In Section 2.1 we outline the principal steps involved in formally deriving the formulas for the homogenized moduli of a periodic solid. The elastostatic problem governing the equilibrium of a Mooney-Rivlin-like material with voids is then formulated in Section 2.2. In Section 3 we discuss the finite element solution of the unit cell problem and also describe the numerical implementation of the homogenization formulas. The results are presented in Section 4.

2 Formulation

The first subsection here deals with the homogenization procedure for an incrementally linear solid with periodic structure while in the second, the elastostatic problem for the porous material is formulated.

2.1 Homogenization Theory for Periodic Structures. The mathematical theory of homogenization for linearly elastic periodic structures has been presented in the applied mathematics literature (and the interested reader is referred to the comprehensive monograph by Bensoussan, Lions, and Papanicolaou [8] and the references cited therein). For reasons of completeness however, a brief review of its principal results is outlined here.

Suppose that the equations of incremental equilibrium for a solid under plane strain conditions, occupying a region $\Omega \subset R^2$ in its undeformed configuration, are given by

$$\frac{\partial}{\partial x_\beta}\left[L^\epsilon{}_{\alpha\beta\gamma\delta}(\mathbf{x})\,\frac{\partial v^\epsilon_\gamma}{\partial x_\delta}(\mathbf{x})\right]+f_\alpha(\mathbf{x})=0,$$
$$\mathbf{x}=(x_1,x_2)\epsilon\Omega. \qquad (2.1)^1$$

Here $\mathbf{v}^\epsilon$ is the incremental displacement vector, $\mathbf{f}$ the body force vector, and $\mathbf{L}^\epsilon$ the (uniformly bounded) incremental modulus tensor of the material which is endowed with the symmetry $L^\epsilon{}_{\alpha\beta\gamma\delta}=L^\epsilon{}_{\gamma\delta\alpha\beta}$. The incremental moduli are assumed to be doubly periodic with periods $2\epsilon a_1$ and $2\epsilon a_2$ with respect to the x_1 and x_2 directions, i.e.,

$$\mathbf{L}^\epsilon(x_1,x_2)=\mathbf{L}^\epsilon(x_1+2\epsilon a_1,x_2)$$
$$=\mathbf{L}^\epsilon(x_1,x_2+2\epsilon a_2),\ \mathbf{x}\epsilon\Omega. \qquad (2.2)$$

The differential equation (2.1), together with certain prescribed boundary conditions on $\partial\Omega$ (which do not concern us here), constitute the governing boundary-value problem for $\mathbf{v}^\epsilon$. Here interest is focused on the behavior of the incremental displacement $\mathbf{v}^\epsilon$ as the periodic structure of the material becomes very fine, i.e., as $\epsilon\to 0$. *The basic result of homogenization theory* asserts that one can construct a set of *constants* $\mathcal{L}_{\alpha\beta\gamma\delta}$ such that $\mathbf{v}^\epsilon\to\mathbf{v}$ (in an appropriate sense) as $\epsilon\to 0$ where $\mathbf{v}$ is the solution of

$$\frac{\partial}{\partial x_\beta}\left[\mathcal{L}_{\alpha\beta\gamma\delta}\,\frac{\partial v_\gamma}{\partial x_\delta}(\mathbf{x})\right]+f_\alpha(\mathbf{x})=0,\mathbf{x}\epsilon\Omega, \qquad (2.3)$$

subject to certain boundary conditions. The coefficients $\mathcal{L}_{\alpha\beta\gamma\delta}$ are termed the "homogenized moduli" of the material and are related to $\mathbf{L}^\epsilon$ (by (2.7) below).

To establish this result and obtain a constructive formula for $\mathcal{L}$ one supposes the $\mathbf{v}^\epsilon(\mathbf{x})$ admits the following asymptotic expansion:

$$\mathbf{v}^\epsilon(\mathbf{x})=\overset{0}{\mathbf{v}}(\mathbf{x},\mathbf{y})+\epsilon\overset{1}{\mathbf{v}}(\mathbf{x},\mathbf{y})+\epsilon^2\overset{2}{\mathbf{v}}(\mathbf{x},\mathbf{y})$$
$$+\ \ldots\ ,\ \mathbf{y}=\mathbf{x}/\epsilon, \qquad (2.4)$$

where $\overset{0}{\mathbf{v}}$, $\overset{1}{\mathbf{v}}$, $\overset{2}{\mathbf{v}}$, . . . are Y periodic[2] functions in $\mathbf{y}$ for each $\mathbf{x}$ in Ω with $Y=[-a_1,a_1]\times[-a_2,a_2]$ being the unit cell of the structure. Assuming that $\mathbf{L}^\epsilon(\mathbf{x})$ depends on ϵ only through $\mathbf{x}/\epsilon$, we may set $\mathbf{L}^\epsilon(\mathbf{x})=\mathbf{L}(\mathbf{x}/\epsilon)$ where according to (2.2), $\mathbf{L}(\mathbf{y})$ is Y periodic. Substituting (2.4) into (2.1) and grouping the terms of like order in ϵ allows one to deduce (from the terms of order ϵ^{-2}, ϵ^{-1} and ϵ^0, respectively,) that

(*a*) $\overset{0}{\mathbf{v}}$ is independent of $\mathbf{y}$, i.e., $\overset{0}{\mathbf{v}}(\mathbf{x},\mathbf{y})\equiv\overset{0}{\mathbf{v}}(\mathbf{x})$,

(*b*) one can represent the function $\overset{1}{\mathbf{v}}$ in the form

$$\overset{1}{\mathbf{v}}(\mathbf{x},\mathbf{y})=\chi^{\gamma\delta}(\mathbf{y})\,\frac{\partial\overset{0}{v}_\gamma}{\partial x_\delta}(\mathbf{x})+\mathbf{c}(\mathbf{x}), \qquad (2.5)$$

where the Y periodic functions $\chi^{\gamma\delta}(\mathbf{y})$ are the unique (up to an additive constant) solutions of

$$\frac{\partial}{\partial y_\beta}\left[L_{\alpha\beta\zeta\eta}(\mathbf{y})\,\frac{\partial\chi_\zeta{}^{\gamma\delta}}{\partial y_\eta}(\mathbf{y})\right]$$
$$=-\frac{\partial}{\partial y_\beta}[L_{\alpha\beta\gamma\delta}(\mathbf{y})],\ \mathbf{y}\epsilon Y, \qquad (2.6)$$

(*c*) a function $\overset{2}{\mathbf{v}}$, consistent with (2.1), (2.4) exists only if (2.3) holds with $\mathbf{v}$ replaced by $\overset{0}{\mathbf{v}}$ and with the constants $\mathcal{L}_{\alpha\beta\gamma\delta}$ given by

$$\mathcal{L}_{\alpha\beta\gamma\delta}=\frac{1}{4a_1a_2}\,Y4\int\left[L_{\lambda\mu\zeta\eta}(\mathbf{y})\,\frac{\partial}{\partial y_\mu}\,(\chi^{\alpha\beta}_\lambda(\mathbf{y})+y_\beta\delta_{\alpha\lambda})\right.$$
$$\left.\frac{\partial}{\partial y_\eta}\,(\chi^{\gamma\delta}_\zeta(\mathbf{y})+y_\delta\delta_{\gamma\zeta})\right]d\mathbf{y}. \qquad (2.7)$$

Therefore in the limit $\epsilon\to 0$ when the structure of the medium becomes very fine, the leading order term of the displacement increment, $\overset{0}{\mathbf{v}}(\mathbf{x})$, obeys the linear differential equation (2.3) with constant coefficients $\mathcal{L}_{\alpha\beta\gamma\delta}$. These coefficients, termed the "homogenized moduli" of the

[1] Greek subscripts range over the integers (1,2) and summation over repeated subscripts is taken for granted.

[2] A function $\mathbf{g}(\mathbf{y})$ is termed Y periodic if it assumes identical values on opposite faces of the boundary of the rectangle Y.

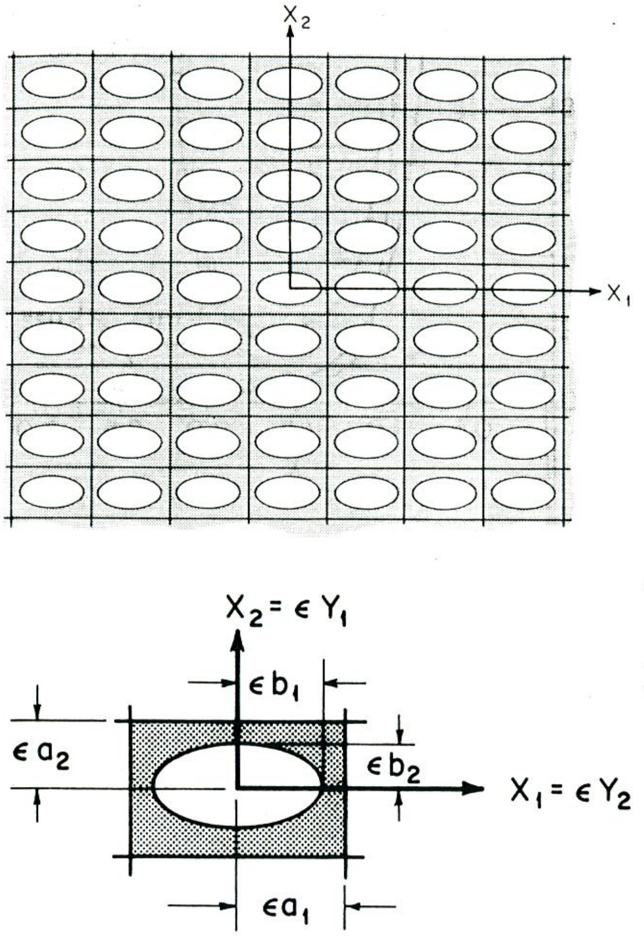

Fig. 1 Cross section of the body and the unit cell

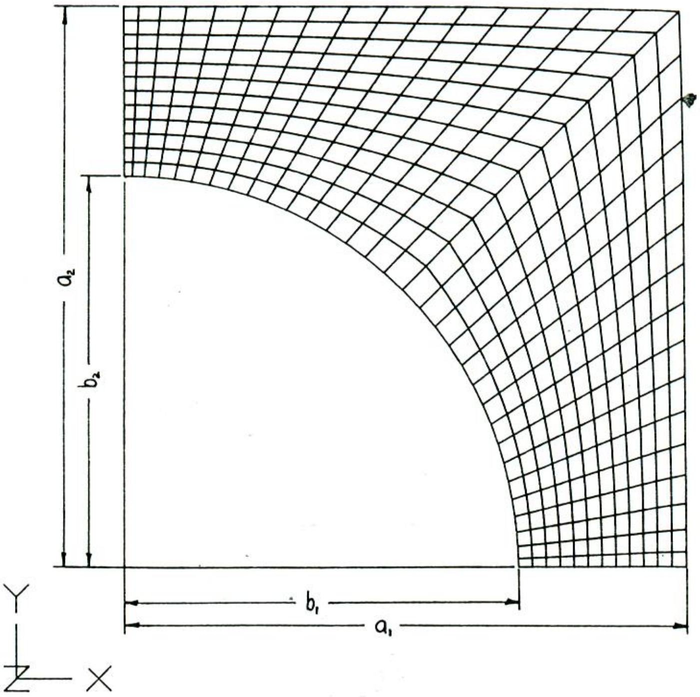

Fig. 2 The finite element grid

material, are given by (2.7) with $\chi^{\alpha\beta}(\mathbf{y})$ being the Y periodic functions satisfying (2.6), or equivalently

$$\int_Y L_{\alpha\beta\gamma\delta} \frac{\partial}{\partial y_\beta}(\chi_\alpha{}^{\zeta\eta} + y_\eta \delta_{\zeta\alpha}) \frac{\partial \phi_\gamma}{\partial y_\delta} dy = 0 \quad \text{for every } Y \text{ periodic function } \phi(\mathbf{y}). \tag{2.8}$$

It should be noted at this point that the preceding results can be rigorously justified (see [8]) under certain restrictive conditions on $\mathbf{L}(\mathbf{y})$. Moreover, it has been shown that when these conditions hold, the homogenized moduli $\mathcal{L}$ are elliptic (see [8], [16]), i.e.,

$$\det[\mathcal{L}_{\alpha\beta\gamma\delta} n_\beta n_\delta] \neq 0 \quad \text{for all unit vectors } \mathbf{n}. \tag{2.9}$$

It is interesting to observe that the boundary conditions of the original problem (i.e. those for $\mathbf{v}^\epsilon$ on $\partial\Omega$) do not affect the calculation of the homogenized moduli.

Finally we turn to the case in which the material contains a periodic array of identical holes so that for some $D \subset Y$ one has $\mathbf{L}(\mathbf{y}) = \mathbf{0}$ for $\mathbf{y} \epsilon D$ and $\mathbf{L}(\mathbf{y}) \neq \mathbf{0}$ on $Y - D$. A formal derivation again leads to (2.7), (2.8) with the integrals now being evaluated over $Y - D$ instead of Y. These modified formulas will be utilized in what follows, though, to the best of our knowledge, a rigorous justification of this result does not (as yet) exist. The strong conditions (previously mentioned) used in [8] for this justification do not hold in the presence of voids. Further, one cannot ensure a priori that the homogenized moduli $\mathcal{L}$ are elliptic. Indeed, the numerical results obtained here indicate that they are not always so.

2.2 Formulation of Problem. Incremental Moduli. Consider a cylindrical block of material whose cross section in its undeformed state is rectangular with dimensions $2A_1 \times 2A_2$. The block contains a periodic array of identical cylindrical cavities of elliptical cross section as shown in Fig. 1. The centers of the ellipses are spaced $2\epsilon a_1$ (and $2\epsilon a_2$) apart in the x_1 (and x_2) directions, and the semimajor and semiminor axes of each ellipse are ϵb_1 and ϵb_2, respectively. In what follows, a Lagrangian formulation of the problem in Cartesian coordinates is adopted.

The lateral boundaries of this block are free of shear tractions and are subjected to prescribed stretches $\lambda_1, \lambda_2 (>0)$. Thus if $\mathbf{x}$ denotes the position vector of a particle in the undeformed body, the displacement field[3] $\mathbf{u}^\epsilon(\mathbf{x})$ and the first Piola-Kirchhoff stress field $\boldsymbol{\sigma}^\epsilon(\mathbf{x})$ obey

$$\begin{aligned} u_1{}^\epsilon(\mathbf{x}) &= (\lambda_1 - 1)x_1, \quad \sigma_{21}{}^\epsilon(\mathbf{x}) = 0 \quad \text{on} \quad x_1 = \pm A_1, \\ u_2{}^\epsilon(\mathbf{x}) &= (\lambda_2 - 1)x_2, \quad \sigma_{12}{}^\epsilon(\mathbf{x}) = 0 \quad \text{on} \quad x_2 = \pm A_2. \end{aligned} \tag{2.10}$$

If $W(\mathbf{F})$ denotes the strain energy density of the material, we have $\sigma^\epsilon{}_{\alpha\beta}(\mathbf{x}) = \partial W(\mathbf{F}^\epsilon(\mathbf{x}))/F_{\alpha\beta}$ and the equilibrium equations in the absence of body forces can be written in the form

$$\frac{\partial}{\partial x_\beta}\left[\frac{\partial W}{\partial F_{\alpha\beta}}(\mathbf{F}^\epsilon(\mathbf{x}))\right] = 0 \quad \text{with} \quad F^\epsilon{}_{\alpha\beta}(\mathbf{x}) = \delta_{\alpha\beta} + \frac{\partial u^\epsilon{}_\alpha}{\partial x_\beta}(\mathbf{x}), \tag{2.11}$$

Lastly, since the cavities are free of traction we must have

$$\frac{\partial W}{\partial F_{\alpha\beta}}(\mathbf{F}^\epsilon(\mathbf{x})) \quad N_\beta = 0 \tag{2.12}$$

at every point on the boundaries of the cavities.

On taking increments in (2.11) (i.e., by linearizing (2.11) about $\mathbf{u}^\epsilon(\mathbf{x})$) one obtains

$$\frac{\partial}{\partial x_\beta}\left[L^\epsilon_{\alpha\beta\gamma\delta}(\mathbf{x}) \frac{\partial v^\epsilon_\gamma}{\partial x_\delta}(\mathbf{x})\right] = 0. \tag{2.13}$$

Similarly (2.12) yeilds $L^\epsilon{}_{\alpha\beta\gamma\delta} \partial v_\gamma{}^\epsilon / \partial x_\delta N_\beta = 0$ on the boundary of the cavities. Here we have denoted the increment of $\mathbf{u}^\epsilon$ by $\mathbf{v}^\epsilon$ and the incremental moduli $\mathbf{L}^\epsilon(\mathbf{x})$ are given by

[3] The problem is presumed to be one of plane strain.

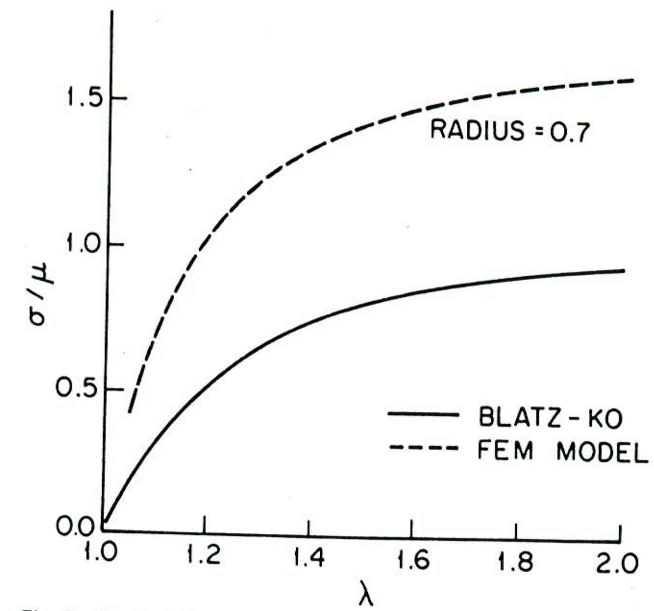

Fig. 3 **Stress-stretch response in plane-strain equibiaxial stretching (homogenized and Blatz-Ko materials)**

$$L^{\epsilon}{}_{\alpha\beta\gamma\delta}(\mathbf{x}) = \frac{\partial^2 W}{\partial F_{\alpha\beta}\partial F_{\gamma\delta}}\,(\mathbf{F}^{\epsilon}(\mathbf{x})). \tag{2.14}$$

In view of the nature of the geometry and loading in the present problem, it is evident that the deformation gradient $\mathbf{F}^{\epsilon}(\mathbf{x})$ depends on ϵ through $\mathbf{x}/\epsilon$ alone, i.e., $\mathbf{F}^{\epsilon}(\mathbf{x}) = \mathbf{F}(\mathbf{x}/\epsilon)$, so that we also have $\mathbf{L}^{\epsilon}(\mathbf{x}) = \mathbf{L}(\mathbf{x}/\epsilon)$. Moreover $\mathbf{L}^{\epsilon}(\mathbf{x})$ clearly satisfies the periodicity condition (2.2). Consequently the mathematical problem governing the displacement increment $\mathbf{v}^{\epsilon}(\mathbf{x})$ here is identical to that stated in Section 2.1.

The overall incremental moduli $\mathcal{L}$ (λ_1, λ_2) of the body may now be calculated from (2.7) after first solving the boundary value problem (2.10)–(2.12) for $\mathbf{u}^{\epsilon}(\mathbf{x})$, then determining $\mathbf{F}^{\epsilon}(\mathbf{x})$ and $\mathbf{L}^{\epsilon}(\mathbf{x})$ from $(2.11)_2$ and (2.14), and finally solving[4] (2.8) for the periodic functions $\chi^{\alpha\beta}$. In what follows, we carry out these calculations in the case when the (matrix) material is isotropic and is characterized by the plane-strain elastic potential

$$W(I,J) = \frac{\mu}{2}\left[(I-2) - 2(J-1) + \frac{1}{1-2\nu}\,(J-1)^2\right], \tag{2.15}$$

where I = trace $(\mathbf{F}\mathbf{F}^T)$, $J = \det \mathbf{F}$. This particular choice for W was made for three reasons. First, in the small strain limit the preceding material reduces to an isotropic linearly elastic one with shear modulus μ and Poisson's ratio ν. Second, for values of ν approaching 0.5 the preceding material presumably approaches[5] a Mooney-Rivlin material which is a suitable model for a class of rubbers under finite strain. Finally, and most importantly, for all values of $\mu > 0$ and $0 < \nu < 0.5$ the material characterized by (2.15) is strongly elliptic in all plane deformations as may be readily verified from (2.15) and equations (2.41) and (2.42) of Knowles and Sternberg [18]. The Blatz-Ko material, with which we will compare the homogenized model, is characterized by the plane strain elastic potential

$$W(I,J) = \frac{\mu}{2}\left(\frac{I+1}{J^2} + 2J - 5\right). \tag{2.16}$$

[4] As mentioned previously, the integrals in (2.7) and (2.8) are taken over $Y-D$. Moreover, note that the boundary conditions for $\chi^{\alpha\beta}$ on ∂D are natural boundary conditions in (2.8).

[5] A rigorous proof of such a claim can be given under certain conditions (not satisfied by (2.15)), e.g., [17]. It is considerably more convenient for numerical reasons, to consider a nearly imcompressible material rather than an incompressible one.

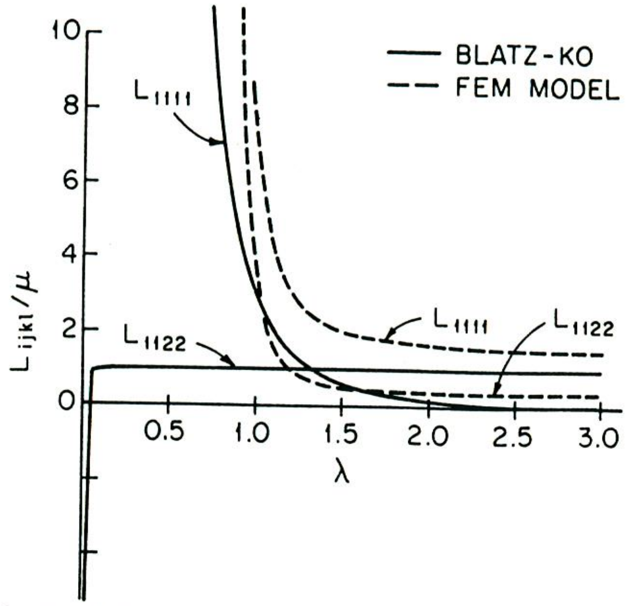

Fig. 4(a) **Variation of the normal incremental moduli in plane-strain equibiaxial stretching (homogenized and Blatz-Ko materials)**

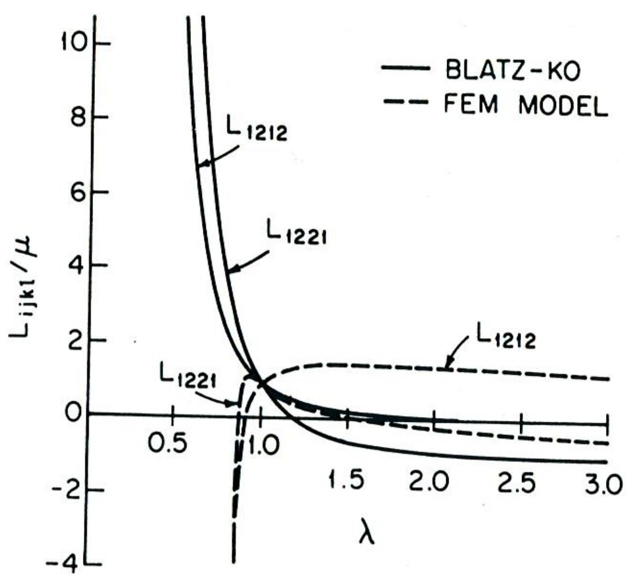

Fig. 4(b) **Variation of the shear incremental moduli in plane-strain equibiaxial stretching (homogenized and Blatz-Ko materials)**

Fig. 4

3 Numerical Method

As seen from the preceding analysis, the computation of the homogenized incremental moduli $\mathcal{L}$ at a given state of deformation requires the fundamental solution of the nonlinear boundary value problem (2.10)–(2.12) for the unit cell Y. One can then calculate the corresponding incremental moduli $\mathbf{L}$ (**y**) based on this solution and subsequently use this information to solve the linear problem (2.8) for $\chi_{\gamma}^{\alpha\beta} + y_{\beta}\delta_{\alpha\gamma}$. These solutions can be obtained most efficiently, in the present problem, by using the finite element method.

The unit cell Y, a rectangle of (undeformed) dimensions $2a_1 \times 2a_2$, has a concentric elliptical hole of major and minor semiaxes b_1 and b_2, respectively. In view of the geometric and loading symmetries involved in this problem, only one quarter of Y need be analyzed. The finite element grid (see Fig. 2)

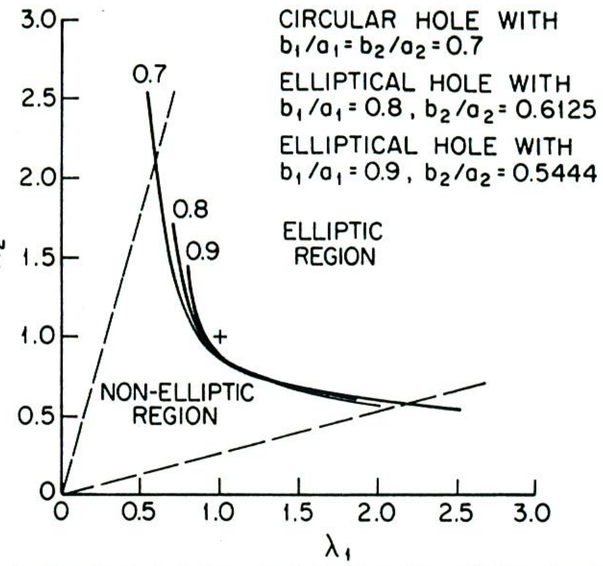

Fig. 5 Domain of ellipticity on the ($\lambda_1 \lambda_2$)-plane. The solid lines denote the boundary of the elliptic region for the homogenized material; the dashed lines correspond to the Blatz-Ko material.

consists of isoparametric bilinear quadrilaterals. Numerical integration over each element was based on a 2×2 Gauss–Legendre scheme. Most of the calculations here were based on a 10 (in the radial direction) by 20 (in the circumferential direction) grid. The mesh was properly refined at the end sections $y_1 = 0$ and $y_2 = 0$ in view of the large strain concentrations occurring at these locations. The adequacy of the mesh was decided on the basis that any further refinement (meshes of up to 18×36 elements have been investigated) should yield displacements no more different than 0.1 percent and homogenized incremental moduli different by less than 1 percent. Finally, the matrix material was taken to be approximately incompressible by using the value $\nu = 0.49$ for the Poisson ratio in (2.15).

An incremental Newton-Raphson method (see [19] for further details) was employed to determine the fundamental solution for the unit cell Y. The calculations were carried out along different "radial" paths through the undeformed state on the (λ_1, λ_2)-plane, i.e., along lines $(\lambda_2 - 1)/(\lambda_1 - 1)$ = constant. Once convergence has been achieved at a given load level, the corresponding discretized incremental stiffness matrix, say $[K]$, could be used to immediately yield the nodal values of $\chi_\gamma^{\alpha\beta}(\mathbf{y}) + y_\beta \, \delta_{\alpha\gamma}$, say $[d^{\alpha\beta}]$ (see (2.8)). Note from (2.7) that the corresponding values of the homogenized incremental moduli $\mathcal{L}$ are simply calculated by the following scalar product:

$$\mathcal{L}_{\alpha\beta\gamma\delta} = [d^{\alpha\beta}]^T [K] [d^{\gamma\delta}]. \tag{3.1}$$

The ellipticity of $\mathcal{L}$ (given by (3.1)) was examined by a direct numerical check of condition (2.9) at every $\pi/720$ radian increment of ϕ where $n_1 = \cos\phi$, $n_2 = \sin\phi$.

4 Results

The selection of the hole size in our calculations was dictated by the requirement of a 50 percent porosity – the same as that observed in the specimens used by Blatz and Ko [1]. Three different void shapes are considered here; a circular one with $b_1/a_1 = b_2/a_2 = 0.7$, an elliptical one with $b_1/a_1 = 0.8$, $b_2/a_2 = 0.6125$, and a second elliptical one with $b_1/a_1 = 0.9$, $b_2/a_2 = 0.5444$. The unit cell was taken to be a square, $a_1 = a_2$.

As mentioned previously, the model used in this work has an overall response that is orthotropic and is thus a somewhat poor approximation to the (isotropic) Blatz-Ko material (2.16). To minimize the influence of this difference, we first carried out a series of calculations for a body with circular cavities under conditions of equibiaxial stretch $\lambda_1 = \lambda_2$. The effective principal true stresses ($\sigma_1 = \sigma_2 = \sigma$) as well as the homogenized incremental moduli $\mathcal{L}_{\alpha\beta\gamma\delta}$ were computed and compared with the corresponding quantities for the Blatz-Ko material.

In Fig. 3 we compare the principal Cauchy stress $\sigma_1 = \sigma_2 = \sigma$ of the Blatz-Ko material with the corresponding average principal stress (defined as the average of the normal Cauchy stress over the side $y_1 = a_1$ or $y_2 = a_2$ of the unit cell) both being evaluated at the same stretch ratio $\lambda_1 = \lambda_2 = \lambda$. The infinitesimal shear modulus of the Blatz-Ko material was chosen to be the same as that of the homogenized material and the results shown are normalized with respect to this common shear modulus μ. It may be observed that the two curves show the same trends, and roughly appear to be scaled versions of each other.

A similar comparison of the homogenized and Blatz-Ko incremental moduli in equibiaxial stretching is made in Figs. 4(*a*) and 4(*b*). These calculations were carried out for circular voids with radius $b_1/a_1 = 0.7$. The qualitative agreement is reasonably good in the case of extension. On the other hand, in the case of contraction, we observe large differences, even to the point of predicting negative incremental shear moduli for the homogenized model. This may be attributed to the fact that the constitutive model for the matrix material (2.15) utilized here is unrealistic at high compressive strains; note for instance that the strain energy density W (see (2.15)) remains bounded as $J \to 0$ for fixed values of the Poisson's ratio ν.

Finally attention is turned to the results pertaining to a loss of ellipticity for the homogenized material and Fig. 5 displays the corresponding elliptic and nonelliptic domains on the (λ_1, λ_2)-plane. The points associated with the onset of a failure of ellipticity lie on a curve which looks roughly like a hyperbola. The corresponding curves (straight lines, in this case) for a Blatz-Ko material are also shown on this figure. One can observe that the homogenized material loses ellipticity after the Blatz-Ko material in extensional deformations while the converse is true in compression. This is not entirely unexpected since, as can be readily verified from (2.15) and (2.16), the response of the matrix material is much stiffer (and softer) than that of the Blatz-Ko material in tension (and compression, respectively).

To estimate the sensitivity of the aforementioned results to the void shape, the preceding calculations were repeated for the case when the voids had an elliptical cross-section while keeping the void ratio constant at 50 percent as in the previous case. The results, for two different elliptical cross-sections, are also shown in Fig. 5. The influence of the void shape appears to be small, and as one might expect (in view of the stress concentration effects) the homogenized material based on elliptical cavities with $b_1 > b_2$, is stiffer than that corresponding to circular cavities in the case when $\lambda_2 < \lambda_1$. The opposite is true when $\lambda_2 > \lambda_1$.

5 Concluding Remarks

This work is not aimed at providing a micromechanical justification for a particular constitutive model but rather is concerned with certain aspects of the influence of voids on the stability properties of an elastic material. The most significant feature of the results of this investigation is that although the matrix material used is stable in the sense that it remains elliptic at all deformations, the resulting homogenized medium (obtained through a formal adaptation of the method of homogenization) exhibits a loss of ellipticity at adequately high strain levels. One possible mechanism that explains this macroscopic instability is that it is a global manifestation of void buckling in a banded pattern at the microstructural level.

Acknowledgments

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Worldwide Mechanics Meetings

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ITEMS FOR THIS CALENDAR SHOULD BE SENT TO PROF. MICHAEL W, HYER, DEPT. OF ENGINEERING SCIENCE AND MECHANICS, VIRGINIA TECH, BLACKSBURG, VA 24061
ITEMS SHOULD INCLUDE: TITLE, DATE, LOCATION OF MEETING; NAME, ADDRESS, AND PHONE NUMBER OF CONTACT; SPONSORING ORGANIZATION; AND ABSTRACT DEADLINE.

DATE : SEPT. 3-7, 1984 LOCATION: BIRMINGHAM, UK ABSTRACT: PAST DUE
TITLE: 2ND INTERNATIONAL CONFERENCE ON FATIGUE AND FATIGUE THRESHOLDS
INFO : C J BEEVERS, METALLURGY AND MATERIALS, UNIVERSITY OF BIRMINGHAM,
BIRMINGHAM B15 2TT, UK

DATE : SEPT. 5-7, 1984 LOCATION: MINNEAPOLIS, MN(USA) ABSTRACT: PAST DUE
TITLE: INTERNATIONAL SYMPOSIUM ON DYNAMIC SOIL-STRUCTURE INTERACTION
INFO : INT SYMP, CIVIL & MINERAL ENGINEERING, UNIV OF MINNESOTA,
500 PILLSBURY DR SE, MINNEAPOLIS, MN 55455

DATE : SEPT. 10-13, 1984 LOCATION: STUTTGART, W. GER. ABSTRACT: PAST DUE
TITLE: 3RD INT CONF ON FINITE ELEMENTS IN NONLINEAR MECHANICS
INFO : FENOMECH 84, ISD, UNIV. OF STUTTGART, PFAFFENWALDRING, 27
D-7000 STUTTGART 80, WEST GERMANY

DATE : SEPT. 11-14, 1984 LOCATION: CACHAN, FRANCE ABSTRACT: WRITE INFO
TITLE: EUROMECH 184, THE INCLUSION OF LOCAL EFFECTS IN THE ANAL OF STRUCTURES
INFO : PROF. P. LADEVEZE, UNIVERSITE PARIS VI/C.N.R.S. 61,
AVENUE DU PRESIDENT WILSON, 94230 CACHAN, FRANCE

DATE : SEPT. 12-14, 1984 LOCATION: NANCY, FRANCE ABSTRACT: WRITE INFO
TITLE: EUROMECH 186, RHEOLOGY OF BIOLOGICAL FLUIDS
INFO : PROF. M. LUCIUS, INSTITUT NATIONAL POLYTECHNIQUE DE LORRAINE,
B.P. 3308, PORTE DE LA CRAFFE, 54014 NANCY CEDEX, FRANCE

DATE : SEPT. 16-23, 1984 LOCATION: VARNA, BULGARIA ABSTRACT: PAST DUE
TITLE: 10TH INTERNATIONAL CONFERENCE ON NONLINEAR OSCILLATIONS
INFO : G. BRANKOV, INST MECHANICS & BIOMECHANICS, ACAD G BONCHEV STR,
BL 8, 1113 SOFIA, BULGARIA

DATE : SEPT. 17-21, 1984 LOCATION: PRAGUE, CZ ABSTRACT: NO INFO
TITLE: IUTAM SYMP ON OPTICAL METHODS IN DYNAMICS OF FLUIDS AND SOLIDS
INFO : MIROSLAV PICHAL(PROFESSOR ING.) CZECHOSLOVAK ACADEMY OF SCIENCES,
PUSKINOVO N.9, CS 160 00 PRAGUE 6 CZECHOSLOVAKIA

DATE : SEPT. 17-21, 1984 LOCATION: LONDON, ENGLAND ABSTRACT: WRITE INFO
TITLE: EUROMECH 185, MATH PROGRAMMING METHODS FOR PLASTIC ANAL. OF STRUCTURES
INFO : PROF. J. MUNRO, CIVIL ENG. DEPT., IMPERIAL COLLEGE, LONDON SW7,UK

DATE : SEPT 17-21, 1984 LOCATION: INTERLAKEN, SWITZ ABSTRACT: PAST DUE
TITLE: 4TH WORLD CONGRESS AND EXHIBITION ON FINITE ELEMENT METHODS
INFO : J ROBINSON, ROBINSON AND ASSOCIATES, HORTON RD, WOODLANDS, WIMBORNE
DORSET BH21 6NB, ENGLAND

DATE : SEPT. 24-27, 1984 LOCATION: VILLETANEUSE, FRANCE ABSTRACT: WRITE INFO
TITLE: EUROMECH 183, PLASTICITY OF CRYSTALLINE MEDIA
INFO : DR. A. ZAOUI, C.N.R.S. UNIVERSITE PARIS-NORD, AVENUE J.B. CLEMENT,
93430 VILLETANEUSE, FRANCE

DATE : SEPT. 24-26, 1984 LOCATION: DAVOS, SWITZ. ABSTRACT: NO INFO
TITLE: 4TH MEETING OF THE EUROPEAN SOCIETY OF BIOMECHANICS
INFO : V GERET, LAB FOR EXPERIMENTAL SURGERY, CH-7270 DAVOS-PLATZ,
SWITZERLAND

DATE : SEPT. 24-26, 1984 LOCATION: LEEDS, U.K. ABSTRACT: WRITE INFO
TITLE: EUROMECH 188, FLUID LOADING AND FLUID-STRUCTURE INTERACTION
INFO : PROF. D.G. CRIGHTON, DEPT. OF APPLIED MATHEMATICAL STUDIES,
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DATE : OCTOBER 1-3, 1984 LOCATION: ROLLA, MO(USA) ABSTRACT: PAST DUE
TITLE: 9TH BIENNIAL SYMPOSIUM ON TURBULENCE
INFO : X. B. REED JR., CHEMICAL ENGINEERING, UNIV. MISSOURI-ROLLA
ROLLA, MO 65401

DATE : OCTOBER 1-4, 1984 LOCATION: HAMBURG-HARBURG, GR ABSTRACT: WRITE INFO
TITLE: EUROMECH 190, DYNAMICAL STABILITY OF INELASTIC STRUCTURES
INFO : PROF. O. MAHRENHOLTZ, TU HAMBURG-HARBURG, EISSENDORFER STRASSE 38,
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DATE : OCTOBER 3-5, 1984 LOCATION: LOS ANGELES, CA(USA) ABSTRACT: NO INFO
TITLE: 8TH ANNUAL MEETING OF THE AMERICAN SOCIETY OF BIOMECHANICS
INFO : R.F. ZERNICKE, DEPT. OF KINESIOLOGY, 2864 SLICHER HALL, UCLA,
LOS ANGELES, CA 90024

DATE : OCTOBER 3-5, 1984 LOCATION: PORTO ALEGRE, BRAZIL ABSTRACT: NO INFO
TITLE: 2ND INT CONF ON STRUCT ANALYSIS AND DESIGN OF NUCLEAR POWER PLANTS
INFO : G J CREUS, EGENH CIVIL, UNIV FED RIO GRANDE DO SUL,
AV OSWALDO ARANHA, 99-3 ANDAR, 90000 PORTO ALEGRE RS, BRAZIL

DATE : OCTOBER 8-13, 1984 LOCATION: ACAPULCO, MEXICO ABSTRACT: NO INFO
TITLE: 9TH INTERNATIONAL CONGRESS ON RHEOLOGY
INFO : B. MENA, INST INVESTIGATIONES EN MATERIALS, APDO POST 70-360,
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DATE : OCTOBER 11-12, 1984 LOCATION: TORINA, ITALY ABSTRACT: NO INFO
TITLE: PROGRESS IN THE FORMULATION OF STRUCTURAL ANALYSIS..SINCE CASTIGLIANO
INFO : DIPARTIMENTO DI INGEGNARIA STRUTTALE, POLITECNICO DI TORINO,
CORSO DUCA DEGLI ABRUZZI 24-10129 TORINO, ITALY

DATE : OCTOBER 15-17, 1984 LOCATION: BLACKSBURG, VA(USA) ABSTRACT: PAST DUE
TITLE: 21ST ANNUAL MEETING OF SOCIETY OF ENGINEERING SCIENCE
INFO : DR. DANIEL FREDERICK, DEPT. OF ENGINEERING SCIENCE AND MECHANICS
SPONS: VIRGINIA TECH BLACKSBURG, VA 24061-4899 PHONE 703-961-6651

DATE : OCTOBER 15-17, 1984 LOCATION: GOTTINGEN, GERMANY ABSTRACT: WRITE INFO
TITLE: EUROMECH 187, ADAPTIVE WALL WIND TUNNELS AND WALL- INTERFERENCE METHODS
INFO : PROF. H. HORNUNG, INSTITUT FUR EXP. STROMUNGSMECHANIK, DFVLR,
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