STABILITY OF SOLIDS:
FROM STRUCTURES TO MATERIALS

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FOREWORD

Stability is a fascinating topic in solid mechanics that has its roots in the celebrated Euler column buckling problem, which first appeared in 1744. Over the years advances in technology have led to the study of ever more complicated structures first in civil and subsequently in mechanical engineering applications. Aerospace applications, most notably failure of solid propellant rockets, led the way in the 1950s. Problems associated with materials and electronics industries came on stage in the 1970s and 1980s, starting with instabilities associated with thin films and phase transformations in shape memory alloys (SMA's), just to name some of the most preeminent examples. In a parallel path, starting in the late 19th century, mathematicians studying nonlinear differential equations, developed the concept of a bifurcation (term coined by Poincare) and created powerful techniques to study the associated singularities. They have also recognized the close association between bifurcation and symmetry in structures. It was for Koiter, beginning with his famous thesis in 1945, to connect the two communities.

Amazing progress has been made since the early days of structural buckling problems and continues to be made in this field, with applications ranging from atomistic to geological scales. With the advent of new materials, the number of applications in this area continues to progress with an ever increasing pace, making it a challenge to present a first course in this topic within the short time available in one semester. The notes that follow are the first attempt to present a comprehensive, modern introduction to the subject of stability of solids. Given the time constraints, only equilibrium configurations of conservative systems will be considered here. These notes start with the introduction of the concepts of stability and bifurcation for conservative elastic systems through finite degree of freedom examples. They continue with the general theory of Lyapunov-Schmidt-Koiter (LSK) asymptotics, followed by examples from continuum mechanics. The presentation subsequently addresses the issue of scale in the stability of solids. In particular the relation between instability at the microstructural level and macroscopic properties of the solid is studied for several types of applications involving different scales: composites (fiber and particle-reinforced), cellular solids and finally SMA's, where temperature- or stress-induced instabilities at the atomic level have macroscopic manifestations visible to the naked eye.

These notes are intended as a complement to lectures given in class. A first draft of these notes was started a long time ago, during my sabbatical leave at the Ecole Polytechnique in 1987. As a researcher in this field, I have learned a lot during the years and for countless times I kept adding, subtracting and modifying. Since better is the worst enemy of good, no comprehensive set of notes has ever been compiled for more than twenty years. Since I recently joined the faculty of the Ecole and wanted to give a new course in this fascinating subject, I finally had to write a set of course notes. I expect that with kind help these notes will soon be completed and relieved from the many mistakes that this first draft inevitably contains (the unbounded kindness of the reader for these deficiencies is greatly appreciated). However, and much more importantly, I hope that this course transmits to the students my enthusiasm about this exciting and vibrant field of solid mechanics.

Nicolas Triantafyllidis, Paris FRANCE, December 2010.

Chapter A

STABILITY AND BIFURCATION - EXAMPLES AND THEORY

Of interest in this chapter is the development of a general theory for the bifurcation and stability of solids. The issues of bifurcation in the equilibrium solutions of nonlinear solids and their stability are closely linked. In addition the stability of the equilibrium solutions near bifurcation points are of great importance to engineering applications. In the first two sections the notions of stability and bifurcation will be introduced with the help of simple finite d.o.f. examples. In the third section the general theory for the bifurcation and stability of the equilibrium solutions of nonlinear elastic systems (discrete or continuum) will be presented.

AA STABILITY OF EQUILIBRIA: DEFINITIONS AND EXAMPLES

The first issue to be addressed in this section is the stability of equilibrium solutions in discrete, nonlinear systems. Following the definition of stability, the two most useful general methods for stability analysis, i.e. the linearization method and Lyapunov's direct method, will be presented along with some simple examples. In addition the case of conservative systems with finite degrees of freedom will be discussed and the energy criterion will be introduced.

AA-1 STABILITY OF AN EQUILIBRIUM - DEFINITIONS

Consider a mechanical system defined by a finite set of real numbers $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ [1] where $\mathbf{p} \in \mathbb{R}^n$. The motion of the system is described by $\mathbf{p}(t)$ and is governed by a set of evolution equations, with respect to time t, of the form:

$$\dot{\mathbf{p}} = \mathbf{f}(\mathbf{p}, t); \quad \text{in component form:} \quad \dot{p}_i = f_i(\mathbf{p}, t), \; i = 1, \cdots, n. \tag{AA-1.1}$$

Here a superimposed dot $\dot{(\ )} \equiv d(\ )/dt$ denotes the differentiation with respect to time t. In addition to the evolution equations, one has also to supply the initial conditions at $t = 0$

$$\mathbf{p}(0) \equiv \mathbf{p}_0. \tag{AA-1.2}$$

By definition, the system is said to be in equilibrium at $\mathbf{p}_e$ if the constant (independent of time) vector $\mathbf{p}_e$ satisfies the evolution equation Eq. (AA-1.1) for all times t, i.e. $\mathbf{f}(\mathbf{p}_e, t) = \mathbf{0}$.

Of interest is the notion of stability of an equilibrium solution. Intuitively, an equilibrium solution is stable if small initial perturbations will generate only small perturbed motions away from it which will remain small for all time. A classical illustration of this concept is given in Fig. AA-1.1, where a ball, under the influence of gravity, is at equilibrium at the bottom of a well or the top of a ridge. The stable equilibrium corresponds to the ball sitting at the bottom of the well, since a small deviation from this position induces small amplitude oscillations about equilibrium. The unstable equilibrium corresponds to the ball resting at the top of the ridge, since a small deviation from the equilibrium position leads to the ball's falling away from the top.

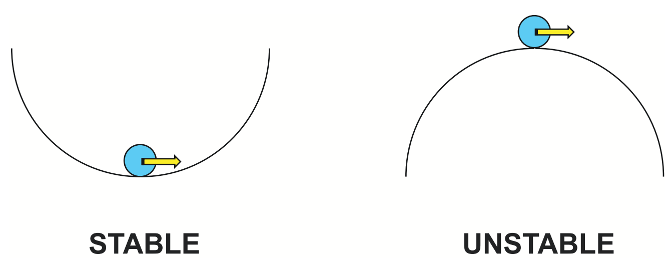

Figure AA-1.1: Intuitive explanation of stability.

In mathematical terms, the stability definition can be stated as follows: Given any small number $\varepsilon > 0$, there exists a number $\eta(\varepsilon) > 0$ such that if the initial conditions are within η from equilibrium, the subsequent motion for all time $t \geq 0$ is within ε from equilibrium

$$\forall \varepsilon > 0 \; \exists \eta(\varepsilon) > 0 \text{ such that:} \quad \| \mathbf{p}(0) - \mathbf{p}_e \| \leq \eta(\varepsilon) \quad \Longrightarrow \quad \| \mathbf{p}(t) - \mathbf{p}_e \| \leq \varepsilon \tag{AA-1.3}$$

where $\| \cdot \|$ denotes the usual Euclidean norm. For the discrete case considered here the norm choice is inconsequential since all norms are equivalent to the Euclidean norm in finite

[1] NOTE Here and subsequently bold symbols are used for vectors or tensors and script symbols for scalars.

dimensional spaces. From the above definition of stability, it is also clear that stability is a local property, since one deals only with small initial perturbations from the equilibrium state in consideration.

A stronger concept of stability, i.e. one that implies Eq. (AA-1.3), is that of an asymptotically stable equilibrium, which states that for an adequately small initial perturbation, the perturbed solution tends, for an adequately large time t, to the equilibrium state

$$\exists \eta > 0 \text{ such that}: \quad \| \mathbf{p}(0) - \mathbf{p}_e \| < \eta \quad \Longrightarrow \quad \lim_{t \longrightarrow \infty} \mathbf{p}(t) = \mathbf{p}_e. \tag{AA-1.4}$$

AA-2 LINEARIZATION METHOD OF STABILITY

According to this general method, the governing equations Eq. (AA-1.1) are linearized near the equilibrium solution $\mathbf{p}_e$. The analysis of the resulting linearized system is the basis for the study of its stability. The method proceeds in three steps:

- i) Linearization of the evolution equations about $\mathbf{p}_e$.
- ii) Stability analysis of the linearized perturbed motions.
- iii) Justification of the results with respect to the actual motion of the initial system.

In the interest of simplicity, it will be assumed here that the the system's motion does not depend explicitly on time, i.e. $\partial\mathbf{f}/\partial t = 0$.

- i) The Linearization of Eq. (AA-1.1) gives

$$\Delta\dot{\mathbf{p}} = \mathbf{A}\bullet\Delta\mathbf{p}; \quad \Delta\mathbf{p} \equiv \mathbf{p}(t) - \mathbf{p}_e, \quad \mathbf{A} \equiv (\partial\mathbf{f}/\partial\mathbf{p})_{\mathbf{p}_e}, \tag{AA-2.1}$$

where $\bullet$ denotes the contraction of $\partial\mathbf{f}/\partial\mathbf{p}$ and $\Delta\mathbf{p}$, i.e. $(\mathbf{A}\bullet\Delta\mathbf{p})_i = \sum_{j=1}^{n}(\partial f_i/\partial p_j)\Delta p_j$. Notice that the matrix $\mathbf{A}$ is constant and recall that $\mathbf{f}(\mathbf{p}_e) = \mathbf{0}$.

- ii) The stability of the linearized system (called autonomous since $\mathbf{A}$ does not depend on time t) depends on the real part of the eigenvalues a_i of $\mathbf{A}$. Lyapunov's theorem states that the linearized system is

a) stable, if $\Re(a_i) \leq 0,\ \forall i\ 1 \leq i \leq n$
b) unstable, if $\exists i,\ \Re(a_i) > 0$

The proof of this statement is straightforward and based on the fact that the solution of the linearized system $\Delta\dot{\mathbf{p}} = \mathbf{A}\bullet\Delta\mathbf{p}$ is given by $\Delta\mathbf{p}(t) = \exp(t\mathbf{A}) \bullet \Delta\mathbf{p}(0)$. Since the solution $\Delta\mathbf{p}(t)$ is a linear combination of the functions $\exp(ta_i)$, the above theorem follows.

- iii) The justification of the stability criterion with respect to the actual motion of the initial system comes next. More specifically we will show that if $\Re(a_i) < 0,\ \forall i\ 1 \leq i \leq n$, the perturbation $\Delta\mathbf{p}(t)$ is asymptotically stable. The proof requires an additional assumption about $\mathbf{g}(\Delta\mathbf{p})$, the remainder from the linear expansion of $\mathbf{f}$ about the equilibrium state $\mathbf{p}_e$, defined in Eq. (AA-2.1). It is assumed that for small $\parallel \Delta\mathbf{p} \parallel$, the remainder grows faster than linearly, i.e.

$$\mathbf{g}(\Delta\mathbf{p}) = \mathrm{o}(\parallel \Delta\mathbf{p} \parallel), \quad \mathbf{g}(\Delta\mathbf{p}) \equiv \mathbf{f}(\mathbf{p}_e + \Delta\mathbf{p}) - \mathbf{A}\bullet\Delta\mathbf{p}, \tag{AA-2.2}$$

and works as follows: From the hypothesis that all eigenvalues of $\mathbf{A}$ have strictly negative real parts, one can find constants $c > 1,\ a > 0$ such that

$$\parallel \exp(t\mathbf{A}) \parallel \leq c\exp(-at), \quad \forall t > 0 \tag{AA-2.3}$$

Also following the hypothesis about the growth of the remainder in Eq. (AA-2.2), one can always find a small positive number $\varepsilon > 0$ such that

$$\parallel \Delta \mathbf{p} \parallel \leq \varepsilon \quad \Longrightarrow \quad \parallel \mathbf{g}(\Delta \mathbf{p}) \parallel \leq \frac{a}{2c} \parallel \Delta \mathbf{p} \parallel \tag{AA-2.4}$$

It will first be shown that

$$\parallel \Delta \mathbf{p}(0) \parallel \leq \frac{\varepsilon}{2c} \quad \Longrightarrow \quad \parallel \Delta \mathbf{p}(t) \parallel < \varepsilon. \tag{AA-2.5}$$

The proof works by contradiction, in which case by time continuity of $\Delta \mathbf{p}(t)$, one could find a time t_ε such that

$$\parallel \Delta \mathbf{p}(t_\varepsilon) \parallel = \varepsilon \quad \parallel \Delta \mathbf{p}(t) \parallel < \varepsilon, \quad 0 \leq t < t_\varepsilon. \tag{AA-2.6}$$

The general expression for the solution of Eq. (AA-2.1) can be shown (by direct substitution) to be

$$\Delta \mathbf{p}(t) = \exp(t\mathbf{A}) \bullet \Delta \mathbf{p}(0) + \int_0^t \exp[(t-s)\mathbf{A}] \bullet \mathbf{g}(\Delta \mathbf{p}(s)) ds \tag{AA-2.7}$$

Taking the norm of both sides of Eq. (AA-2.7) and using of the inequalities in Eq. (AA-2.2) - (AA-2.4) one obtains the following estimate

$$\parallel \Delta \mathbf{p}(t_\varepsilon) \parallel \leq \frac{\varepsilon}{2} \exp(-a t_\varepsilon) + c \int_0^{t_\varepsilon} \exp[-a(t_\varepsilon - s)] \frac{a}{2c} \varepsilon ds = \frac{\varepsilon}{2}, \tag{AA-2.8}$$

which is a contradiction of Eq. (AA-2.6).

Taking norms of both sides of the solution for $\Delta \mathbf{p}(t)$ in Eq. (AA-2.7) and recalling the second inequality in Eq. (AA-2.4) one obtains the following estimate

$$\exp(at) \parallel \Delta \mathbf{p}(t) \parallel \leq \frac{\varepsilon}{2} + \frac{a}{2} \int_0^t \exp(as) \parallel \Delta \mathbf{p}(s) \parallel ds, \quad \forall t > 0. \tag{AA-2.9}$$

The last piece of the proof follows by rewriting the above inequality Eq. (AA-2.9) in terms of an auxiliary function $F(t)$

$$\dot{F}(t) - \frac{a}{2} F(t) \leq \frac{\varepsilon}{2}, \quad \forall t > 0; \quad F(t) \equiv \int_0^t \exp(as) \parallel \Delta \mathbf{p}(s) \parallel ds. \tag{AA-2.10}$$

Multiplying both sides of the above equation Eq. (AA-2.10) by $\exp(-at/2)$ and integrating in the interval $[0, t]$ one obtains the following inequality for $F(t)$

$$F(t) \leq \frac{\varepsilon}{a} [\exp(at/2) - 1], \tag{AA-2.11}$$

which combined with the previous inequality for $F(t)$ in Eq. (AA-2.10) yields

$$\dot{F}(t) \leq \frac{\varepsilon}{2} \exp(at/2). \tag{AA-2.12}$$

By substituting in Eq. (AA-2.12) the definition for $F(t)$ from Eq. (AA-2.10) one finally has

$$\parallel \Delta \mathbf{p}(t) \parallel \leq \frac{\varepsilon}{2} \exp(-at/2), \quad \forall t > 0, \tag{AA-2.13}$$

thus proving the asymptotic stability of the system.

A word of caution: It is always possible to linearize any system. For the case of autonomous systems, it is not difficult to find the eigenvalues of $\mathbf{A}$ and check stability of the linearized perturbations. However, guaranteeing stability for the initial, nonlinear system is not trivial, as we have seen. Without this final step, the stability analysis is incomplete.

AA-3 LYAPUNOV'S DIRECT METHOD OF STABILITY

For certain mechanical systems, and especially the conservative systems or systems with simple dissipative mechanisms (exactly the cases that are of interest in this work) the stability issue can be assessed directly without the employment of a linearization process. The method consists of finding a functional $L(\mathbf{p}(t))$, termed Lyapunov's functional since its independent variable is the function $\mathbf{p}(t)$, and which in general depends on the history of the motion from time $t = 0$ up to the present time and which has the following properties:

- $L(\mathbf{p}(t))$ is non increasing function of t:

$$dL/dt \leq 0. \tag{AA-3.1}$$

- $L(\mathbf{p}(t))$ is a measure of the distance from the equilibrium for each t:

$$L(\mathbf{p}(t)) \geq c \parallel \mathbf{p}(t) - \mathbf{p}_e \parallel^2 . \tag{AA-3.2}$$

- $L(\mathbf{p}(0))$ is a measure of the initial perturbation at $t = 0$:

$$L(\mathbf{p}(0)) \leq d \parallel \mathbf{p}(0) - \mathbf{p}_e \parallel^2, \tag{AA-3.3}$$

where c and d are arbitrary positive constants and $\mathbf{p}(t)$ satisfies the evolution equations Eq. (AA-1.1).

The existence of such a functional ensures the stability of the equilibrium solution $\mathbf{p}_e$. Indeed from Eqs. (AA-3.1) - (AA-3.3) one has:

$$c \parallel \mathbf{p}(t) - \mathbf{p}_e \parallel^2 \leq L(\mathbf{p}(t)) \leq L(\mathbf{p}(0)) \leq d \parallel \mathbf{p}(0) - \mathbf{p}_e \parallel^2, \tag{AA-3.4}$$

which ensures that the inequality $\parallel \mathbf{p}(t) - \mathbf{p}_e \parallel \leq \varepsilon$ is satisfied if η is chosen to be $\eta^2 \leq c\varepsilon^2/d$ thus ensuring stability according to the definition given in Eq. (AA-1.3).

AA-4 FINITE D.O.F. EXAMPLES

Analyzing the finite d.o.f. examples given here will need some elements of Lagrangian mechanics. It will be assumed that the mechanical system in question can be fully described by $m = n/2$ generalized displacement coordinates $\mathbf{q} = (q_1,\ q_2, ...,\ q_m)$, while the system's complete d.o.f. consist of the generalized displacement coordinates $\mathbf{q}$ and their derivatives $\dot{\mathbf{q}}$. This way we are consistent with the notation of subsection AA-1 by noting that $\mathbf{p} \equiv (\mathbf{q}, \dot{\mathbf{q}})$.

It is assumed that the system has a kinetic energy $\mathcal{K}$ and an internal (elastic) energy $\mathcal{E}$. The Lagrangian of the system is defined as their difference, i.e.

$$\mathcal{L} = \mathcal{K} - \mathcal{E}. \tag{AA-4.1}$$

The system's equation of motion takes the form

$$\mathbf{F} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}}; \quad \text{in component form :} \quad F_i = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i},\ i = 1, \cdots, m \tag{AA-4.2}$$

where $F_i(\mathbf{q}, \dot{\mathbf{q}})$[2] are the generalized external forces associated with the generalized coordinate q_i. As mentioned before, the second order system of Eqs. (AA-4.1) - (AA-4.2) is amenable to the general first order form considered in Eq. (AA-1.1) by defining $\mathbf{p} \equiv (\mathbf{q}, \dot{\mathbf{q}})$.

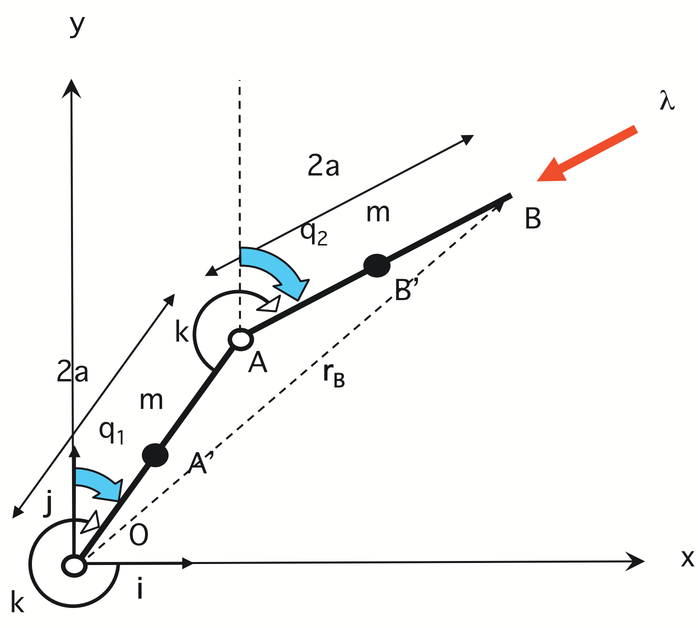

Figure AA-4.1: Two-bar model with follower force.

To illustrate the linearization method of stability for the study of finite degree of freedom mechanical systems, one can consider the mechanism in Fig. AA-4.1. Two rigid bars OA and AB each of length $2a$ have a mass m attached to their respective mid points A' and B'.

[2]NOTE For the case of conservative systems, the external forces are derivable from a potential, which can be added to the internal (elastic) energy. In this case $F_i = 0$ and $\mathcal{E}$ is the system's total (potential) energy

Two identical torsional springs of stiffness k are attached at the frictionless hinges O and A. The system is loaded by a follower force of magnitude λ acting always on the direction of the rigid bar AB. The two degrees of freedom q_1 and q_2 are the angles between OA, AB and the Oy axis respectively.

The system's kinetic energy $\mathcal{K}$ is given by:

$$\mathcal{K} = \frac{1}{2}\ mv^2_{A'} + \frac{1}{2}\ mv^2_{B'} = \frac{1}{2}\ m(\dot{\mathbf{r}}_{A'} \bullet \dot{\mathbf{r}}_{A'} + \dot{\mathbf{r}}_{B'} \bullet \dot{\mathbf{r}}_{B'}) \tag{AA-4.3}$$

where the position vectors of points A' and B' are:

$$\begin{aligned} \mathbf{r}_{A'} &= (a \sin q_1)\mathbf{i} + (a \cos q_1)\mathbf{j} \\ \mathbf{r}_{B'} &= (2a \sin q_1 + a \sin q_2)\mathbf{i} + (2a \cos q_1 + a \cos q_2)\mathbf{j} \end{aligned} \tag{AA-4.4}$$

By using Eq. (AA-4.4) into Eq. (AA-4.3) the total kinetic energy of the system is:

$$\mathcal{K} = ma^2\ [\frac{5}{2}(\dot{q}_1)^2 + \frac{1}{2}(\dot{q}_2)^2 + 2\dot{q}_1\dot{q}_2 \cos(q_1 - q_2)] \tag{AA-4.5}$$

For the calculation of the external forces, one can proceed through the principle of virtual work. Consequently

$$-\lambda\ (\sin q_2\ \mathbf{i} + \cos q_2\ \mathbf{j}) \bullet \delta\mathbf{r}_B = F_1\delta q_1 + F_2\delta q_2 \tag{AA-4.6}$$

Since from geometry the position vector of point B is found to be:

$$\mathbf{r}_B = 2a\ [(\sin\ q_1 + \sin\ q_2)\mathbf{i} + (\cos\ q_1 + \cos\ q_2)\mathbf{j}] \tag{AA-4.7}$$

which combined with Eq. (AA-4.6) gives for the generalized external forces F_1 and F_2:

$$F_1 = 2a\lambda \sin(q_1 - q_2)\ , \quad F_2 = 0 \tag{AA-4.8}$$

The system's internal energy $\mathcal{E}$ is the elastic energy stored in the springs at O and A

$$\mathcal{E} = \frac{1}{2}k(q_1)^2 + \frac{1}{2}k(q_1 - q_2)^2 \tag{AA-4.9}$$

Consequently, by employing Eqs. (AA-4.9), (AA-4.8) and (AA-4.5) into Eq. (AA-4.2) one obtains the following nonlinear equations governing the motion of the mechanism in Fig. AA-4.1:

$$\begin{aligned} &2kq_1 - kq_2 - 2a\lambda \sin(q_1 - q_2) + ma^2[5\ddot{q}_1 + 2\ddot{q}_2 \cos(q_1 - q_2) \\ &-2\dot{q}_2(\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2) + 2\dot{q}_1\dot{q}_2 \sin(q_1 - q_2)] = 0\ , \\ &-kq_1 + kq_2 + ma^2[\ddot{q}_2 + 2\ddot{q}_1 \cos(q_1 - q_2) - 2\dot{q}_1(\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2) \\ &-2\dot{q}_1\dot{q}_2 \sin(q_1 - q_2)] = 0\ . \end{aligned} \tag{AA-4.10}$$

It is not difficult to see that $\mathbf{q}(t) = \mathbf{0} \equiv \mathbf{q}_e$ is an equilibrium solution. To investigate its stability, one obtains from the linearization of Eq. (AA-4.10), i.e. by substituting $\mathbf{q}(t) = \Delta\mathbf{q}(t) + \mathbf{q}_e$ into Eq. (AA-4.10) and keeping only the linear terms in $\Delta\mathbf{q}$

$$\mathbf{M} \bullet \Delta\ddot{\mathbf{q}} + \mathbf{K} \bullet \Delta\mathbf{q} = \mathbf{0}; \quad \mathbf{M} \equiv ma^2 \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{K} \equiv \begin{bmatrix} 2k - 2a\lambda & 2a\lambda - k \\ -k & k \end{bmatrix}. \tag{AA-4.11}$$

In Eq. (AA-4.11), $\mathbf{M}$ is the structure's mass matrix and $\mathbf{K}$ is the structure's stiffness matrix. The solution of the above, constant coefficient linear system is:

$$\Delta\mathbf{q}(t) = \sum_{I=1}^{4} \Delta\mathbf{Q}^I \exp(s_I t), \tag{AA-4.12}$$

where $\Delta\mathbf{Q}^I$ are constants and s_I are the four roots of the characteristic equation which results by substituting Eq. (AA-4.12) into Eq. (AA-4.11), namely

$$\det[\mathbf{M}(s_I)^2 + \mathbf{K}] = 0, \quad \Longrightarrow \quad m^2a^4(s_I)^4 + ma^2(11k - 6a\lambda)(s_I)^2 + k^2 = 0, \; I = 1\cdots 4. \tag{AA-4.13}$$

The study of the four roots of Eq. (AA-4.13) yields the following results:

- If $\lambda \leq 3k/2a$, then $\Re(s_I) = 0,\; (I = 1, 2, 3, 4)$, i.e. all four roots of the biquadratic are purely imaginary. The linearized system is stable.
- If $3k/2a < \lambda < 13k/6a$, then $\Re(s_I) > 0,\; \Im(s_I) \neq 0,\; (I = 1, 2)$, i.e. two of the four roots of the biquadratic have positive real part. Hence the linearized system is unstable. By definition the instability where $\Re(s_I) > 0,\; \Im(s_I) \neq 0$ is called a flutter type instability.
- If $13k/6a \leq \lambda$, then $\Re(s_I) > 0,\; \Im(s_I) = 0,\; (I = 1, 2, 3, 4)$, i.e. all four roots of the biquadratic are real and positive. Hence the linearized system is unstable. By definition the instability where $\Re(s_I) > 0,\; \Im(s_I) = 0$ is called a divergence type instability.

When $0 \leq \lambda < 3k/2a$, any initial perturbation produces a constant amplitude periodic motion in the linearized system, since no dissipation is considered. Consequently the system is not asymptotically stable according to the definition in Eq. (AA-1.4). If a small linear viscous damping is introduced, a viscous term $\mathbf{C} \bullet \Delta\mathbf{q}$ – where $\mathbf{C}$ is a small positive definite viscosity matrix – is added to the linearized perturbation Eq. (AA-4.11). Consequently one can show that for $\lambda \leq 3k/2a$ all roots have negative real parts, i.e. $\Re(s_I) < 0,\; (I = 1, 2, 3, 4)$ and hence that the more realistic, linearized viscous system is also asymptotically stable.

Some final remarks are in order for the mechanism in Fig. AA-4.1. In view of its follower force loading, a potential energy does not exist, i.e. external forces cannot be derived from a potential, and the only equilibrium solution is the trivial one $q_1(t) = q_2(t) = 0$.

The second example to be studied is one that does have a potential energy. For this a small change of the external loading of the mechanism in Fig. AA-4.1 is considered, namely a load λ acting in the **-j** direction as shown in Fig. AA-4.2. The only change in the model pertains to the generalized external forces F_1 and F_2 which in this case are:

$$(-\lambda\, \mathbf{j}) \bullet \delta\mathbf{r}_B = F_1 \delta q_1 + F_2 \delta q_2 \tag{AA-4.14}$$

which in conjunction with Eq. (AA-4.7) yields for the external forces:

$$F_1 = 2a\lambda \sin q_1, \quad F_2 = 2a\lambda \sin q_2 \tag{AA-4.15}$$

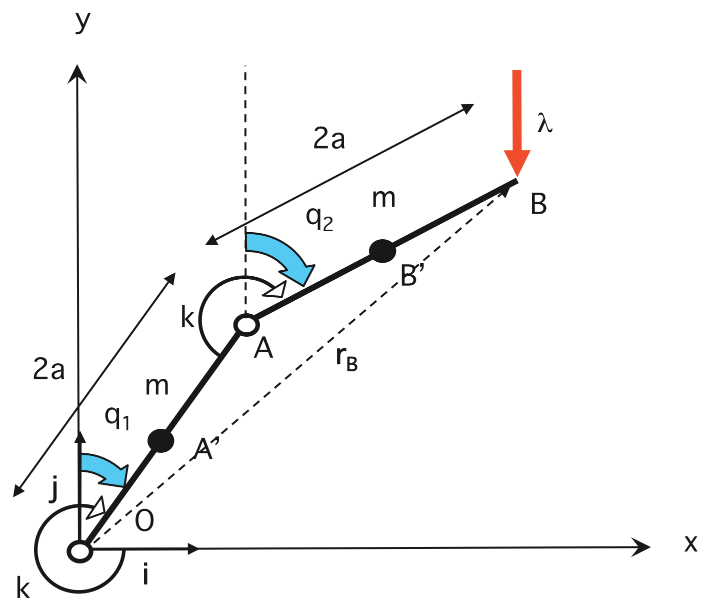

Figure AA-4.2: Two-bar model with conservative force.

Consequently from Eqs. (AA-4.1), (AA-4.2), (AA-4.5), (AA-4.9) and (AA-4.15), the corresponding nonlinear equation of motion takes the form:

$$\begin{aligned}
&2kq_1 - kq_2 - 2a\lambda \sin q_1 + ma^2[5\ddot{q}_1 + 2\ddot{q}_2 \cos(q_1 - q_2) \\
&-2\dot{q}_2(\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2) + 2\dot{q}_1\dot{q}_2 \sin(q_1 - q_2)] = 0 \\
&-kq_1 + kq_2 - 2a\lambda \sin q_2 + ma^2[\ddot{q}_2 + 2\ddot{q}_1(q_1 - q_2)\cos(q_1 - q_2) \\
&-2\dot{q}_1(\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2) - 2\dot{q}_1\dot{q}_2 \sin(q_1 - q_2)] = 0
\end{aligned} \tag{AA-4.16}$$

One can easily verify that $q_1(t) = q_2(t) = 0$ is again an equilibrium solution. Upon linearization, one obtains from (AA-4.16):

$$\mathbf{M} \bullet \Delta\ddot{\mathbf{q}} + \mathbf{K} \bullet \Delta\mathbf{q} = \mathbf{0}; \quad \mathbf{M} \equiv ma^2 \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{K} \equiv \begin{bmatrix} 2k - 2a\lambda & -k \\ -k & k - 2a\lambda \end{bmatrix}, \tag{AA-4.17}$$

which is similar to Eq. (AA-4.11), save for a new stiffness matrix $\mathbf{K}$. Once again, the solution to the above constant coefficient linear system is given by Eq. (AA-4.12), thus leading to the following characteristic equation for s_I

$$\det[\mathbf{M}(s_I)^2 + \mathbf{K}] = 0, \quad \Longrightarrow \quad m^2a^4(s_I)^4 + ma^2(11k - 12a\lambda)(s_I)^2 + k^2 - 6a\lambda k + 4a^2\lambda^2 = 0, \ I = 1\cdots 4. \tag{AA-4.18}$$

It is not difficult to verify that:

- If $\lambda \leq (3-\sqrt{5})k/4a$, then $\Re(s_I) = 0, \ \Im(s_I) \neq 0, \ (I = 1, 2, 3, 4)$ and hence the linearized system is stable.

• If $(3-\sqrt{5})k/4a < \lambda < (3+\sqrt{5})k/4a$, then $\Re(s_I) > 0$, $\Im(s_I) = 0$, $(I = 1, 2)$ and hence the linearized system exhibits a divergence type instability.

• If $(3+\sqrt{5})k/4a \leq \lambda$, then $\Re(s_I) > 0$, $\Im(s_I) = 0$, $(I = 1, 2, 3, 4)$ and hence the linearized system again exhibits a divergence type instability.

When $0 \leq \lambda < (3-\sqrt{5})k/4a$, any initial perturbation produces a constant amplitude periodic motion in the linearized system, since no dissipation is considered. Consequently the system is not asymptotically stable according to the definition in Eq. (AA-1.4). If a small linear viscous damping is introduced, a viscous term $\mathbf{C} \bullet \Delta\mathbf{q}$ – where $\mathbf{C}$ is a small positive definite viscosity matrix – is added to the linearized perturbation Eq. (AA-4.17). Consequently one can show that for $\lambda \leq (3-\sqrt{5})k/4a$ all roots have negative real parts, i.e. $\Re(s_I) < 0$, $(I = 1, 2, 3, 4)$ and hence that the more realistic, linearized viscous system is also asymptotically stable.

Some final remarks are in order for the mechanism in Fig. AA-4.2. In view of its fixed direction loading, a potential energy does exist, i.e. external forces can be derived from a potential. Moreover, for loads $0 \leq \lambda < (3-\sqrt{5})k/4a$, it can be shown that the equilibrium solution $\mathbf{q}_e = \mathbf{0}$ minimizes the system's potential energy $\mathcal{E}$, which is the sum of the internal (elastic) energy in Eq. (AA-4.9) and the potential energy of the applied load $(-\lambda\mathbf{j})\bullet\mathbf{r}_B$, namely

$$\mathcal{E} = \frac{1}{2}\ k(q_1)^2 + \frac{1}{2}\ k(q_1 - q_2)^2 + 2\lambda a(\cos q_1 + \cos q_2). \tag{AA-4.19}$$

It follows from Eq. (AA-4.19), by taking the second derivative of $\mathcal{E}$ with respect to $\mathbf{q}$ on the equilibrium state $\mathbf{q}_e$ that

$$\left[\frac{\partial^2\mathcal{E}}{\partial\mathbf{q}\partial\mathbf{q}}\right]_{\mathbf{q}_e} = \mathbf{K}, \tag{AA-4.20}$$

where $\mathbf{K}$ is the linearized conservative system's stiffness matrix defined in Eq. (AA-4.17). It can easily be checked that for loads $0 \leq \lambda < (3-\sqrt{5})k/4a$, the stiffness matrix $\mathbf{K}$ is positive definite, thus showing that loads for which the linearized system is stable, also minimize the structure's potential energy. The connection found in this example between stability and minimization of potential energy, is a general property of conservative systems and will be discussed in a more general setting in the next subsection.

AA-5 ENERGY CRITERION OF STABILITY

By definition, a conservative system is a system with a constant total energy which, for the case of the mechanical systems considered here, means that the sum of its potential energy $\mathcal{E}$ and its kinetic energy $\mathcal{K}$ is a constant. We will show that for a finite d.o.f. conservative system, stability of an equilibrium is equivalent to minimization of the potential energy by the equilibrium solution in question. To achieve this, we will show that if an equilibrium solution $\mathbf{q}_e$ minimizes the potential energy of a system, then one can define a Lyapunov functional by

$$L(\mathbf{p}) \equiv \mathcal{E}(\mathbf{q}) + \mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) - \mathcal{E}(\mathbf{q}_e), \quad \mathbf{p} \equiv (\mathbf{q}, \dot{\mathbf{q}}). \tag{AA-5.1}$$

The above defined functional is a constant (since the system is conservative) and hence non-increasing, thus satisfying the first requirement for a Lyapunov functional according to Eq. (AA-3.1).

The fact that $\mathcal{E}(\mathbf{q}_e)$ is a strict local minimum of the function $\mathcal{E}(\mathbf{q})$ implies the existence of a constant c_1 such that $\mathcal{E}(\mathbf{q}) - \mathcal{E}(\mathbf{q}_e) \geq c_1 \parallel \mathbf{q} - \mathbf{q}_e \parallel^2$ $(c_1 > 0)$. For any realistic system the kinetic energy $\mathcal{K} \geq 0$, being a sum of squares of velocities, implies that $\mathcal{K} \geq c_2 \parallel \dot{\mathbf{q}} \parallel^2$ $(c_2 \geq 0)$ and hence one can find $c > 0$ such that:

$$\mathcal{E}(\mathbf{q}) - \mathcal{E}(\mathbf{q}_e) + \mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) \geq c \parallel (\mathbf{q} - \mathbf{q}_e, \dot{\mathbf{q}} - \dot{\mathbf{q}}_e) \parallel^2 = c \parallel \mathbf{p} - \mathbf{p}_e \parallel^2, \tag{AA-5.2}$$

hence ensuring the second property required of a Lyapunov functional according to Eq. (AA-3.2).

Similarly, the continuity of $\mathcal{E}(\mathbf{q})$ as well as the finite dimensionality of the space of $\mathbf{q}$ ensure a $d_1 > 0$ such that $\mathcal{E}(\mathbf{q}) - \mathcal{E}(\mathbf{q}_e) \leq d_1 \parallel \mathbf{q} - \mathbf{q}_e \parallel^2$ while for the kinetic energy $\mathcal{K}$ one can also find a $d_2 > 0$ such that $\mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) \leq d_2 \parallel \dot{\mathbf{q}} \parallel^2$. Consequently, and for the same reasons as before one can find a $d > 0$ with the property:

$$\mathcal{E}(\mathbf{q}) - \mathcal{E}(\mathbf{q}_e) + \mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) \leq d \parallel \mathbf{q} - \mathbf{q}_e, \dot{\mathbf{q}} - \dot{\mathbf{q}}_e \parallel^2 = d \parallel \mathbf{p} - \mathbf{p}_e \parallel^2, \tag{AA-5.3}$$

thus ensuring the third property of a Lyapunov functional according to Eq. (AA-3.3) when above inequality is evaluated at $t = 0$

The above discussion proves that a local minimum of the potential energy at equilibrium is a sufficient condition for stability, a statement which is known in the literature under the name of Lejeune-Dirichlet (or minimum potential energy) stability theorem.

For the case of finite d.o.f. conservative systems, the strict minimum of the potential energy is also a necessary condition for stability. It is not difficult to see that if $\mathbf{q}_e$ is not a minimum of the potential energy $\mathcal{E}(\mathbf{q})$ then in view of the energy conservation $\mathcal{E}(\mathbf{q}_e) - \mathcal{E}(\mathbf{q}) = \mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) > 0$ for some $\mathbf{q}$ in the neighborhood of $\mathbf{q}_e$. Moreover if the initial perturbation is $\mathbf{q}(0) \neq \mathbf{0}, \dot{\mathbf{q}}(0) = \mathbf{0}$ with $\parallel \mathbf{q}(0) \parallel < \eta$ for any small η, one can show that $\parallel \mathbf{p} - \mathbf{p}_e \parallel$ will be an increasing function of time in view of the energy conservation equation mentioned above, thus proving instability. Consequently, (assuming appropriate continuity conditions), for a finite d.o.f., conservative system to be stable, a local minimum of the system's energy $\mathcal{E}$ at an equilibrium state $\mathbf{p}_e$ is equivalent with:

$$\left[\frac{\partial^2 \mathcal{E}}{\partial \mathbf{q} \partial \mathbf{q}}\right]_{\mathbf{q}_e} \quad \text{positive definite} \tag{AA-5.4}$$

The above necessary and sufficient criterion for stability in finite d.o.f. conservative systems will be employed in the rest of this work for the stability discussion of all discrete elastic systems. A generalization of the above criterion for continuous conservative systems will be discussed subsequently. It is interesting to note that although the criterion of stability is a dynamical concept and involves the equations of motion and hence the mass distribution in the system, for the case of conservative systems it has just been shown that the stability criterion involves only the properties of the potential energy, thus considerably simplifying the task of stability investigations.

AB BIFURCATION CONCEPTS AND FINITE D.O.F. SIMPLE EXAMPLES

In this chapter the notion of a *bifurcation* in the solution of a nonlinear system is presented and illustrated by means of simple examples that permit analytical solutions.

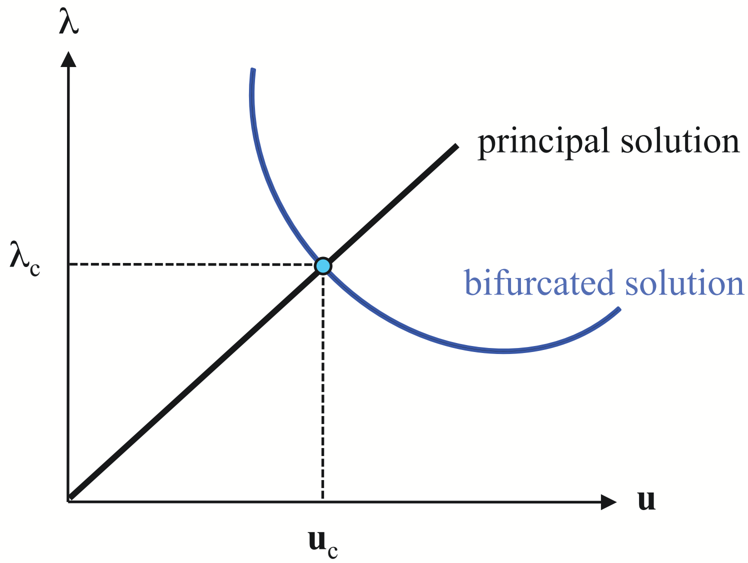

Figure AB-0.1: Schematic representation of a bifurcation.

The concept is explained with the help of Fig. AB-0.1. Assume that we want to solve the nonlinear system (representing the equilibrium of a finite d.o.f. structure)

$$\mathbf{f}(\mathbf{u}, \lambda) = \mathbf{0}, \tag{AB-0.1}$$

where $\mathbf{u} \in \mathbb{R}^n$ are the system's unknowns (typically displacements) and λ a scalar parameter characterizing the system's load. To indicate that the system's d.o.f. are time-independent, from here on use the symbol $\mathbf{u}$ instead of $\mathbf{p}$ that has been used before. Since we deal here with conservative systems that have a potential $\mathcal{E}$, the above equilibrium equations are the stationarity conditions for $\mathcal{E}$, namely:

$$\mathbf{f}(\mathbf{u}, \lambda) = \frac{\partial \mathcal{E}(\mathbf{u}, \lambda)}{\partial \mathbf{u}} \equiv \mathcal{E}_{,\mathbf{u}}\,. \tag{AB-0.2}$$

We are interested in the solutions $\mathbf{u}(\lambda)$ of Eq. (AB-0.1) as functions of the load parameter λ. One solution, termed "*principal* " solution is the (usually straightforward) solution of Eq. (AB-0.1) which starts at zero load with $\mathbf{u} = \mathbf{0}$ at $\lambda = 0$. Due to the system's nonlinearity the principal solution is not necessarily unique and as the load increases away from zero there is a certain point in the load versus displacement graph shown in Fig. AB-0.1 where another solution, termed "*bifurcated*" solution (in view of the fork shape of the graph) emerges at point $(\lambda_c, \mathbf{u}_c)$ termed respectively "*critical* " load and displacement.

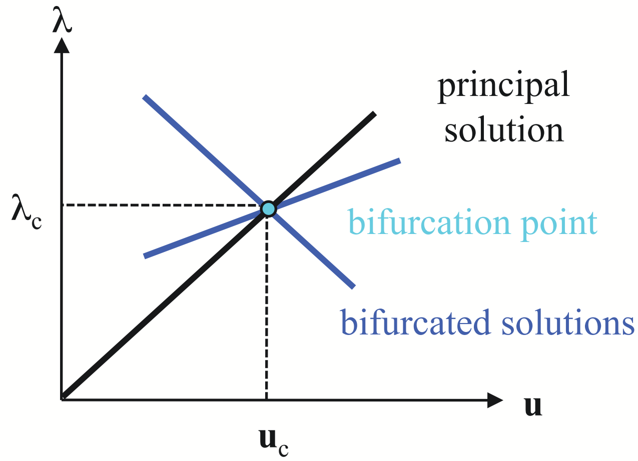

Figure AB-0.2: Schematic representation of a multiple bifurcation.

A rigid T model with two degrees of freedom will provide the example of a "*simple bifurcation*", the case where just one branch emerges from the principal path, as seen in Fig. AB-0.1. Many applications exist where several equilibrium branches emerge from the principal path, in which case we talk about a "*multiple bifurcation*", a situation depicted in Fig. AB-0.2. As such an example the study a rigid plate with three degrees of freedom will provide the example of a multiple (double) bifurcation.

An important feature of bifurcation is that it is non-robust, i.e. changes its character, under perturbations. By means of the simple examples introduced here, it will be shown that the bifurcation point either becomes a *limit load* (or *limit point*) (see Fig. AB-0.3) or a bifurcation point of lower order in the presence of imperfections.

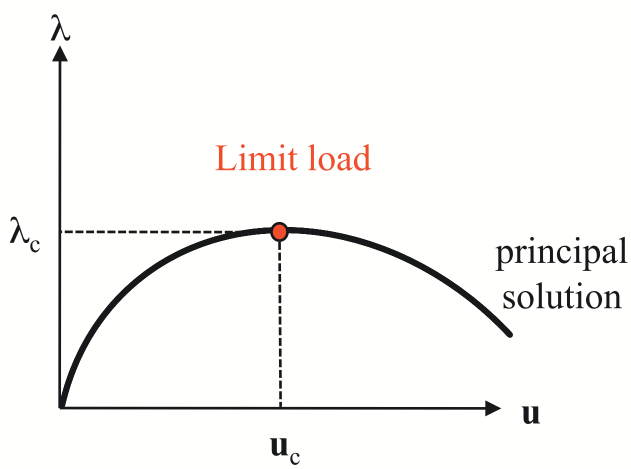

Figure AB-0.3: Schematic representation of a limit point.

Finally to illustrate the concept of a limit load, a two bar truss model is presented.

AB-1 PERFECT RIGID T MODEL

To illustrate the concept of bifurcation, we use a two degrees of freedom rigid T model shown in FIG. AB-1.1. For the perfect model the part OC of length L is attached perpendicularly to the middle of the segment AB of length $2l$. The midpoint O of the segment AB can only move vertically by a distance v while the entire structure can rotate about O by a small angle θ. Two identical, vertical linear springs with restoring force f proportional to their change of length d, ($f = -Ed,\ E > 0$, are attached to ends A and B. At the end C a horizontal nonlinear spring is attached, with a force-displacement relation given by $F = -[kd + md^2 + nd^3]$. The structure is subjected to a vertical load $\lambda \geq 0$ at C.

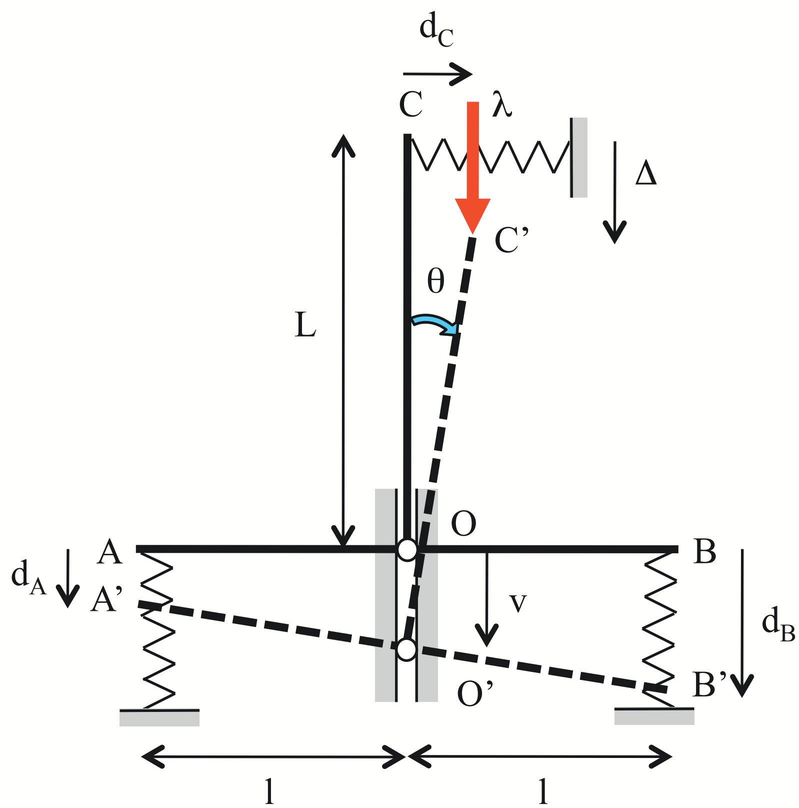

Figure AB-1.1: Perfect rigid T model.

From kinematics $d_A,\ d_B$, the vertical displacements at points A and B, d_C, Δ the horizontal and vertical displacements at point C, are:

$$d_A = v - l\theta, \quad d_B = v + l\theta, \quad d_C = L\theta, \quad \Delta = v + L(1 - \cos\theta) \approx v + L\frac{\theta^2}{2}, \tag{AB-1.1}$$

where the dependence of all the displacements on θ are correct up to $O(\theta^2)$.

The total potential energy $\mathcal{E}$ of the system, consisting of the energies stored in springs at

A, B and C, plus the potential energy of the applied load, is found to be:

$$\mathcal{E}(v,\theta,\lambda) = E[v^2 + (l\theta)^2] + \frac{k}{2}\,(L\theta)^2 + \frac{m}{3}\,(L\theta)^3 + \frac{n}{4}\,(L\theta)^4 - \lambda\,[v + L\frac{\theta^2}{2}]. \qquad \text{(AB-1.2)}$$

Extremizing $\mathcal{E}$ with respect to the degrees of freedom v and θ, one obtains two equilibrium equations (respectively, the force equilibrium along the vertical direction and the moment equilibrium about point O, as one can also verify by direct calculations)

$$\begin{aligned} &\mathcal{E},_v = 2Ev - \lambda = 0, \\ &\mathcal{E},_\theta = (2El^2 + kL^2)\theta + mL^3\theta^2 + nL^4\theta^3 - \lambda L\theta = 0. \end{aligned} \qquad \text{(AB-1.3)}$$

One solution to the above system is obviously:

$$\overset{0}{v}(\lambda) = \lambda/2E, \quad \overset{0}{\theta}(\lambda) = 0, \qquad \text{(AB-1.4)}$$

which is the principal solution, for it satisfies equilibrium at the unloaded state, i.e. for $\lambda = 0$ the displacements $(v, \theta) = (0, 0)$.

For $\theta \neq 0$, the same system of equilibrium equations admits the solutions:

$$\begin{aligned} &v(\lambda) = \lambda/2E, \quad \lambda = \lambda_c + \theta^2 nL^3 \quad \text{if } n \neq 0,\ m = 0, \quad \text{symmetric}, \\ &v(\lambda) = \lambda/2E, \quad \lambda = \lambda_c + \theta mL^2 \quad \text{if } n = 0,\ m \neq 0, \quad \text{asymmetric}, \\ &\text{where}: \ \lambda_c \equiv (2El^2 + kL^2)/L. \end{aligned} \qquad \text{(AB-1.5)}$$

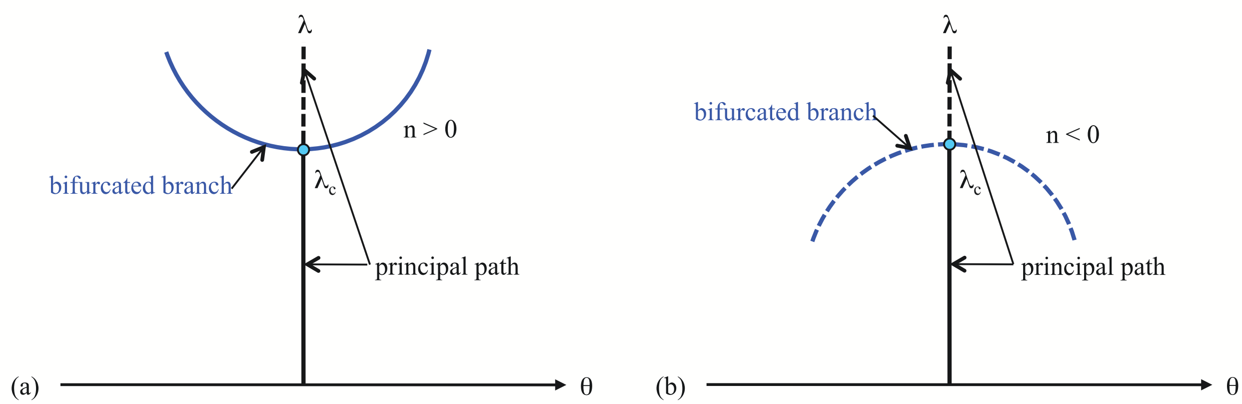

Figure AB-1.2: Symmetric bifurcation of perfect rigid T model. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

The above solutions are the bifurcated solutions and pass at $\theta = 0$ through the load $\lambda = \lambda_c$, which, according to the definition at the beginning of this section, is the critical load. The bifurcation shown for the symmetric case $(m = 0, n \neq 0)$ in Fig. AB-1.2 where the bifurcated branch emerges perpendicular to the principal branch is called "*symmetric*" bifurcation. The bifurcation shown for the asymmetric case $(m \neq 0, n = 0)$ in Fig. AB-1.3, where the bifurcated branch intersects the principal one at an angle different from a right angle, is called "*transverse*" or "*asymmetric*" bifurcation.

Notice that for the symmetric bifurcation in Fig. AB-1.2, the load λ of the bifurcated solution is higher (if $n > 0$) or lower (if $n < 0$) than the critical load in the neighborhood of λ_c. The corresponding bifurcations are termed "*supercritical*" and "*subcritical*" respectively. For the asymmetric bifurcation in Fig. AB-1.3, the load λ of the bifurcated solution can be either higher or lower than λ_c in the neighborhood of the critical load depending on the sign of θ and the corresponding bifurcation is called "*transcritical*".

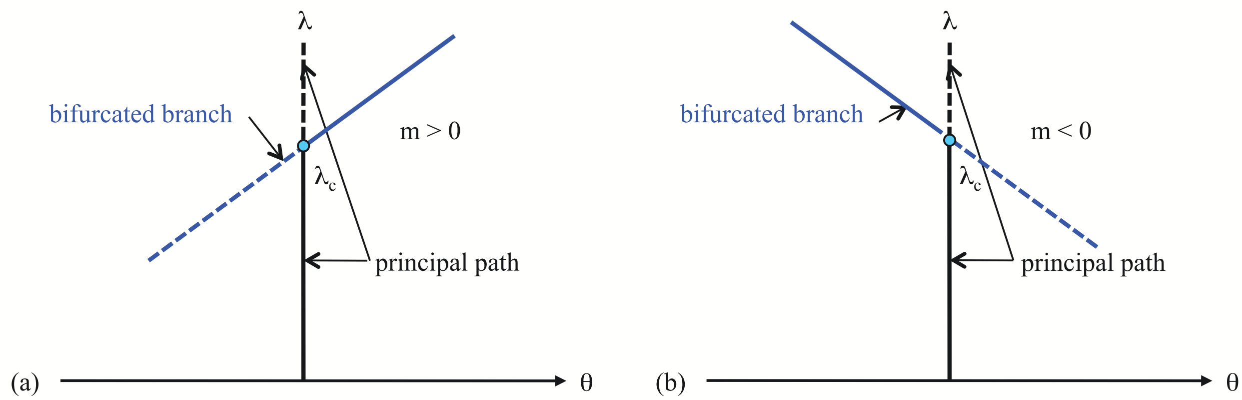

Figure AB-1.3: Transverse (or asymmetric) bifurcation of perfect rigid T model. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

An important feature of the problem is the stability of the different equilibrium paths. According to the discussion in subsection AA-5, an equilibrium path is stable if it corresponds to a local minimum of the system's potential energy. For the conservative two degree of freedom system here, a point $v(\lambda), \theta(\lambda)$ of an equilibrium path is a local minimum of the energy if, according to Eq. (AA-5.4), the matrix $\mathcal{E}_{,\mathbf{uu}}$ defined below is positive definite:

$$\mathcal{E}_{,\mathbf{uu}} = \begin{bmatrix} \mathcal{E}_{,vv} & \mathcal{E}_{,v\theta} \\ \mathcal{E}_{,\theta v} & \mathcal{E}_{,\theta\theta} \end{bmatrix} = \begin{bmatrix} 2E & 0 \\ 0 & (\lambda_c - \lambda)L + 2mL^3\theta + 3nL^4\theta^2 \end{bmatrix}. \tag{AB-1.6}$$

It is not difficult to see from Eq. (AB-1.6) that the principal branch $\theta = 0$ is stable for $0 \leq \lambda < \lambda_c$ and unstable for $\lambda \geq \lambda_c$ (since $E > 0$ and the positive definiteness of the diagonal matrix $\mathcal{E}_{,\mathbf{uu}}$ is determined by the sign of $\mathcal{E}_{,\theta\theta}$). For the symmetric bifurcation case $(m = 0, n \neq 0)$ substituting Eq. (AB-1.5) into Eq. (AB-1.6) one obtains $\mathcal{E}_{,\theta\theta} = 2(\lambda - \lambda_c)L$. Hence, for the supercritical bifurcation where $\lambda > \lambda_c$ the bifurcated branch is stable, while for the subcritical bifurcation where $\lambda < \lambda_c$, the bifurcated branch is unstable. For the asymmetric bifurcation $(m \neq 0, n = 0)$, the same reasoning as before leads to $\mathcal{E}_{,\theta\theta} = (\lambda - \lambda_c)L$ which shows that one part of the bifurcated branch is stable (the one with $\lambda > \lambda_c$), while the other (in which $\lambda < \lambda_c$) is not, regardless of the sign of m. In Fig. AB-1.2 and Fig. AB-1.3 the stable equilibrium paths are drawn using a continuous line while the unstable equilibrium paths are drawn using a dashed line.

Since at any given load the system has several possible equilibrium paths, it is also of interest to compare the energy levels associated with these different solutions.

Using Eq. (AB-1.4), Eq. (AB-1.5) into Eq. (AB-1.1) the total potential energy on the principal branch, is found to be:

$$\mathcal{E} = -\frac{1}{4E}\lambda^2, \tag{AB-1.7}$$

while the energy associated with each bifurcated branch is given by:

$$\begin{aligned} \mathcal{E} &= -\frac{1}{4E}\lambda^2 - \frac{(\lambda-\lambda_c)^2}{4nL^2} \quad \text{if } m = 0, \quad n \neq 0, \\ \mathcal{E} &= -\frac{1}{4E}\lambda^2 - \frac{(\lambda-\lambda_c)^3}{6m^2L^3} \quad \text{if } m \neq 0, \quad n = 0. \end{aligned} \tag{AB-1.8}$$

Given a load level λ, one can see from Eq. (AB-1.8) that for a symmetric bifurcation $(m = 0, n \neq 0)$, the stable bifurcation branch of the supercritical case $(n > 0)$ has less energy than the principal branch, while for the subcritical case $(n < 0)$ the situation is reversed. For the asymmetric bifurcation $(m \neq 0, n = 0)$ one observes that for loads $\lambda > \lambda_c$ the bifurcated branch has lower energy than the principal branch, while for loads $\lambda < \lambda_c$ the situation is reversed.

This simple example shows that for a given load level λ, the minimum energy always corresponds to the stable equilibrium solution (principal or bifurcated). Also note that for the transcritical asymmetric or the supercritical symmetric bifurcation, one can always find a stable equilibrium branch for any load level λ, while for the subcritical symmetric bifurcation a stable equilibrium solution exists only for $\lambda < \lambda_c$.

AB-2 IMPERFECT RIGID T MODEL

The introduction of an imperfection into the rigid T model investigated in the previous section, provides the physically plausible mechanism which determines uniquely the structure's equilibrium path in a loading process starting from $\lambda = 0$. Of all the many possible ways to introduce an imperfection, the one considered here is geometric and is in the form of a slight defect in the normality of the part OC to the part OA by angle δ, as seen in Fig. AB-2.1. All the other elements of the model (dimensions, stiffnesses of linear and nonlinear springs, etc.) remain the same as in the perfect model. The imperfect model reduces to the perfect model when the imperfection angle $\delta = 0$.

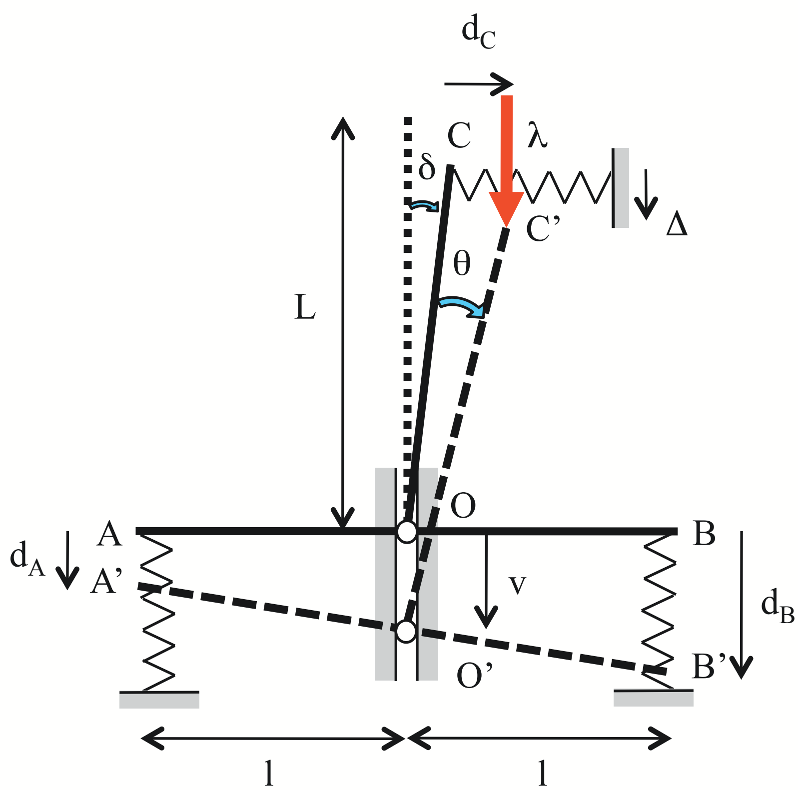

Figure AB-2.1: Imperfect rigid T model.

The kinematics of the imperfect rigid T model are the same as for its perfect counterpart (see Eq. (AB-1.1)), save for the vertical displacement Δ of point C, which, due to the small values of the angles θ and δ, is now given by

$$\Delta = v + L[\cos\delta - \cos(\theta + \delta)] \approx v + L(\frac{\theta^2}{2} + \theta\delta). \tag{AB-2.1}$$

The potential energy $\overline{\mathcal{E}}$ of the imperfect system is:

$$\overline{\mathcal{E}}(v, \theta, \lambda, \delta) = E[v^2 + (l\theta)^2] + \frac{k}{2}(L\theta)^2 + \frac{m}{3}(L\theta)^3 + \frac{n}{4}(L\theta)^4 - \lambda[v + L(\frac{\theta^2}{2} + \theta\delta)]. \tag{AB-2.2}$$

As expected the imperfect energy reduces to its perfect counterpart when the imperfection vanishes, i.e. $\overline{\mathcal{E}}(v,\theta,\lambda,0) = \mathcal{E}(v,\theta,\lambda)$.

Extremizing $\overline{\mathcal{E}}$ with respect to the two degrees of freedom of the system v and θ, the two equilibrium equations obtained are:

$$\begin{aligned} &\overline{\mathcal{E}}_{,v} = 2Ev - \lambda = 0 \\ &\overline{\mathcal{E}}_{,\theta} = (2El^2 + kL^2)\theta + mL^3\theta^2 + nL^4\theta^3 - \lambda L(\theta + \delta) = 0 \end{aligned} \tag{AB-2.3}$$

Recalling from Eq. (AB-1.5) that $\lambda_c \equiv (2El^2 + kL^2)/L$, the solution of the above system is:

$$\begin{aligned} &v(\lambda,\delta) = \lambda/2E, \quad \lambda = [\lambda_c\theta + n(L\theta)^3]/(\theta+\delta) \qquad \text{if } n \neq 0,\ m = 0, \\ &v(\lambda,\delta) = \lambda/2E, \quad \lambda = [\lambda_c\theta + m(L\theta)^2]/(\theta+\delta) \qquad \text{if } n = 0,\ m \neq 0. \end{aligned} \tag{AB-2.4}$$

and the results are depicted in Fig. AB-2.2 and Fig. AB-2.3.

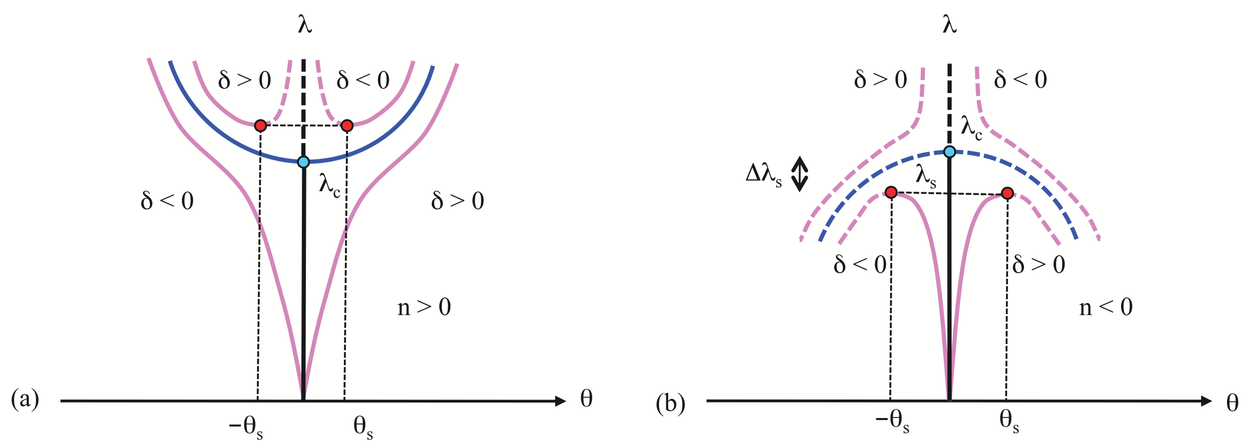

Figure AB-2.2: Equilibrium solutions for symmetric imperfect rigid T model. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

The most important observation made about the imperfect structure is the absence of bifurcation. Notice in Fig. AB-2.2 and Fig. AB-2.3 that the equilibrium branches no longer intersect as in the perfect case. Moreover, for a given value of the imperfection parameter δ, only one equilibrium branch passes through the initial, stress free state $(v,\theta,\lambda) = (0,0,0)$. The behavior of this simple imperfect model is representative of what occurs in a real structure. For values of the externally applied load parameter sufficiently lower than the perfect structure's critical load λ_c, the equilibrium solution of the real imperfect structure differs little from the principal solution of the perfect one. For load values near the critical load λ_c of the perfect structure, the equilibrium solution of the imperfect structure deviates considerably from the principal solution of its perfect counterpart, since it starts following one of the perfect structures bifurcated equilibrium paths.

Only the equilibrium paths passing through the origin have a physical meaning, for they are the ones accessible to the structure starting from rest $(v,\theta,\lambda) = (0,0,0)$. The other equi-

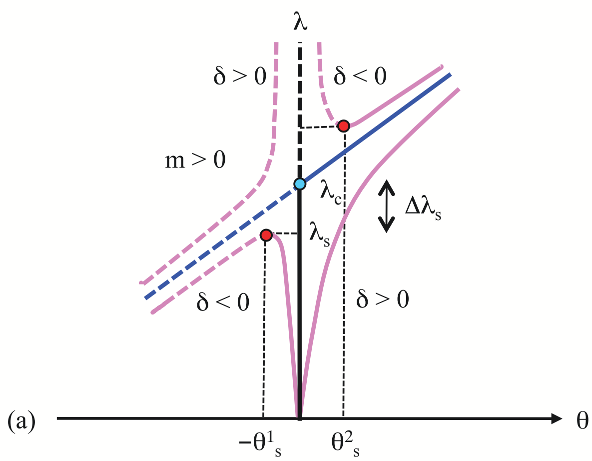

Figure AB-2.3: Equilibrium solutions for asymmetric imperfect rigid T model. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

librium paths cannot be reached in a continuing loading process starting from the unloaded state.

The next important question to be addressed concerns the stability of the aforedescribed equilibrium branches. Reasoning in a similar fashion as in the perfect case, an equilibrium point $v(\lambda,\delta)$, $\theta(\lambda,\delta)$ is stable if it is a local minimum of the system's energy $\overline{\mathcal{E}}$, which according to Eq. (AA-5.4), requires the matrix $\overline{\mathcal{E}}_{,\mathbf{uu}}$ to be positive definite:

$$\overline{\mathcal{E}}_{,\mathbf{uu}} = \begin{bmatrix} \overline{\mathcal{E}}_{,vv}\,, & \overline{\mathcal{E}}_{,v\theta} \\ \overline{\mathcal{E}}_{,\theta v}\,, & \overline{\mathcal{E}}_{,\theta\theta} \end{bmatrix} = \begin{bmatrix} 2E, & 0 \\ 0, & (\lambda_c-\lambda)L + 2mL^3\theta + 3nL^4\theta^2 \end{bmatrix}, \tag{AB-2.5}$$

or equivalently (since $E > 0$) if $\overline{\mathcal{E}}_{,\theta\theta} > 0$. Two cases are distinguished:

In case $m = 0, n \neq 0$ substituting Eq. (AB-2.4)$_1$ into the expression for $\overline{\mathcal{E}}_{,\theta\theta}$ in Eq. (AB-2.5) one obtains $\overline{\mathcal{E}}_{,\theta\theta} = [nL^4\theta^2(2\theta+3\delta) + \lambda_c L\delta]/(\theta+\delta)$. When $n > 0$, $\overline{\mathcal{E}}_{,\theta\theta} > 0$ for all θ if $\theta\delta > 0$, and $\overline{\mathcal{E}}_{,\theta\theta} < 0$ for $|\delta| < |\theta| < \theta_s$, $\overline{\mathcal{E}}_{,\theta\theta} > 0$ for $|\theta| < |\delta|$ or $|\theta| > \theta_s$ if $\theta\delta < 0$ where θ_s is the positive root of $2nL^3(\theta_s)^3 + 3nL^3(\theta_s)^2\delta + \lambda_c\delta = 0$ for $\delta > 0$. (It can easily be shown that only one admissible root exists for adequately small δ). When $n < 0$, $\overline{\mathcal{E}}_{,\theta\theta} < 0$ for $|\theta| > |\delta|$ and $\overline{\mathcal{E}}_{,\theta\theta} > 0$ for $|\theta| < |\delta|$ if $\theta\delta < 0$ and $\overline{\mathcal{E}}_{,\theta\theta} > 0$ for $|\theta| < \theta_s$, $\overline{\mathcal{E}}_{,\theta\theta} < 0$ for $|\theta| > \theta_s$ if $\theta\delta > 0$ where θ_s is again the unique positive root of the cubic equation $2nL^3(\theta_s)^3 + 3nL^3(\theta_s)^2\delta + \lambda_c\delta = 0$ for $\delta > 0$.

It is interesting to notice that at $\theta = \theta_s$ for $\delta > 0$ (or $\theta = -\theta_s$ for $\delta < 0$), the $\lambda - \theta$ curves have an extremum of λ. Indeed from (AB-2.4)$_1$:

$$\frac{d\lambda}{d\theta} = \frac{2nL^3\theta^3 + 3nL^3\theta^2\delta + \lambda_c\delta}{(\theta+\delta)^2} \tag{AB-2.6}$$

which shows that the extremum of λ is reached for $|\theta| = \theta_s$.

For the case $m \neq 0, n = 0$ substituting Eq. (AB-2.4)$_2$ into the expression for $\overline{\mathcal{E}}_{,\theta\theta}$ from Eq. (AB-2.5) one obtains $\overline{\mathcal{E}}_{,\theta\theta} = [mL^3\theta(\theta+2\delta) + \lambda_c L\delta]/(\theta+\delta)$. When $m\delta > 0$, $\overline{\mathcal{E}}_{,\theta\theta} > 0$ if $\theta\delta > 0$ and

$\overline{\mathcal{E}}_{,\theta\theta} < 0$ for $|\theta| > |\delta|$ if $\theta\delta < 0$ (assuming adequately small values of δ). When $m\delta < 0$, $\overline{\mathcal{E}}_{,\theta\theta} < 0$ for $|\theta| > \theta_s^1$ or $|\delta| < |\theta| < \theta_s^2$ and $\overline{\mathcal{E}}_{,\theta\theta} > 0$ for $|\theta| < \theta_s^1$ or $|\theta| > \theta_s^2$ where $\theta_s^1 > 0$ and $-\theta_s^2 < 0$ are the two roots of the quadratic $mL^2(\theta_s)^2 + 2mL^2\theta_s\delta + \lambda_c\delta = 0$ with $\theta_s^1 < \theta_s^2$.

Once more the points $\theta = -\theta_s^1,\ \theta_s^2$ for $\delta < 0$ and $\theta = -\theta_s^2,\ \theta_s^1$ for $\delta > 0$ are the extrema of the corresponding $\lambda - \theta$ curves. Indeed from Eq. (AB-2.4)$_2$ one has:

$$\frac{d\lambda}{d\theta} = \frac{mL^2\theta^2 + 2mL^2\theta\delta + \lambda_c\delta}{(\theta+\delta)^2} \tag{AB-2.7}$$

In Fig. AB-2.2 and Fig. AB-2.3 the stable equilibrium branches of the imperfect structure are drawn using a solid line while the unstable equilibrium branches are drawn using a dotted line.

Only for the case where the perfect structure exhibits a supercritical bifurcation $(m = 0, n > 0)$ the (physically admissible) equilibrium paths of the corresponding imperfect structure are stable for all possible values of the imperfection δ (assumed small). In all the other cases at least one (physically admissible) equilibrium path of the imperfect structure exhibits a load maximum beyond which the equilibrium is unstable. Such an instability is called in the literature a "*snap through*" instability. The snap through instability that happens in realistic imperfect structures explains the experimental observation that the critical load calculated for the perfect structure is always higher than the one actually measured in a lab test.

Using Eq. (AB-2.4) in Eq. (AB-2.6) and Eq. (AB-2.7) evaluated at θ_s one finds for $\Delta\lambda_s \equiv \lambda(\theta_s) - \lambda_c$:

$$\begin{aligned} \Delta\lambda_s &= 3nL^3(\theta_s)^2 \quad \text{if } m = 0,\ n < 0 \\ \Delta\lambda_s &= 2mL^2\theta_s \qquad \text{if } m \neq 0,\ n = 0 \end{aligned} \tag{AB-2.8}$$

For small values of δ, by expanding Eq. (AB-2.6) and Eq. (AB-2.7) in terms of powers of δ one obtains the following expressions for θ_s and $\Delta\lambda_s$:

$$\begin{aligned} \theta_s &= \frac{1}{2^{1/3}}\ \frac{(\lambda_c)^{1/3}(-n)^{-1/3}}{L}\ |\delta|^{1/3} + O(|\delta|^{2/3}), & \\ & & \text{if } m = 0,\ n < 0 \\ \Delta\lambda_s &= \frac{3}{2^{2/3}} L(\lambda_c)^{2/3}(n)^{1/3}\ |\delta|^{2/3} + O(|\delta|) & \\ \theta_s &= \frac{(\lambda_c)^{1/2}(-m\ \text{sgn}\ \delta)^{-1/2}}{L}(\text{sgn}\ \delta)\ |\delta|^{1/2} + O(|\delta|), & \\ & & \text{if } m \neq 0,\ n = 0 \\ \Delta\lambda_s &= -2L(\lambda_c)^{1/2}(-m\ \text{sgn}\ \delta)^{1/2}\ |\delta|^{1/2} + O(|\delta|) & \end{aligned} \tag{AB-2.9}$$

which shows that even a small imperfection δ can account for a rather substantial load reduction $\Delta\lambda_s$ given that $\Delta\lambda_s$ is proportional to the fractional powers $|\delta|^{1/2}$ or $|\delta|^{2/3}$.

AB-3 PERFECT SQUARE PLATE MODEL

The perfect rigid T model examined in subsection AB-1 has only one bifurcated equilibrium branch emerging from the principal path at $\lambda = \lambda_c$. This situation occurs in structures where the matrix $\mathcal{E}_{,\mathbf{uu}}$ has a simple eigenvalue at the critical load λ_c, as seen in Eq. (AB-1.6). Often in applications one encounters structures whose $\mathcal{E}_{,\mathbf{uu}}$ matrix has a multiple eigenvalue at λ_c. In such structures one typically finds more than one bifurcated equilibrium emerging from the critical load. This case is illustrated by the simple rigid plate model given below.

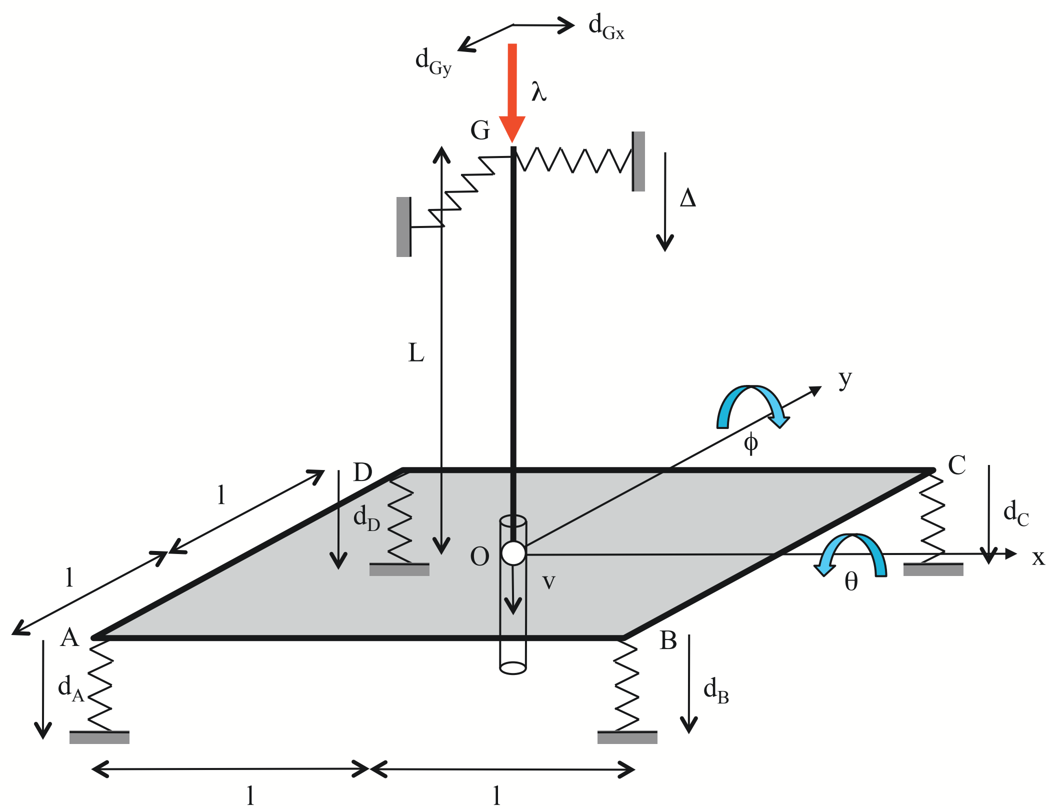

Figure AB-3.1: Perfect rigid plate model.

The rigid square plate ABCD shown in Fig. AB-3.1 has dimensions $2l \times 2l$. A rigid rod OG of length L is attached perpendicularly to the plate at its midpoint O. The center O of the rigid plate can only move vertically by a distance v, while the entire structure can rotate about the axes x and y by small angles θ and ϕ respectively. Four identical linear springs with restoring force f proportional to their length change d, ($f = -Ed$) are attached to the ends A, B, C, and D. At the end G two nonlinear springs are attached in the x and y directions with corresponding force–displacement relations $F_x = -[kd_x + m(d_x^2 + d_y^2) + n(2d_x^3 + 6d_x^2 d_y - 3d_x d_y^2 + 2d_y^3)]$ and $F_y = -[kd_y + 2md_x d_y + n(2d_y^3 + 6d_y^2 d_x - 3d_y d_x^2 + 2d_x^3)]$ respectively, where $d_x = d_{Gx}$ and $d_y = d_{Gy}$ are the two horizontal displacements of G. A vertical load λ is applied on OG at point G and the corresponding vertical displacement of G is Δ.

From kinematics d_A, d_B, d_C, d_D, the vertical displacements at points A, B, C and B, d_{Gx}, d_{Gy}, Δ the horizontal and vertical displacements at point G, are for small values of angles θ and ϕ (again taking approximations that are correct up to $O(\theta^2)$ and $O(\phi^2)$):

$$\begin{gathered} d_A = v + l\theta - l\phi, \quad d_B = v + l\theta + l\phi, \quad d_C = v - l\theta + l\phi, \quad d_D = v - l\theta - l\phi, \\ d_{Gx} = L\phi, \quad d_{Gy} = L\theta, \quad \Delta = v + L[1 - (1 - \sin^2\theta - \sin^2\phi)^{1/2}] \approx v + \frac{L}{2}(\theta^2 + \phi^2). \end{gathered} \tag{AB-3.1}$$

The total energy $\mathcal{E}$ of the system, consisting of the energy stored in the springs and the potential energy of the applied load, is found to be:

$$\begin{gathered} \mathcal{E}(v,\theta,\phi,\lambda) = 2E[v^2 + l^2(\theta^2+\phi^2)] + \frac{kL^2}{2}(\theta^2+\phi^2) + \frac{mL^3}{3}(3\theta^2\phi + \phi^3) + \\ + \frac{nL^4}{4}(2\theta^4 + 8\theta^3\phi - 6\theta^2\phi^2 + 8\theta\phi^3 + 2\phi^4) - \lambda[v + \frac{L}{2}(\theta^2+\phi^2)]. \end{gathered} \tag{AB-3.2}$$

By extremizing $\mathcal{E}$ with respect to its degrees of freedom, the three equilibrium equations of the system are:

$$\begin{aligned} &\mathcal{E}_{,v} = 4Ev - \lambda = 0, \\ &\mathcal{E}_{,\theta} = (4El^2 + kL^2 - \lambda L)\theta + 2mL^3\phi\theta + nL^4(2\theta^3 + 6\theta^2\phi - 3\theta\phi^2 + 2\phi^3) = 0, \\ &\mathcal{E}_{,\phi} = (4El^2 + kL^2 - \lambda L)\phi + mL^3(\phi^2 + \theta^2) + nL^4(2\phi^3 + 6\phi^2\theta - 3\phi\theta^2 + 2\theta^3) = 0. \end{aligned} \tag{AB-3.3}$$

The principal solution of the above system, i.e. the one passing at zero load $\lambda = 0$ through the origin $(v,\theta,\phi) = (0,0,0)$ is:

$$\overset{0}{v}(\lambda) = \lambda/4E, \quad \overset{0}{\theta}(\lambda) = \overset{0}{\phi}(\lambda) = 0. \tag{AB-3.4}$$

For $\phi \neq 0$, the same system of equilibrium equations has the following solutions:

$$\begin{aligned} &N1: \quad v(\lambda) = \lambda/4E, \quad \theta(\lambda) = \phi(\lambda), \quad \lambda = \lambda_c + 7nL^3\phi^2 \\ &N2: \quad v(\lambda) = \lambda/4E, \quad \theta(\lambda) = -\phi(\lambda), \quad \lambda = \lambda_c - 9nL^3\phi^2 \qquad m = 0,\ n \neq 0, \\ &N3: \quad v(\lambda) = \lambda/4E, \quad \theta(\lambda) = 2\phi(\lambda), \quad \lambda = \lambda_c + 18nL^3\phi^2 \\ &N4: \quad v(\lambda) = \lambda/4E, \quad \theta(\lambda) = \phi(\lambda)/2, \quad \lambda = \lambda_c + (9/2)nL^3\phi^2 \\ \\ &M1: \quad v(\lambda) = \lambda/4E, \quad \theta(\lambda) = 0, \quad \lambda = \lambda_c + mL^2\phi \\ &M2: \quad v(\lambda) = \lambda/4E, \quad \theta(\lambda) = \phi(\lambda), \quad \lambda = \lambda_c + 2mL^2\phi \qquad m \neq 0,\ n = 0, \\ &M3: \quad v(\lambda) = \lambda/4E, \quad \theta(\lambda) = -\phi(\lambda), \quad \lambda = \lambda_c + 2mL^2\phi \end{aligned} \tag{AB-3.5}$$

with $\quad \lambda_c \equiv (4El^2 + kL^2)/L.$

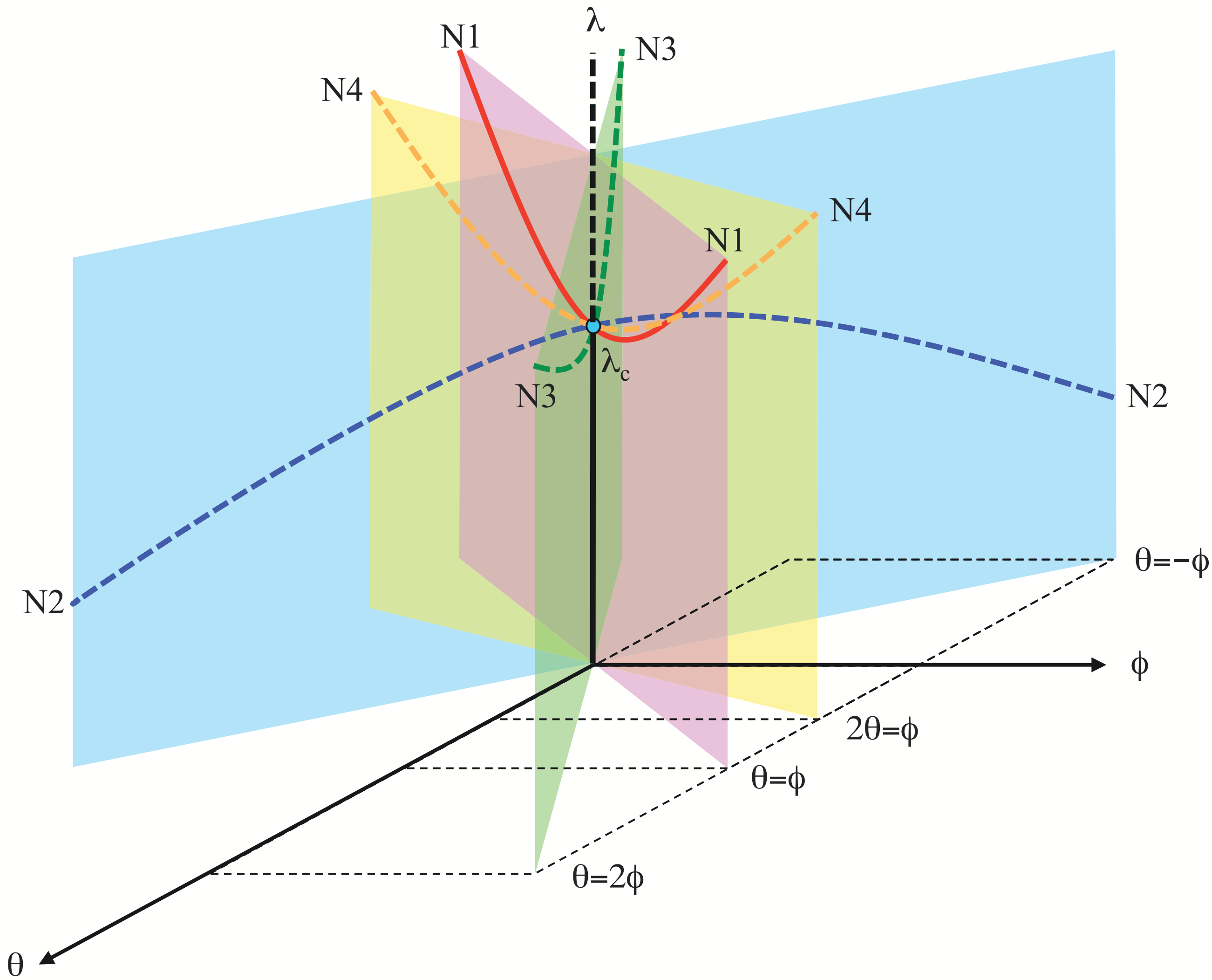

Figure AB-3.2: Bifurcated solutions for symmetric perfect rigid plate model. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

Note that in the case $m = 0, n \neq 0$ four equilibrium paths go through the critical load $\lambda = \lambda_c$ while for the case $m \neq 0, n = 0$ three equilibrium paths go through the critical load. Also notice that all the bifurcated equilibrium paths in case $m = 0, n \neq 0$ are symmetric (i.e. they intersect the λ axis at a right angle) while the corresponding equilibrium paths for case $m \neq 0, n = 0$ are asymmetric (i.e. they intersect the λ axis obliquely). One can also observe from Eq. (AB-3.5), where it was tacitly assumed that $n > 0$, that in the symmetric bifurcation case some bifurcated solutions (N1, N3, N4) are supercritical (i.e. $\lambda > \lambda_c$) while the remaining one (N2) is subcritical (i.e. $\lambda < \lambda_c$). The bifurcated paths for the asymmetric bifurcation are found from Eq. (AB-3.5) to be all transcritical. The equilibrium solutions in $\lambda - \theta - \phi$ space are depicted in Fig. AB-3.2 and Fig. AB-3.3 for the symmetric and asymmetric bifurcations respectively. Only $n > 0$ and $m > 0$ have been considered here. The discussion

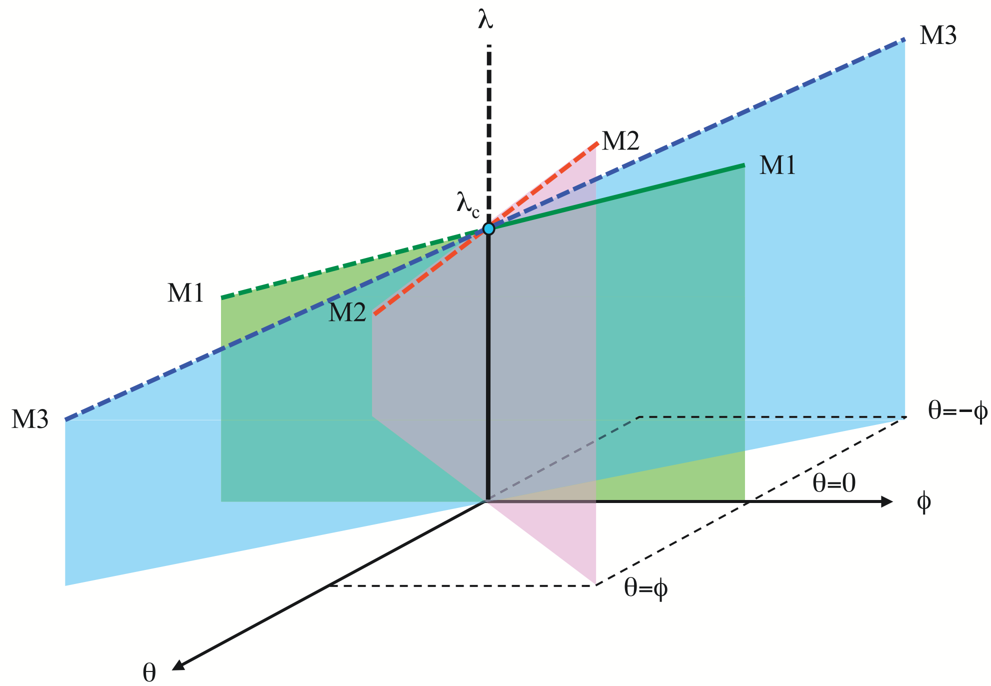

Figure AB-3.3: Bifurcated solutions for asymmetric perfect rigid plate model. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

of $n < 0$ and $m < 0$ will be omitted as completely analogous.

Similar to the discussion of the perfect rigid T model of subsection AB-1, the next issue to be addressed is that of stability of the principal and bifurcated equilibrium paths of the plate given respectively by Eq. (AB-3.4) and Eq. (AB-3.5). Recall that an equilibrium path is stable if it corresponds to a local minimum of the system's potential energy. For the present three degree of freedom system, an equilibrium point $v(\lambda), \theta(\lambda), \phi(\lambda)$ is stable if it corresponds to local minimum of the energy $\mathcal{E}$, which according to Eq. (AA-5.4) requires the matrix $\mathcal{E}_{,\mathbf{uu}}$ to be positive definite:

$$\mathcal{E}_{,\mathbf{uu}} = \begin{bmatrix} \mathcal{E}_{vv}, & \mathcal{E}_{v\theta}, & \mathcal{E}_{,v\phi} \\ \mathcal{E}_{,\theta v}\,, & \mathcal{E}_{,\theta\theta}\,, & \mathcal{E}_{,\theta\phi} \\ \mathcal{E}_{,\phi v}\,, & \mathcal{E}_{,\phi\theta}\,, & \mathcal{E}_{,\phi\phi} \end{bmatrix} =$$

$$\begin{bmatrix} 4E, & 0, & 0 \\ 0, & (\lambda_c - \lambda)L + 2mL^3\phi + nL^4(6\theta^2 + 12\phi\theta - 3\phi^2), & 2mL^3\theta + nL^4(6\theta^2 - 6\phi\theta + 6\phi^2) \\ 0, & 2mL^3\theta + nL^4(6\phi^2 - 6\theta\phi + 6\theta^2), & (\lambda_c - \lambda)L + 2mL^3\phi + nL^4(6\phi^2 + 12\theta\phi - 3\theta^2) \end{bmatrix} \tag{AB-3.6}$$

One can see from Eq. (AB-3.4) that on the principal branch $\mathcal{E}_{,\mathbf{uu}} = \mathrm{diag}[4E, (\lambda_c - \lambda)L, (\lambda_c - \lambda)L]$ and hence the principal solution is stable for $0 \leq \lambda < \lambda_c$ and unstable for $\lambda \geq \lambda_c$.

For the symmetric bifurcation case $(m = 0, n > 0)$ substitution of Eq. (AB-3.5)$_1$ into Eq. (AB-3.6) yields:

$$
\begin{aligned}
\text{For } N1: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & (8/7)(\lambda-\lambda_c)L, & (6/7)(\lambda-\lambda_c)L \\ 0, & (6/7)(\lambda-\lambda_c)L, & (8/7)(\lambda-\lambda_c)L \end{bmatrix}, \\
\text{For } N2: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & 0, & 2(\lambda_c-\lambda)L \\ 0, & 2(\lambda_c-\lambda)L, & 0 \end{bmatrix}, \\
\text{For } N3: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & (3/2)(\lambda-\lambda_c)L, & (\lambda-\lambda_c)L \\ 0, & (\lambda-\lambda_c)L, & 0 \end{bmatrix}, \\
\text{For } N4: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & 0, & (\lambda-\lambda_c)L \\ 0, & (\lambda-\lambda_c)L, & (3/2)(\lambda-\lambda_c)L \end{bmatrix}.
\end{aligned}
\tag{AB-3.7}
$$

From Eq. (AB-3.7) and (AB-3.5) follows that the only positive definite matrix $\mathcal{E}_{,\mathbf{uu}}$ is found for the N1 equilibrium solution. Thus of the four possible equilibrium paths in the symmetric multiple bifurcation only one supercritical path (N1) is stable.

For the asymmetric bifurcation case $(m > 0, n = 0)$, substitution of Eq. (AB-3.5)$_2$ into Eq. (AB-3.6) yields:

$$
\begin{aligned}
\text{For } M1: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & (\lambda-\lambda_c)L, & 0 \\ 0, & 0, & (\lambda-\lambda_c)L \end{bmatrix}, \\
\text{For } M2: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & 0, & (\lambda-\lambda_c)L \\ 0, & (\lambda-\lambda_c)L, & 0 \end{bmatrix}, \\
\text{For } M3: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & 0, & (\lambda_c-\lambda)L \\ 0, & (\lambda_c-\lambda)L, & 0 \end{bmatrix}.
\end{aligned}
\tag{AB-3.8}
$$

From Eq. (AB-3.8) and Eq. (AB-3.5) follows that the only positive definite matrix $\mathcal{E}_{,\mathbf{uu}}$ is found for the supercritical $(\lambda > \lambda_c)$ part of the M1 equilibrium solution.

For both the symmetric $(m = 0, n > 0)$ and asymmetric $(m > 0, n = 0)$ models, the stable and unstable equilibrium paths, denoted by a solid line and a dashed line respectively, are depicted in Fig. AB-3.2 and Fig. AB-3.3. It is interesting to note the exchange of stability of the principal branch at the critical load λ_c exactly as in the rigid T model analyzed in subsection AB-1 (see Fig. AB-1.2 and Fig. AB-1.3).

Finally of interest is the total potential energy $\mathcal{E}$, for a given load parameter λ, associated with each equilibrium path. Using Eq. (AB-3.4) into Eq. (AB-3.2) the potential energy of the principal branch is:

$$\mathcal{E} = -\frac{\lambda^2}{8E} \tag{AB-3.9}$$

For the bifurcated equilibrium branches of the symmetric structure $m = 0, n > 0$ the corresponding potential energies are from Eq. (AB-3.2) and Eq. (AB-3.5):

$$\begin{aligned}
\text{For } N1: \quad \mathcal{E} &= -\frac{\lambda^2}{8E} - \frac{(\lambda - \lambda_c)^2}{14nL^2}, \\
\text{For } N2: \quad \mathcal{E} &= -\frac{\lambda^2}{8E} + \frac{(\lambda - \lambda_c)^2}{18nL^2}, \\
\text{For } N3: \quad \mathcal{E} &= -\frac{\lambda^2}{8E} - \frac{5(\lambda - \lambda_c)^2}{72nL^2}, \\
\text{For } N4: \quad \mathcal{E} &= -\frac{\lambda^2}{8E} - \frac{5(\lambda - \lambda_c)^2}{72nL^2}.
\end{aligned} \tag{AB-3.10}$$

Notice that for $\lambda > \lambda_c$ the equilibrium branch with the least energy is the stable bifurcated branch N1. On the other hand for $\lambda < \lambda_c$ the stable principal equilibrium branch has less energy than the bifurcated subcritical branch. Hence for all values of the load parameter λ, the stable equilibrium path is the one with least energy. This situation is similar to the behavior of the simple perfect rigid T model. Indeed recall from Eq. (AB-1.7) and Eq. (AB-1.8) that for a given λ the stable equilibrium branch always corresponds to the lowest potential energy.

For the bifurcated equilibrium branches of the asymmetric structure $(m > 0, n = 0)$ the corresponding potential energies are from Eq. (AB-3.2) and Eq. (AB-3.5)$_2$

$$\begin{aligned}
\text{For } M1: \quad \mathcal{E}_b &= -\frac{\lambda^2}{8E} - \frac{(\lambda - \lambda_c)^3}{6m^2L^3}, \\
\text{For } M2: \quad \mathcal{E}_b &= -\frac{\lambda^2}{8E} - \frac{(\lambda - \lambda_c)^3}{12m^2L^3}, \\
\text{For } M3: \quad \mathcal{E}_b &= -\frac{\lambda^2}{8E} - \frac{(\lambda - \lambda_c)^3}{12m^2L^3}.
\end{aligned} \tag{AB-3.11}$$

From Eq. (AB-3.11) follows that for $\lambda < \lambda_c$ the minimum potential energy corresponds to the stable principal branch while for $\lambda > \lambda_c$ the minimum potential energy corresponds to the stable bifurcated branch M1. This situation is again similar to the behavior of the asymmetric rigid T model (see Eq. (AB-1.7) and Eq. (AB-1.8)).

AB-4 IMPERFECT SQUARE PLATE MODEL

As in the case of the rigid T model, the introduction of imperfections in the perfect rigid plate model investigated in subsection AB-3 provides a physically realistic model for the determination of the plate's actual equilibrium path in a loading process starting at $\lambda = 0$.

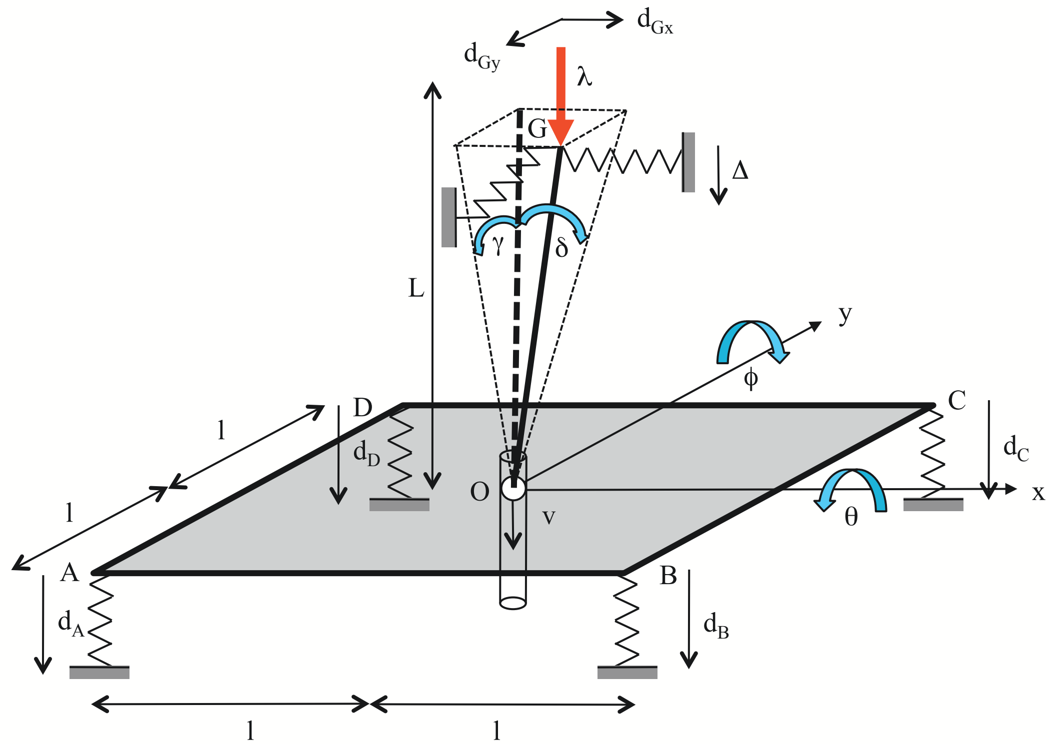

Figure AB-4.1: Imperfect square plate model.

Once again, a geometric type imperfection is considered. The rod OG is no longer normal to the plate but deviates from the ideal normal by the small angles γ and δ in the θ and ϕ directions respectively as shown in Fig. AB-4.1. All the remaining elements of the model (dimensions, stiffness of springs, etc.) remain the same as for the perfect structure shown in Fig. AB-3.1.

The kinematics of the imperfect rigid plate model are the same as for its perfect counterpart (see Eq. (AB-3.1)), save for the vertical displacement Δ of point G, which, due to the small values of the angles $\gamma,\ \delta,\ \theta,\ \phi$, is now given by

$$\Delta = v + L[(1-\sin^2\gamma - \sin^2\delta)^{1/2} - (1-\sin^2(\theta+\gamma) - \sin^2(\phi+\delta))^{1/2}] \approx v + L(\frac{\theta^2+\phi^2}{2} + \theta\gamma + \phi\delta), \quad \text{(AB-4.1)}$$

where, as in the previous sections, only terms up to the second order with respect to the small angles $\theta,\ \phi,\ \gamma,\ \delta$ are kept in the kinematic relations.

The total potential energy $\overline{\mathcal{E}}$ for the imperfect plate is:

$$\overline{\mathcal{E}}(v,\theta,\phi\lambda\gamma,\delta) = 2E[v^2 + l^2(\theta^2+\phi^2)] + \frac{kL^2}{2}(\theta^2+\phi^2) + \frac{mL^3}{3}(3\theta^2\phi+\phi^3)+ \\ +\frac{nL^4}{4}(2\theta^4+8\theta^3\phi-6\theta^2\phi^2+8\theta\phi^3+2\phi^4) - \lambda[v+\frac{L}{2}(\theta^2+\phi^2+2\gamma\theta+2\delta\phi)]. \tag{AB-4.2}$$

As expected, for the case of zero values for the imperfections, the imperfect energy yields back its perfect counterpart, i.e. $\overline{\mathcal{E}}(v,\theta,\phi,\lambda,0,0) = \mathcal{E}(v,\theta,\phi,\lambda)$ (see (AB-3.2)).

Extremizing $\overline{\mathcal{E}}$ with respect to its degrees of freedom, one obtains the following equilibrium equations:

$$\begin{aligned} &\overline{\mathcal{E}}_{,v} = 4Ev - \lambda = 0, \\ &\overline{\mathcal{E}}_{,\theta} = -\lambda L\gamma + (4El^2+kL^2-\lambda L)\theta + 2mL^3\theta\phi + nL^4(2\theta^3+6\theta^2\phi-3\theta\phi^2+2\phi^3) = 0, \\ &\overline{\mathcal{E}}_{,\phi} = -\lambda L\delta + (4El^2+kL^2-\lambda L)\phi + mL^3(\phi^2+\theta^2) + nL^4(2\phi^3+6\phi^2\theta-3\phi\theta^2+2\theta^3) = 0. \end{aligned} \tag{AB-4.3}$$

Recalling from Eq. (AB-3.5) the definition of the critical load λ_c, the solution of the above system is found to be:

$$\begin{aligned} &v = \lambda/4E \\ &\lambda = [\lambda_c\theta + nL^3(2\theta^3+6\theta^2\phi-3\theta\phi^2+2\phi^3)]/(\theta+\gamma) \qquad m=0,\ n\neq 0, \\ &\lambda = [\lambda_c\phi + nL^3(2\phi^3+6\phi^2\theta-3\phi\theta^2+2\theta^3)]/(\phi+\delta) \\ \\ &v = \lambda/4E \\ &\lambda = [\lambda_c\theta + 2mL^2\theta\phi]/(\theta+\gamma) \qquad m\neq 0,\ n=0. \\ &\lambda = [\lambda_c\phi + mL^2(\phi^2+\theta^2)]/(\phi+\delta) \end{aligned} \tag{AB-4.4}$$

Determining the imperfect structure's equilibrium paths is a rather cumbersome matter that will not be pursued here. Suffices to say that of all possible imperfections with a given amplitude $\varepsilon \equiv (\delta^2+\gamma^2)^{1/2}$, the imperfection with the "*worst* " shape, i.e. the one with the maximum load drop from the critical load, is the one with equilibrium paths that are on the the N2 plane for the symmetric case or on the M1 plane for the asymmetric one (see Fig. AB-1.2 and Fig. AB-1.3).

AB-5 TWO-BAR PLANAR TRUSS MODEL

The perfect rigid T and plate models discussed in the previous subsections all share the common feature of a trivial principal solution with a linear force-displacement response. Many applications of interest have non-trivial principal solutions with non-monotonic force-displacement responses, which exhibit limit loads. In these applications bifurcated equilibrium paths can also emerge from the non-monotonic principal solutions. The discrete two-bar plan truss model presented below is an illustrative example of structures with a non-trivial principal solution that exhibit snap-trough instabilities (associated with limit loads) as well as buckling instabilities (associated with bifurcations).

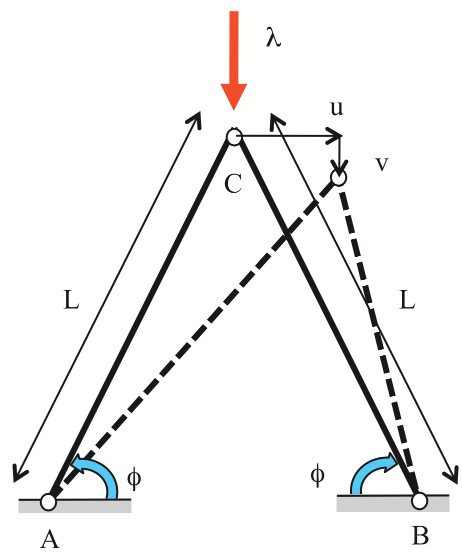

Figure AB-5.1: Two-bar truss model.

The two-bar planar truss shown in FIG. AB-5.1 consists of two elastic bars, AC (bar 1) and BC (bar 2) each of cross-sectional area A, Young's modulus E and initial length L. A vertical load λ is applied at node C which can move by u and v respectively along the horizontal and vertical directions. The response of each bar is linearly elastic $(\sigma = E\varepsilon)$ where σ and ε are the stress and strain in the bar, which deforms only in the axial direction (no bending occurs in the bars).

The axial strain in each bar is given in terms of its final (ℓ) and initial (L) lengths by

$$\varepsilon = \frac{1}{2}\left[\left(\frac{\ell}{L}\right)^2 - 1\right]. \tag{AB-5.1}$$

From the geometry of the deformed configuration, one can see that the deformed lengths of the two bars are

$$\begin{aligned} (\ell_1)^2 &= (L\cos\phi + u)^2 + (L\sin\phi - v)^2, \\ (\ell_2)^2 &= (L\cos\phi - u)^2 + (L\sin\phi - v)^2. \end{aligned} \tag{AB-5.2}$$

and hence the potential energy $\mathcal{E}$ of the truss, which consists of the elastic energy stored in the two bars plus the potential energy of the external load, takes the form

$$\mathcal{E}(u, v; \lambda) = \frac{1}{2} EAL[(\varepsilon_1)^2 + (\varepsilon_2)^2] - \lambda v. \tag{AB-5.3}$$

By extremizing $\mathcal{E}$ with respect to its degrees of freedom, the two equilibrium equations are

$$\begin{aligned} \mathcal{E}_{,u} &= \frac{EA}{L}[\varepsilon_1(u + L\cos\phi) + \varepsilon_2(u - L\cos\phi)] = 0, \\ \mathcal{E}_{,v} &= \frac{EA}{L}[\varepsilon_1(v - L\sin\phi) + \varepsilon_2(v - L\sin\phi)] - \lambda = 0. \end{aligned} \tag{AB-5.4}$$

The principal solution of this model, i.e. the one starting with zero displacements at zero load, is the symmetric solution with horizontal displacement $\overset{0}{u}(\lambda) = 0$, since as one can easily check, Eq. (AB-5.4) is always satisfied for $u = 0$. However this principal solution has a vertical displacement $\overset{0}{v}(\lambda)$ which is a non-monotonic function of λ, given from Eq. (AB-5.4)

$$\frac{EA}{L^3} \overset{0}{v}(\lambda)[\overset{0}{v}(\lambda) - 2L\sin\phi][\overset{0}{v}(\lambda) - L\sin\phi] - \lambda = 0. \tag{AB-5.5}$$

The dimensionless displacement $\overset{0}{v}(\lambda)/L$ versus dimensionless load $\lambda/(EA)$ for the principal solution is given by the cubic in $\overset{0}{v}/L$ equation Eq. (AB-5.5) and plotted in Fig. AB-5.2.

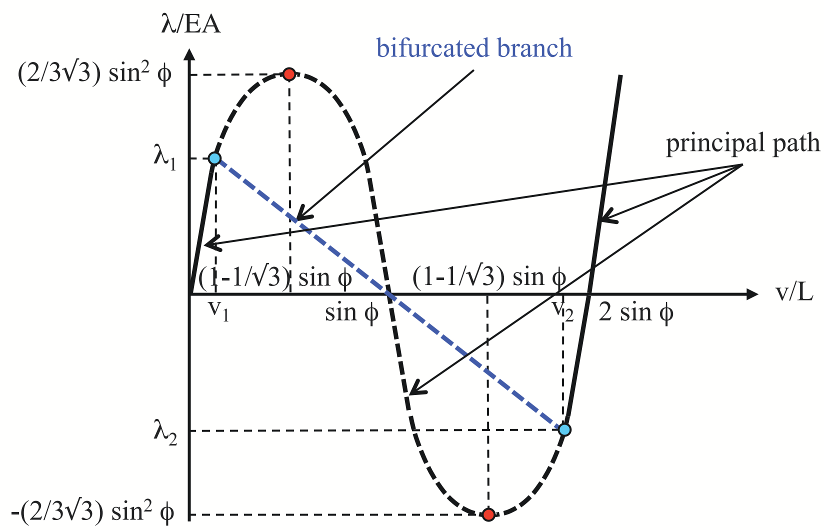

Figure AB-5.2: Principal and bifurcated solutions of the two–bar truss model in dimensionless load λ/EA vs. dimensionless vertical displacement v/L space. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

In addition to the above described principal solution, the structure also admits a bifurcated equilibrium branch with $u(\lambda) \neq 0$ which can also be found by solving Eq. (AB-5.4). Indeed by using Eq. (AB-5.1) and Eq. (AB-5.2) into Eq. (AB-5.4), one obtains for the bifurcated equilibrium branch

$$\begin{aligned} & u^2 + v^2 - 2Lv\sin\phi + 2L^2\cos^2\phi = 0, \\ & \frac{EA}{L^3}(u^2 + v^2 - 2vL\sin\phi)(v - L\sin\phi) = \lambda. \end{aligned} \tag{AB-5.6}$$

Notice that the equilibrium equation Eq. (AB-5.6)$_1$ can be restated as

$$u^2 + (v - L\sin\phi)^2 = L^2(3\sin^2\phi - 2), \tag{AB-5.7}$$

which admits a real solution for $\sin\phi > (2/3)^{\frac{1}{2}}$, i.e. for angles $\phi > 54.7°$. In this case the bifurcated solution is a circle in the dimensionless displacements (u/L) vs (v/L) plane with center at $v/L = \sin\phi$ and radius $(3\sin^2\phi - 2)^{1/2}$ as shown in Fig. AB-5.3.

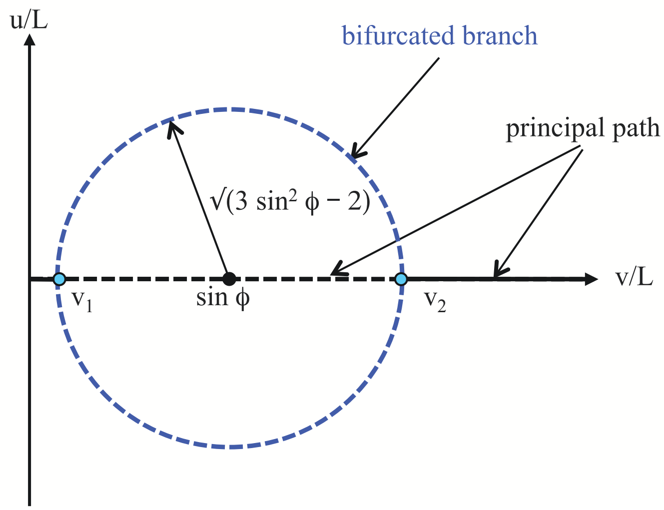

Figure AB-5.3: Principal and bifurcated solutions of the two–bar truss model in dimensionless horizontal displacement v/L vs. dimensionless vertical displacement v/L space. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

The bifurcated solution in the (λ/EA) vs (v/L) plane is found, by substituting Eq. (AB-5.7) into the equilibrium equation Eq. (AB-5.6)$_2$, to be

$$\frac{\lambda}{EA} = 2\cos^2\phi(\sin\phi - \frac{v}{L}), \tag{AB-5.8}$$

which appears as the straight line with negative slope in Fig. AB-5.2.

Notice that the straight line described by Eq. (AB-5.8) is the projection of the bifurcated equilibrium path given by Eq. (AB-5.6) into the $(\lambda/(EA))$ vs (v/L) plane, while the projection of the same bifurcated equilibrium path in the (u/L) vs (v/L) space is the circle depicted in Fig. AB-5.3. The bifurcated solution connects the with the principal branch of the two points depicted by a small blue circle in Fig. AB-5.2 and Fig. AB-5.3. The points have coordinates

$$\begin{aligned} & (\lambda_1/(EA), v_1/L) = (2\cos^2\phi(3\sin^2\phi - 2)^{\frac{1}{2}}, \sin\phi - (3\sin^2\phi - 2)^{\frac{1}{2}}) \\ & (\lambda_2/(EA), v_2/L) = (-2\cos^2\phi(3\sin^2\phi - 2)^{\frac{1}{2}}, \sin\phi + (3\sin^2\phi - 2)^{\frac{1}{2}}). \end{aligned} \tag{AB-5.9}$$

Having found the principal as well as the bifurcated equilibrium paths of the two-bar planar truss model, the next topic to be addressed pertains to the stability of these solutions, which is given by analyzing the matrix $\mathcal{E}_{,\mathbf{uu}}$, found with the help of Eq. (AB-5.4) to be

$$\mathcal{E}_{,\mathbf{uu}} = \begin{bmatrix} \mathcal{E}_{,uu} & \mathcal{E}_{,uv} \\ \mathcal{E}_{,vu} & \mathcal{E}_{,vv} \end{bmatrix}. \tag{AB-5.10}$$

By evaluating $\mathcal{E}_{,\mathbf{uu}}$ on the principal solution $\overset{0}{u}(\lambda) = 0$ Eq. (AB-5.10) yields principal solution.

$$\mathcal{E}_{,\mathbf{uu}} = \frac{EA}{L}\begin{bmatrix} (\overset{0}{v}/L)^2 - 2(\overset{0}{v}/L)\sin\phi + 2\cos^2\phi, & 0 \\ 0, & 3[(\overset{0}{v}/L)^2 - 2(\overset{0}{v}/L)\sin\phi + \frac{2}{3}\sin^2\phi] \end{bmatrix} \tag{AB-5.11}$$

The roots of $\mathcal{E}_{,uu}$ are $\overset{0}{v}/L = \sin\phi \pm (3\sin^2\phi - 2)^{1/2}$ which correspond to the displacements $(\overset{0}{v}/L)_1$ and $(\overset{0}{v}/L)_2$ given in Eq. (AB-5.9) which are the displacements corresponding to the start and finish of the bifurcated equilibrium branch as seen in Fig. AB-5.2. The roots of $\mathcal{E}_{,vv}$ are $\overset{0}{v}/L = \sin\phi(1 \pm 1/\sqrt{3})$ which are the displacements $(\overset{0}{v}/L)_{max}$ and $(\overset{0}{v}/L)_{min}$ corresponding to the maximum and minimum loads of the principal solution as seen in Fig. AB-5.2. Hence in the absence of a bifurcated solution, i.e. for $\phi \leq 54.7°$. $\mathcal{E}_{,uu} > 0$ and the stability of the principal solution is determined solely by the sign of $\mathcal{E}_{,vv}$ which is < 0 in the interval of $\overset{0}{v}/L \in [(\overset{0}{v}/L)_{max}, (\overset{0}{v}/L)_{min}]$ and > 0 outside. Consequently, in the absence of a bifurcated solution, the principal branch is stable in the two load increasing branches, i.e. from zero load up to the maximum load and from the minimum load and beyond, while it is unstable in the load decreasing branch.

In the presence of a bifurcation ($\phi > 54.7°$) the principal solution is stable when both $\mathcal{E}_{,uu} > 0$ and $\mathcal{E}_{,vv} > 0$. Notice that the roots of $\mathcal{E}_{,uu}$ from Eq. (AB-5.11) are $(\overset{0}{v}/L) = \sin\phi \pm (3\sin^2\phi - 2)^{1/2}$, which are the displacements $(\overset{0}{v}/L)_1$ and $(\overset{0}{v}/L)_2$ where the bifurcated path connects with the principal solution. Since $\mathcal{E}_{,uu} < 0$ in the interval $\overset{0}{v}/L \in [(\overset{0}{v}/L)_1, (\overset{0}{v}/L)_2]$, the principal solution, in the case of existence of a bifurcated path, as seen in FIG. AB-5.2, is unstable (dashed line) for those displacements $(\overset{0}{v}/L)$ for which a bifurcated solution exists and is stable (solid line) otherwise. It should be noted at this point that in plotting Fig. AB-5.2 it was tacitly assumed that $(\overset{0}{v}/L)_1 < (\overset{0}{v}/L)_{max}$, or equivalently from Eq. (AB-5.9) $\sin^2\phi > 3/4$, ($\phi > 60°$), in which case bifurcation occurs prior to reaching maximum load, as load increases away from zero. For $60° < \phi < 54.7°$, the first bifurcation occurs past the maximum load on the descending part of the principal branch and ends prior to reaching the minimum load, also on the descending part of the principal branch.

Finally, the stability of the bifurcated path is investigated. Evaluating $\mathcal{E}_{,\mathbf{uu}}$ on the bifurcated equilibrium branch, one obtains from Eq. (AB-5.10) with the help of Eq. (AB-5.7)

$$\mathcal{E}_{,\mathbf{uu}} = \frac{EA}{L}\begin{bmatrix} 2(u/L)^2 & 2(u/L)[(v/L) - \sin\phi] \\ 2(u/L)[(v/L) - \sin\phi] & 2[-(u/L)^2 + 4\sin^2\phi - 3] \end{bmatrix}, \tag{AB-5.12}$$

The determinant of the above matrix evaluated on the bifurcated equilibrium path Eq. (AB-5.7) is found to be

$$\text{Det } \mathcal{E}_{,\mathbf{uu}} = \frac{4EA}{L}\left(\frac{u}{L}\right)^2 [\sin^2\phi - 1] < 0, \qquad \text{(AB-5.13)}$$

thus establishing that the entire bifurcated equilibrium path is unstable.

AC BIFURCATION AND STABILITY - LSK ASYMPTOTICS FOR ELASTIC CONTINUA

In the first two sections of this chapter, the notions of stability and bifurcation of equilibrium solutions in nonlinear elastic solids have been introduced and illustrated by simple examples, which admitted analytical solutions. Unfortunately this is not the case for more realistic models, discrete as well as continuum.

To this end, an asymptotic method comes to rescue. As it turns out, a powerful asymptotic technique, termed "*Lyapunov – Schmidt – Koiter*" method (LSK) can be applied to track the equilibrium solutions of perfect or imperfect systems near critical points and check their stability. This method is presented in this section for the case of continuum, conservative elastic systems (structures or solids).

AC-1 FUNCTIONALS AND THEIR DERIVATIVES

The treatment of the bifurcation, post-bifurcation and imperfection sensitivity behavior of continuous elastic systems requires some elements from the calculus of variations. The purpose of this brief section is not the presentation of a mathematically oriented introduction to the subject. The goal here is to illustrate the technique for calculating the required derivatives of the system's potential energy involved in the corresponding bifurcation and stability analyses.

For an elastic system, its potential energy is given by a real valued functional $\mathcal{E}(u, \lambda)$, where the term "*functional* " is used to indicate that the function's independent variable is itself a function defined at each point of the elastic system in question. For elastic systems, the independent variable is $u \equiv \mathbf{u}(\mathbf{x})$, typically the displacement field $\mathbf{u}$ at point $\mathbf{x}$ of the solid in question. The independent variable u is a scalar or vector valued function of position in the system depending on the application. The scalar parameter λ, usually termed the "*load parameter*", controls the externally applied loads or displacements to the system. In general, a continuous structure has (countably) infinite possibilities to deform and hence the displacement field $u \in U$, with U some appropriate, infinite dimensional vector space. Often in applications U is the simplest possible such space, namely a Hilbert space (or a Cartesian product of such spaces). For the case of a discrete structure $U = \mathbb{R}^n$ where n is the total number of the degrees of freedom. Henceforth, it will be assumed that U possesses an inner product. If $u_1, u_2 \in U$ their inner product is denoted by (u_1, u_2). The norm of an element $u \in U$ will be the inner product induced one, namely $\| u \| \equiv (u, u)^{1/2}$.

Of interest is the notion of the derivative of a real valued functional, say $\mathcal{E}(u)$, with respect to its argument u. To this end one defines the first derivative of $\mathcal{E}$ with respect to u, denoted by $\mathcal{E}_{,u}$, to be a linear operator on U, i.e. a linear function assigning a real scalar $\mathcal{E}_{,u}\, \delta u$ to every element $\delta u \in U$. The mathematical definition for this derivative, also termed the "*Frechet derivative*" is:

$$\lim_{\|\delta u\| \to 0} \mid \mathcal{E}(u + \delta u) - \mathcal{E}(u) - \mathcal{E}_{,u}\, \delta u \mid / \| \delta u \| = 0 \qquad \text{(AC-1.1)}$$

Since the δu in the above definition is arbitrary, one can fix in U the direction of δu and consider in the definition Eq. (AC-1.1) the special case $\delta u = \epsilon v$ with $\| v \| = 1$ but $\epsilon \to 0$. Thus, if $\mathcal{E}_{,u}$ exists it will satisfy:

$$\lim_{\epsilon \to 0} \mid \mathcal{E}(u + \epsilon v) - \mathcal{E}(u) - \epsilon \mathcal{E}_{,u}\, v \mid / \epsilon = 0 \qquad \text{(AC-1.2)}$$

Recalling the standard definition for the partial derivative, the above equation can alternately be rewritten:

$$\mathcal{E}_{,u}\, v = [\partial \mathcal{E}(u + \epsilon v) / \partial \epsilon]_{\epsilon = 0} \qquad \text{(AC-1.3)}$$

thus providing the practical method for the computation of the derivative $\mathcal{E}_{,u}$ of $\mathcal{E}$. Strictly speaking the existence of the special (weaker) derivative defined in Eq. (AC-1.2) or Eq. (AC-

1.3), also termed the "*Gateau derivative*" of the functional, does not imply the existence of the stronger Frechet derivative defined in Eq. (AC-1.1). In all the applications of interest, however, such pathological cases will not arise and the computationally simpler Gateau definition will be used for calculations.

One can in a similar fashion define the derivative of the linear operator $\mathcal{E}_{,u}$, considered as a function of u, with respect to u. The result $\mathcal{E}_{,uu}$ is a symmetric, bilinear operator, operating on arbitrary elements $v, w \in U$ which, in analogy to Eq. (AC-1.3), is given by:

$$(\mathcal{E}_{,uu}\, v)w = (\mathcal{E}_{,uu}\, w)v = [\partial^2 \mathcal{E}(u + \epsilon v + \zeta w)/\partial\epsilon\partial\zeta]_{\epsilon=\zeta=0} \tag{AC-1.4}$$

The p^{th} order derivative of $\mathcal{E}(u)$ is a completely symmetric, with respect to any two arguments, p-linear operator, computed by a straightforward generalization of Eq. (AC-1.4). It should be noticed for future use that, like in the case of a real valued function of a real argument, one can have under suitable conditions a converging Taylor series expansion for the functional $\mathcal{E}(u)$, i.e.:

$$\mathcal{E}(u + \delta u) = \mathcal{E}(u) + \frac{1}{1!}\mathcal{E}_{,u}\, \delta u + \frac{1}{2!}(\mathcal{E}_{,uu}\, \delta u)\delta u + \frac{1}{3!}((\mathcal{E}_{,uuu}\, \delta u)\delta u)\delta u + ... \tag{AC-1.5}$$

It is perhaps useful to illustrate the above described definitions by applying them to a concrete example, namely the functional:

$$\mathcal{E}(u) = \int_0^1 [\exp(w) + (2 + \cos(x))w^2 + 3(v_{,x}^2 + v^2)]dx \tag{AC-1.6}$$

In this example $u \equiv (v(x), w(x))$. Moreover, for the above integral to make sense v and w are required to belong to some appropriate functional space. Although from the mathematical standpoint it is important to provide the spaces to which the displacement functions belong, in the engineering applications considered in this work it is tacitly assumed that all the integrals are finite and that all the displacement functions are adequately smooth.

Employing Eq. (AC-1.3), the first derivative of $\mathcal{E}$ is found to be:

$$\begin{aligned} \mathcal{E}_{,u}\, u_1 &= \frac{\partial}{\partial \epsilon_1}\Bigg\{\int_0^1 [\exp(w + \epsilon_1 w_1) + (2 + \cos(x))(w + \epsilon_1 w_1)^2 \\ &\quad + 3((v_{,x} + \epsilon_1 v_{1,x})^2 + (v + \epsilon_1 v_1)^2)]dx\Bigg\}_{\epsilon_1 = 0} \\ &= \int_0^1 [\exp(w)w_1 + 2(2 + \cos(x))ww_1 + 6(v_{,x}\, v_{1,x} + vv_1)]dx \end{aligned} \tag{AC-1.7}$$

where $u_1 = (v_1, w_1)$ and v_1, w_1 are arbitrary functions belonging to the same spaces as v, w respectively on which the linear operator $\mathcal{E}_{,u}$ acts. The linearity of $\mathcal{E}_{,u}$ is easily checked.

Similarly, from Eq. (AC-1.4) the second derivative of $\mathcal{E}$ is:

$$
\begin{aligned}
(\mathcal{E}_{,uu}\, u_1)u_2 \quad &= \frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}\Bigg\{\int_0^1 [\exp(w+\epsilon_1 w_1+\epsilon_2 w_2)+(2+\cos(x))(w+\epsilon_1 w_1+\epsilon_2 w_2)^2 \\
&+3((v_{,x}+\epsilon_1 v_{1,x}+\epsilon_2 v_{2,x})^2+(v+\epsilon_1 v_1+\epsilon_2 v_2)^2)]dx\Bigg\}_{\epsilon_1=\epsilon_2=0} \qquad \text{(AC-1.8)} \\
&= \int_0^1 [\exp(w)w_1 w_2 + 2(2+\cos(x))w_1 w_2 + 6(v_{1,x}v_{2,x}+v_1 v_2)]dx
\end{aligned}
$$

It is not difficult to verify that $(\mathcal{E}_{,uu}\, u_1)u_2$ is linear with respect to both arguments u_1 and u_2 as well as symmetric i.e. $(\mathcal{E}_{,uu}\, u_1)u_2 = (\mathcal{E}_{,uu}\, u_2)u_1$.

One can continue in a similar fashion with the higher order of derivatives of $\mathcal{E}$. Thus the third derivative of $\mathcal{E}$ is a completely symmetric trilinear operator namely:

$$
((\mathcal{E}_{,uuu}\, u_1)u_2)u_3 = \int_0^1 [\exp(w)w_1 w_2 w_3]dx \qquad \text{(AC-1.9)}
$$

In general, the p^{th} order derivative of $\mathcal{E}$ can be computed by:

$$
(((\mathcal{E}_{,uuu...u}\,)u_1)u_2...)u_p = \frac{\partial^p}{\partial\epsilon_1\partial\epsilon_2...\partial\epsilon_p}\Big[\mathcal{E}\big(u+\sum_{i=1}^{p}\epsilon_i u_i\big)\Big]_{\epsilon_1=\epsilon_2=...\epsilon_p=0} \qquad \text{(AC-1.10)}
$$

For the special case where the displacement space is finite dimensional i.e. $U=\mathbb{R}^n$ and noting that $\{\mathbf{e}_i\}_{i=1}^n$ is the corresponding orthonormal basis ($u=\sum_{i=1}^n u_i\ \mathbf{e}_i$), from Eq. (AC-1.3) the first derivative of $\mathcal{E}$ is the vector:

$$
\mathcal{E}_{,u} = \sum_{i=1}^{n}\frac{\partial\mathcal{E}}{\partial u_i}\mathbf{e}_i \qquad \left(\mathcal{E}_{,u}\, v = \sum_{i=1}^{n}\frac{\partial\mathcal{E}}{\partial u_i}v_i\right) \qquad \text{(AC-1.11)}
$$

From Eq. (AC-1.4) the second derivative of $\mathcal{E}$ is the symmetric matrix:

$$
\mathcal{E}_{,uu} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\mathcal{E}}{\partial u_i\partial u_j}\mathbf{e}_i\mathbf{e}_j \qquad \left((\mathcal{E}_{,uu}\, v)w = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\mathcal{E}}{\partial u_i\partial u_j}v_i w_j\right) \qquad \text{(AC-1.12)}
$$

In general the p^{th} derivative of $\mathcal{E}$ is the symmetric rank p tensor:

$$
\mathcal{E}_{,uu..u} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^p\mathcal{E}}{\partial u_i\partial u_j...\partial u_k}\mathbf{e}_i\mathbf{e}_j...\mathbf{e}_k \qquad \text{(AC-1.13)}
$$

AC-2 CRITICAL POINTS: LIMIT VERSUS BIFURCATION POINTS

As discussed in the previous section, an elastic system is completely described by its potential energy $\mathcal{E}(u,\lambda)$, where the admissible displacement $u \in U$ and where λ is the load parameter. The potential energy is arbitrarily set to zero for zero displacements:

$$\mathcal{E}(0,\lambda) = 0 \tag{AC-2.1}$$

The system evolves from its initial stress-free equilibrium configuration at which $\lambda = 0$ and $u = 0$, to a loaded configuration with $\lambda \neq 0,\ u \neq 0$. For a given load level, the corresponding equilibrium equations are found by extremizing $\mathcal{E}$ with respect to u[3], namely:

$$\mathcal{E}_{,u}(u,\lambda)\delta u = 0, \quad \forall\ \delta u \in U \tag{AC-2.2}$$

For all physically realistic problems, for a given load parameter λ one expects the solution to the equilibrium equation Eq. (AC-2.2) to be unique in some neighborhood of $(u,\lambda) = (0,0)$. This solution denoted by $\overset{0}{u}(\lambda)$ and termed the "*principal branch*", in addition to identically satisfying the equilibrium equation for all values of λ, is the only equilibrium branch that passes trough the origin $(u,\lambda) = (0,0)$, i.e.

$$\mathcal{E}_{,u}(\overset{0}{u}(\lambda),\lambda)\delta u = 0, \quad \overset{0}{u}(0) = 0 \tag{AC-2.3}$$

One further assumes that for a physically meaningful problem the principal equilibrium solution $\overset{0}{u}(\lambda)$ has to be stable in some neighborhood of $\lambda = 0$, which implies the existence of a positive number $\overset{0}{\beta}(\lambda)$[4] such that:

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u \geq \overset{0}{\beta}(\lambda) \parallel \delta u \parallel^2, \quad \overset{0}{\beta}(\lambda) > 0 \tag{AC-2.4}$$

A symmetric operator satisfying the above property is termed in mathematics a "*strongly elliptic*" operator. Recalling that for the finite dimensional case, strong ellipticity coincides with positive definiteness of the matrix corresponding to the operator in question, by abuse of language a symmetric operator satisfying the above stability condition will subsequently be called "*positive definite*". It will further be assumed that the quantity $\overset{0}{\beta}(\lambda)$ is the minimum eigenvalue of the stability operator $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)$ and satisfies

$$\overset{0}{\beta}(\lambda) = \min_{\|\delta u\|=1} [(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u]. \tag{AC-2.5}$$

This eigenvalue will play an important role in the subsequent stability investigations.

Assuming that $\overset{0}{u}(\lambda)$ is an adequately smooth function of λ, one obtains by differentiating the equilibrium equation Eq. (AC-2.3) with respect to λ:

$$\big[\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)(d\overset{0}{u}/d\lambda) + \mathcal{E}_{,u\lambda}(\overset{0}{u}(\lambda),\lambda)\big]\delta u = 0 \tag{AC-2.6}$$

[3]NOTE: From here and subsequently δu denotes an arbitrary element of the space of admissible functions U.

[4]NOTE: From here on all quantities associated with the principal solution will be surmounted by a $\overset{0}{(\ )}$.

For as long as the stability operator $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda)$ is positive definite, it is also invertible with inverse $(\mathcal{E}_{,uu})^{-1}$. Hence, equation Eq. (AC-2.6) is solvable and produces a unique solution $d\overset{0}{u}/d\lambda$. This way, by starting at load $\lambda = 0$, one can visualize a constructive method for finding the equilibrium path $\overset{0}{u}(\lambda)$. Assuming that at a given load λ the solution $\overset{0}{u}(\lambda)$ is known, then at $\lambda + \Delta\lambda$ Eq. (AC-2.6) gives $\overset{0}{u}(\lambda + \Delta\lambda) \approx \overset{0}{u}(\lambda) + (d\overset{0}{u}/d\lambda)\Delta\lambda$ since the existence of $d\overset{0}{u}/d\lambda = -(\mathcal{E}_{,uu})^{-1}\mathcal{E}_{,u\lambda}$ is guaranteed by Eq. (AC-2.4). This is exactly the approach that forms the basis for most numerical algorithms (variations of the incremental Newton - Raphson method) that calculate the equilibrium paths in nonlinear elastic systems.

Suppose now that as the load λ increases (without loss of generality it is tacitly assumed that $\lambda \geq 0$), it reaches a value λ_c termed "*critical load*" for which the stability operator loses its positive definiteness and becomes singular. More specifically there exist a unit vector $\overset{1}{u} \in U$ such that:

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c), \lambda_c)\overset{1}{u})\delta u = 0, \quad \| \overset{1}{u} \| = 1 \tag{AC-2.7}$$

At this point in the interest of simplicity it will be assumed that the eigenmode $\overset{1}{u}$ is unique (up to sign), a condition which will be relaxed subsequently when structures with multiple bifurcation points are considered. Under this assumption $\overset{1}{u}$ is the only direction in which $\mathcal{E}^c_{,uu} \equiv \mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c), \lambda_c)$ loses its positive definiteness while in all other directions δv of the space U, the operator $\mathcal{E}^c_{,uu}$ continues to be positive definite. Hence, it is assumed that a constant $c > 0$ exists such that:

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c), \lambda_c)\delta v)\delta v \geq c \| \delta v \|^2, \quad c > 0, \quad \forall \delta v \in \mathcal{N}^{\perp} \tag{AC-2.8}$$

where the set $\mathcal{N} \equiv \{u \in U \mid u = \alpha\overset{1}{u}, \ \forall \alpha \in \mathbb{R}\}$ is called the "*null space*" of the stability operator $\mathcal{E}^c_{,uu}$ and is defined as the set containing all elements $u \in U$ for which the expression $((\mathcal{E}^c_{,uu})u)u = 0$. The set $\mathcal{N}^{\perp}$ is called the "*orthogonal complement of $\mathcal{N}$ with respect to U*" and is the subset of all elements δv [5] of U whose projection on $\overset{1}{u}$ is zero, i.e. $\mathcal{N}^{\perp} \equiv \{v \in U | (v, \overset{1}{u}) = 0\}$. Thus any element $u \in U$ can be uniquely decomposed into a sum of two parts, one in $\mathcal{N}$ and the other in $\mathcal{N}^{\perp}$ ($\mathcal{N} \oplus \mathcal{N}^{\perp} = U$). From continuity, it is expected that for $0 \leq \lambda < \lambda_c$, the stability operator $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda)$ is strictly positive definite, i.e. satisfies Eq. (AC-2.4).

To investigate the equilibrium around the critical point (u_c, λ_c), where $u_c \equiv \overset{0}{u}(\lambda_c)$, one employs the "*Lyapunov – Schmidt – Koiter*" decomposition method. According to this method, illustrated in Fig. AC-2.1, the increment of the displacement $u - u_c$ due to an increment in the load $\Delta\lambda \equiv \lambda - \lambda_c$ is decomposed in two components: One component $\xi\overset{1}{u}$ is on the null space $\mathcal{N}$ of $\mathcal{E}^c_{,uu}$ and the other component v is in $\mathcal{N}^{\perp}$, i.e. orthogonal to the first, namely:

$$u = u_c + \xi\overset{1}{u} + v, \quad v \in \mathcal{N}^{\perp}, \quad \xi \in \mathbb{R} \tag{AC-2.9}$$

Since the unknown displacement u is in essence replaced by the equivalent pair (ξ, v), the equilibrium equation Eq. (AC-2.2) can also be replaced by the following two ones: An

[5] NOTE: Here and subsequently δv denotes any element of $\mathcal{N}^{\perp}$.

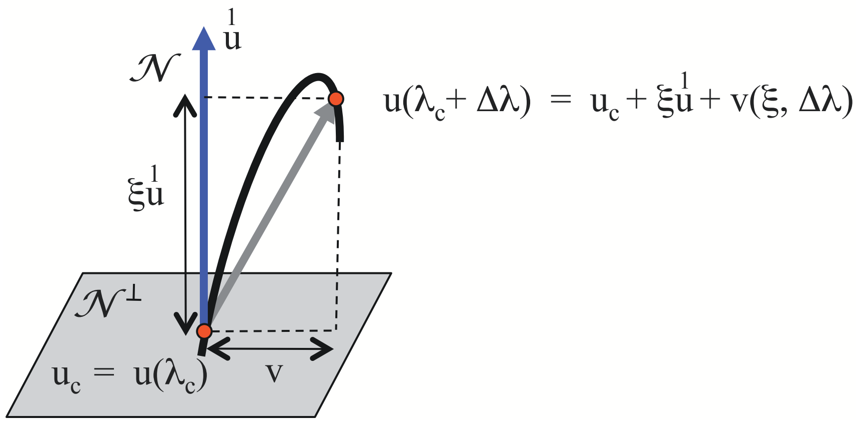

Figure AC-2.1: Schematics of the Lyapunov–Schmidt-Koiter (LSK) decomposition.

equilibrium equation in $\mathcal{N}^\perp$, obtained by extremizing $\mathcal{E}$ with respect to v:

$$\mathcal{E}_{,v}\,\delta v = 0 \implies \mathcal{E}_{,u}\,(u_c + \xi\overset{1}{u} + v, \lambda_c + \Delta\lambda)\delta v = 0, \quad \forall \delta v \in \mathcal{N}^\perp, \tag{AC-2.10}$$

and an equilibrium equation in $\mathcal{N}$, obtained by extremizing $\mathcal{E}$ with respect to ξ:

$$\mathcal{E}_{,\xi} = 0 \implies \mathcal{E}_{,u}\,(u_c + \xi\overset{1}{u} + v, \lambda_c + \Delta\lambda)\overset{1}{u} = 0. \tag{AC-2.11}$$

Both equilibrium equations will be expanded about the critical point (u_c, λ_c). Starting with Eq. (AC-2.10), one first observes that this equation will provide v as a function of ξ and $\Delta\lambda$. Without loss of generality it can be assumed that $v(\xi, \Delta\lambda)$ defined in Eq. (AC-2.9) has a regular expansion about $(\xi, \Delta\lambda) = (0, 0)$:

$$v(\xi, \Delta\lambda) = \xi v_\xi + \Delta\lambda v_\lambda + \frac{1}{2!}[\xi^2 v_{\xi\xi} + 2\xi\Delta\lambda v_{\xi\lambda} + (\Delta\lambda)^2 v_{\lambda\lambda}] + ... \tag{AC-2.12}$$

Upon substitution of Eq. (AC-2.12) into the equilibrium equation Eq. (AC-2.10) and expansion of the result about $(\xi, \Delta\lambda) = (0, 0)$, one obtains the following results: the $O(1)$ term gives $\mathcal{E}_{,u}^{c}\,\delta v = 0$ [6], which is automatically satisfied in view of Eq. (AC-2.2) since $\delta v \in U$. Continuing with the linear in ξ and $\Delta\lambda$ terms and recalling Eq. (AC-2.7) one obtains:

$$O(\xi) \ : \ (\mathcal{E}_{,uu}^{c}\, v_\xi)\delta v = 0 \tag{AC-2.13}$$

In view of the positive definiteness of $\mathcal{E}_{,uu}^{c}$ on $\mathcal{N}^\perp$ according to Eq. (AC-2.8), the only solution to Eq. (AC-2.13) is:

$$v_\xi = 0 \tag{AC-2.14}$$

for had this not been the case, Eq. (AC-2.13) for $\delta v = v_\xi$ would have been in contradiction with Eq. (AC-2.8).

[6]NOTE: From here and subsequently, a superscript (^{c}) or a subscript ($_c$) denotes evaluation of the quantity in question at the critical point (u_c, λ_c), following the convention introduced for Eq. (AC-2.8, AC-2.9).

The fact that the operator $\mathcal{E},^c_{vv}$, which is defined as the operator $\mathcal{E},^c_{uu}$ restricted on $\mathcal{N}^\perp$ is positive definite, assures the existence of a unique solution a to any equation of the type $(\mathcal{E},^c_{vv} a)\delta v = (b, \delta v)$ for $a, b, \delta v \in \mathcal{N}^\perp$. The reason for the existence of such a unique solution $a = (\mathcal{E},^c_{vv})^{-1} b$ is the existence of the unique inverse $(\mathcal{E},^c_{vv})^{-1}$ on $\mathcal{N}^\perp$, a property guaranteed by the positive definiteness condition satisfied by $\mathcal{E},^c_{vv}$ on the same space $\mathcal{N}^\perp$.

The $O(\Delta\lambda)$ term of the equilibrium equation Eq. (AC-2.10) yields:

$$O(\Delta\lambda) \ : \ (\mathcal{E},^c_{uu} v_\lambda + \mathcal{E},^c_{u\lambda})\delta v = 0 \tag{AC-2.15}$$

which in view of Eq. (AC-2.6) admits a unique solution $v_\lambda \in \mathcal{N}^\perp$.

The quadratic in $\xi, \Delta\lambda$ terms in the expansion of the equilibrium equation Eq. (AC-2.10) are found, with the help of Eq. (AC-2.14) to be:

$$O(\xi^2) \ : \ (\mathcal{E},^c_{uu} v_{\xi\xi} + (\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\delta v = 0 \tag{AC-2.16}$$

$$O(\xi\Delta\lambda) \ : \ (\mathcal{E},^c_{uu} v_{\xi\lambda} + (\mathcal{E},^c_{uuu} v_\lambda + \mathcal{E},^c_{uu\lambda})\overset{1}{u})\delta v = 0 \tag{AC-2.17}$$

$$O((\Delta\lambda)^2) \ : \ (\mathcal{E},^c_{uu} v_{\lambda\lambda} + (\mathcal{E},^c_{uuu} v_\lambda)v_\lambda + 2\mathcal{E},^c_{uu\lambda} v_\lambda + \mathcal{E},^c_{u\lambda\lambda})\delta v = 0 \tag{AC-2.18}$$

The existence and uniqueness of $v_{\xi\xi}, v_{\xi\lambda}, v_{\lambda\lambda}$ is assured from Eq. (AC-2.8) for the reasons already explained in detail in the discussion of Eq. (AC-2.13). The calculation of any term in the expansion Eq. (AC-2.12) of v proceeds in the straightforward fashion illustrated above.

Having thus constructed the solution $v(\xi, \Delta\lambda)$ of the equilibrium equation Eq. (AC-2.10), attention is next focused on the remaining equilibrium equation Eq. (AC-2.11), which in turn is expanded about the critical point (u_c, λ_c), or equivalently about $(\xi, \Delta\lambda) = (0, 0)$, yielding with the help of Eq. (AC-2.2), Eq. (AC-2.7) and Eq. (AC-2.14):

$$\begin{aligned} 0 = \Delta\lambda(\mathcal{E},^c_{u\lambda} \overset{1}{u}) + \ & \frac{1}{2}[\xi^2((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} + 2\xi\Delta\lambda((\mathcal{E},^c_{uuu} v_\lambda + \mathcal{E},^c_{uu\lambda})\overset{1}{u})\overset{1}{u} + \\ & (\Delta\lambda)^2((\mathcal{E},^c_{uuu} v_\lambda)v_\lambda + 2\mathcal{E},^c_{uu\lambda} v_\lambda + \mathcal{E},_{u\lambda\lambda})^c \overset{1}{u}] + ... \end{aligned} \tag{AC-2.19}$$

The above equation provides the wanted relation between ξ and $\Delta\lambda$ along the equilibrium path (or paths) through the critical point (u_c, λ_c). Assume that $\Delta\lambda$, in the neighborhood of the critical point, can be put in a Taylor series expansion in terms of ξ:

$$\Delta\lambda = \lambda_1\xi + \lambda_2\frac{\xi^2}{2!} + \lambda_3\frac{\xi^3}{3!} + ... \tag{AC-2.20}$$

By introducing the above expansion into Eq. (AC-2.19) one can find the wanted coefficients λ_i.

Two cases are distinguished:

$$\text{case (i)} \ : \quad \mathcal{E},^c_{u\lambda} \overset{1}{u} \neq 0 \tag{AC-2.21}$$

in which case all the coefficients λ_i in Eq. (AC-2.20) can be determined uniquely; the first two coefficients are:

$$\lambda_1 = 0, \quad \lambda_2 = -((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} / \mathcal{E},^c_{u\lambda} \overset{1}{u} \tag{AC-2.22}$$

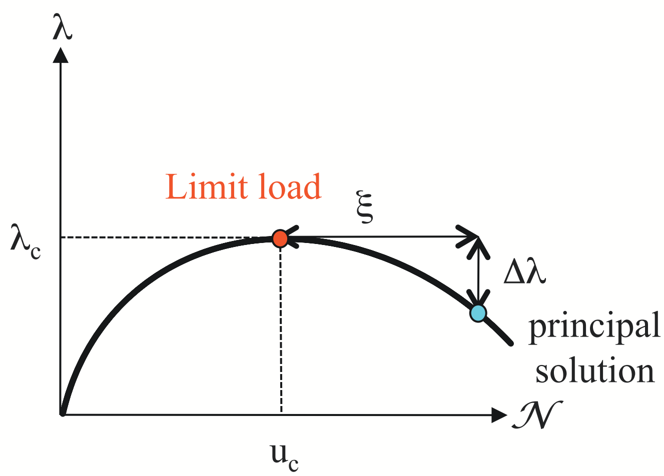

Figure AC-2.2: Schematics of the limit load case, where the solution about the critical load is unique.

One can conclude that if Eq. (AC-2.21) is satisfied, there is only one equilibrium branch through the critical point (u_c, λ_c). Moreover, and under the tacit assumption that $((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0$, (assumption satisfied in most applications) one also concludes from Eq. (AC-2.22) that the unique equilibrium branch through the critical point has a load extremum there, since the sign of $\Delta\lambda$ is independent on the sign of ξ. A schematic representation of this situation is given in Fig. AC-2.2

Once the uniqueness of the equilibrium path through the critical point has been established, attention is focused on its stability. To this end assume that $\beta(\xi)$ denotes the minimum eigenvalue of the stability matrix $\mathcal{E},_{uu}(u(\xi), \lambda(\xi))$ where ξ has been employed as a convenient parameter to describe the above found unique equilibrium path. The eigenvector corresponding to $\beta(\xi)$ is denoted by $x(\xi)$. Since β is an eigenvalue of the stability operator, by definition:

$$(\mathcal{E},_{uu}(u(\xi), \lambda(\xi) x(\xi))\delta u = \beta(\xi)(x(\xi), \delta u), \quad \| x(\xi) \| = 1 \tag{AC-2.23}$$

Since $\beta(\xi)$ is the lowest eigenvalue of $\mathcal{E},^c_{uu}$, as expected from Eq. (AC-2.7), at the critical point $\xi = 0$:

$$\beta_c = \beta(0) = 0, \quad x_c = x(0) = \overset{1}{u} \tag{AC-2.24}$$

Expansion of Eq. (AC-2.23) about $\xi = 0$ yields the following results: The $O(1)$ term reduces in view of Eq. (AC-2.24) to Eq. (AC-2.7). The next term is the $O(\xi)$ term, which in conjunction with Eq. (AC-2.9), Eq. (AC-2.12), Eq. (AC-2.14), Eq. (AC-2.20), Eq. (AC-2.22) and Eq. (AC-2.24) gives:

$$O(\xi) \ : \ ((\mathcal{E}{,}^{c}_{uuu}\overset{1}{u})\overset{1}{u} + \mathcal{E}{,}^{c}_{uu}(dx/d\xi)_c)\delta u = (d\beta/d\xi)_c(\overset{1}{u}, \delta u) \tag{AC-2.25}$$

which for $\delta u = \overset{1}{u}$, and recalling Eq. (AC-2.7), implies:

$$(d\beta/d\xi)_c = ((\mathcal{E}{,}^{c}_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0 \tag{AC-2.26}$$

The above result provides a physical explanation for the hypothesis $((\mathcal{E}{,}^{c}_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0$ adopted in the discussion of Eq. (AC-2.22), since it links it with the stability of the equilibrium path near the critical point. Since $\beta(\xi) = \xi(d\beta/d\xi)_c + O(\xi^2) = \xi((\mathcal{E}{,}^{c}_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} + O(\xi^2)$ in the neighborhood of the critical point, a stability change for the equilibrium branch is implied by crossing λ_c in view of the dependence of the sign of β on the sign of ξ. It has thus been shown that when Eq. (AC-2.21) holds (and $((\mathcal{E}{,}^{c}_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0$), the critical point (u_c, λ_c) corresponds to a load extremum of the equilibrium path $\overset{0}{u}(\lambda)$ and that the stability of the path changes by crossing λ_c. Given that the limit point in question is the first one encountered under increasing load , one concludes from continuity that this critical point should be a load maximum. Moreover, and since the principal equilibrium path is initially stable, one also concludes from continuity that the principal branch becomes unstable after it crosses the limit point.

The second possibility for the $\Delta\lambda - \xi$ relation in Eq. (AC-2.19) is:

$$\text{case (ii)} \ : \quad \mathcal{E}{,}^{c}_{u\lambda}\overset{1}{u} = 0 \tag{AC-2.27}$$

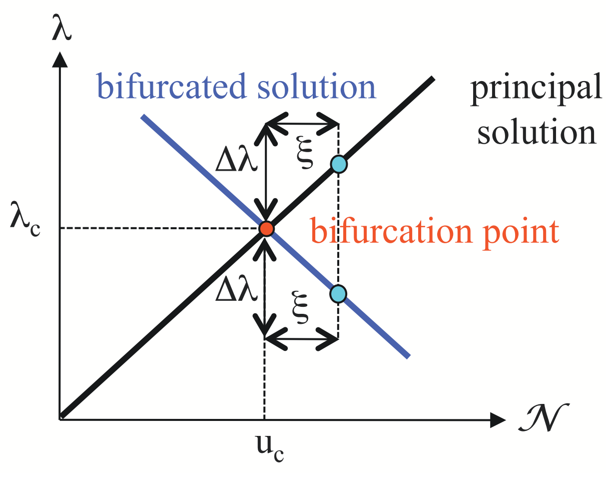

Figure AC-2.3: Schematics of the simple bifurcation case, where the solution about the critical load is not unique.

By introducing Eq. (AC-2.20) into Eq. (AC-2.19) one observes that, in view of Eq. (AC-2.27), the $\Delta\lambda - \xi$ relationship is no longer unique. Indeed, λ_1, the first term in the expansion of $\Delta\lambda$, satisfies:

$$(\lambda_1)^2((\mathcal{E},^c_{uuu} v_\lambda)v_\lambda + 2\mathcal{E},^c_{uu\lambda} v_\lambda + \mathcal{E},^c_{u\lambda\lambda})\overset{1}{u} + 2\lambda_1((\mathcal{E},^c_{uuu} v_\lambda + \mathcal{E},^c_{uu\lambda})\overset{1}{u})\overset{1}{u} + ((\mathcal{E},^c_{uuu})\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0 \quad \text{(AC-2.28)}$$

The above equation has in general two real solutions in λ_1 and hence from Eq. (AC-2.19) and Eq. (AC-2.20) one can construct two functions $\Delta\lambda(\xi)$, as seen in the schematics of Fig. AC-2.3. The fact that there already exists the principal equilibrium path through (u_c, λ_c) guarantees the existence of at least one real root λ_1 of Eq. (AC-2.28). Consequently two real roots for the quadratic in λ_1 equation Eq. (AC-2.28) must exist and which in general are different. Thus the condition Eq. (AC-2.27) implies that the critical point (u_c, λ_c) of the stability operator $\mathcal{E},_{uu}$ is a bifurcation point (some additional conditions are also needed in order to ensure the existence of such a bifurcated path, as it will be seen in the next section). Since in most applications one equilibrium branch, the principal branch $\overset{0}{u}(\lambda)$ is known explicitly, this information facilitates enormously the resulting algebraic manipulations. In the remaining parts of this section, it is assumed that a principal solution $\overset{0}{u}(\lambda)$ is known and that the critical point (u_c, λ_c) satisfies Eq. (AC-2.27).

AC-3 PERFECT SYSTEM - SIMPLE MODE

Of interest in this section is the behavior of elastic systems near critical points which are simple bifurcation points, i.e. have a unique eigenvalue $\overset{1}{u}$ at λ_c. To this end it is assumed that the system's potential energy satisfies Eqs. (AC-2.1), (AC-2.3), (AC-2.4), (AC-2.7), (AC-2.8) and (AC-2.27), and that its principal solution $\overset{0}{u}(\lambda)$ is a known and adequately smooth function of λ.

At the neighborhood of the critical point (u_c, λ_c) the solution u of the equilibrium Eq. (AC-2.2) can be written with the help of an LSK decomposition:

$$u = \overset{0}{u}(\lambda) + \xi \overset{1}{u} + v, \quad v \in \mathcal{N}^{\perp}, \quad \xi \in \mathbb{R}, \tag{AC-3.1}$$

where the space $\mathcal{N}^{\perp}$ is defined in the discussion of Eq. (AC-2.8). The projection ξ of $u - \overset{0}{u}$ along the eigenmode $\overset{1}{u}$, defined as $\xi \equiv (u - \overset{0}{u}, \overset{1}{u})$, is termed the "*(bifurcation) amplitude parameter*".

Splitting as before the equilibrium equation Eq. (AC-2.2) into two components on the subspaces $\mathcal{N}^{\perp}$ and $\mathcal{N}$, one obtains by extremizing $\mathcal{E}$ with respect to v the equilibrium equation along $\mathcal{N}^{\perp}$:

$$\mathcal{E}_{,v}\, \delta v = 0 \implies \mathcal{E}_{,u}(\overset{0}{u}(\lambda) + \xi \overset{1}{u} + v, \lambda_c + \Delta\lambda)\delta v = 0 \quad \forall \delta v \in \mathcal{N}^{\perp}, \tag{AC-3.2}$$

while by extremizing $\mathcal{E}$ with respect to ξ one has the equilibrium equation in the null space $\mathcal{N}$:

$$\mathcal{E}_{,\xi} = 0 \implies \mathcal{E}_{,u}(\overset{0}{u}(\lambda) + \xi \overset{1}{u} + v, \lambda_c + \Delta\lambda)\overset{1}{u} = 0. \tag{AC-3.3}$$

As discussed in the previous section, the positive definiteness of $\mathcal{E}^{c}_{,uu}$ on $\mathcal{N}^{\perp}$ assumed in Eq. (AC-2.8) implies its invertibility there and hence ensures the existence of a unique solution $v(\xi, \Delta\lambda)$ to Eq. (AC-3.2). Consider once more the Taylor series expansion of v about $(\xi, \Delta\lambda) = (0, 0)$ according to Eq. (AC-2.12). Upon substitution of Eq. (AC-2.12) into Eq. (AC-3.2) and expansion of the result in a Taylor series about $(\xi, \Delta\lambda) = (0, 0)$ one obtains the following results: The $O(1)$ term gives $\mathcal{E}^{c}_{,u}\, \delta v = 0$ which is automatically satisfied in view of the equilibrium equation Eq. (AC-2.2). The $O(\xi)$ term yields in view of Eq. (AC-2.27) the same results as in Eq. (AC-2.13):

$$O(\xi) \ : \ \mathcal{E}^{c}_{,uu}\, v_{\xi} \delta v = 0. \tag{AC-3.4}$$

which for the same reasons admits as a unique solution:

$$v_{\xi} = 0. \tag{AC-3.5}$$

On the other hand, the $O(\Delta\lambda)$ term in Eq. (AC-3.2) gives:

$$O(\Delta\lambda) \ : \ (\mathcal{E}^{c}_{,uu}\, v_{\lambda} + \mathcal{E}^{c}_{,uu}(d\overset{0}{u}/d\lambda)_c + \mathcal{E}^{c}_{,u\lambda})\delta v = (\mathcal{E}^{c}_{,uu}\, v_{\lambda})\delta v = 0. \tag{AC-3.6}$$

In deriving Eq. (AC-3.6) use was made of the fact that $(\mathcal{E}^{c}_{,uu}(d\overset{0}{u}/d\lambda)_c + \mathcal{E}^{c}_{,u\lambda})\delta u = 0$, for it is obtained by differentiation with respect to λ of the equilibrium equation Eq. (AC-2.3) along the principal branch. Since Eq. (AC-3.6) is the same with Eq. (AC-3.4), it admits as its unique solution:

$$v_\lambda = 0. \tag{AC-3.7}$$

Continuing with the quadratic order terms in the expansion of the equilibrium Eq. (AC-3.2):

$$O(\xi^2) \ : \ (\mathcal{E},^c_{uu} v_{\xi\xi} + (\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\delta v = 0, \tag{AC-3.8}$$

$$O(\xi\Delta\lambda) \ : \ (\mathcal{E},^c_{uu} v_{\xi\lambda} + (\mathcal{E},^c_{uuu} (d \overset{0}{u} /d\lambda)_c + \mathcal{E},^c_{uu\lambda})\overset{1}{u})\delta v = 0, \tag{AC-3.9}$$

$$\begin{aligned} O((\Delta\lambda)^2) \ : \ & (\mathcal{E},^c_{uu} v_{\lambda\lambda} + (\mathcal{E},^c_{uuu} (d \overset{0}{u} /d\lambda)_c)(d \overset{0}{u} /d\lambda)_c + 2\mathcal{E},^c_{uu\lambda} (d \overset{0}{u} /d\lambda)_c + \\ & \mathcal{E},^c_{u\lambda\lambda} + \mathcal{E},^c_{uu} (d^2 \overset{0}{u} /d\lambda^2)_c)\delta v = (\mathcal{E},^c_{uu} v_{\lambda\lambda})\delta v = 0. \end{aligned} \tag{AC-3.10}$$

Note that as in Eq. (AC-3.6), Eq. (AC-3.10) contains the second derivative with respect to λ of the equilibrium equation Eq. (AC-2.3) evaluated on the principal branch and hence it simplifies to $(\mathcal{E},^c_{uu} v_{\lambda\lambda})\delta v = 0$. For the same reasons as in Eq. (AC-3.4) and Eq. (AC-3.6) one obtains from Eq. (AC-3.10):

$$v_{\lambda\lambda} = 0. \tag{AC-3.11}$$

It is worth mentioning at this point that all the coefficients of $(\Delta\lambda)^n$ in the expansion of $v(\xi, \Delta\lambda)$ (see Eq. (AC-2.12)), satisfy $v,_\lambda = v,_{\lambda\lambda} = v,_{\lambda\lambda\lambda} ... = 0$. This result had to be expected, since by the definition of the LSK decomposition of u in Eq. (AC-3.1), $\xi = 0$ corresponds to the principal solution and hence one should have $v(0, \Delta\lambda) = 0$.

The results from the solution of Eq. (AC-3.2) for $v(\xi, \Delta\lambda)$ are employed in the remaining equilibrium equation Eq. (AC-3.3) which expanded about $(\xi, \Delta\lambda) = (0, 0)$ yields with the help of Eqs. (AC-2.3), (AC-2.7), (AC-2.27), (AC-3.5), (AC-3.7):

$$\begin{aligned} 0 = \ & \frac{1}{2}[\xi^2((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} + 2\xi\Delta\lambda((d\mathcal{E},_{uu} /d\lambda)_c\overset{1}{u})\overset{1}{u}] + \\ & \frac{1}{6}[\xi^3((((\mathcal{E},^c_{uuuu} \overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E},^c_{uuu} v_{\xi\xi})\overset{1}{u})\overset{1}{u}) + ...] + ... \end{aligned} \tag{AC-3.12}$$

In the above equation, which provides the $\Delta\lambda - \xi$ relation for both the principal and the bifurcated equilibrium paths through the critical point (u_c, λ_c), one can easily verify that all the $O((\Delta\lambda)^n)$ terms vanish identically. Consequently, and as expected from Eq. (AC-3.1) and Eq. (AC-3.12) admits two different solutions: One solution with $\xi = 0, \ \Delta\lambda \neq 0$ corresponds to the principal branch $\overset{0}{u}(\lambda)$, while the other solution with $\xi \neq 0, \ \Delta\lambda = \Delta\lambda(\xi)$ corresponds to the bifurcated path.

For the bifurcated path, by employing the same Taylor series expansion of $\Delta\lambda$ about $\xi = 0$ as in Eq. (AC-2.20), one obtains for the asymmetric bifurcation case, defined as $((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0$:

$$\lambda_1 = -\frac{1}{2}((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u}/((d\mathcal{E},_{uu} /d\lambda)_c\overset{1}{u})\overset{1}{u}, \tag{AC-3.13}$$

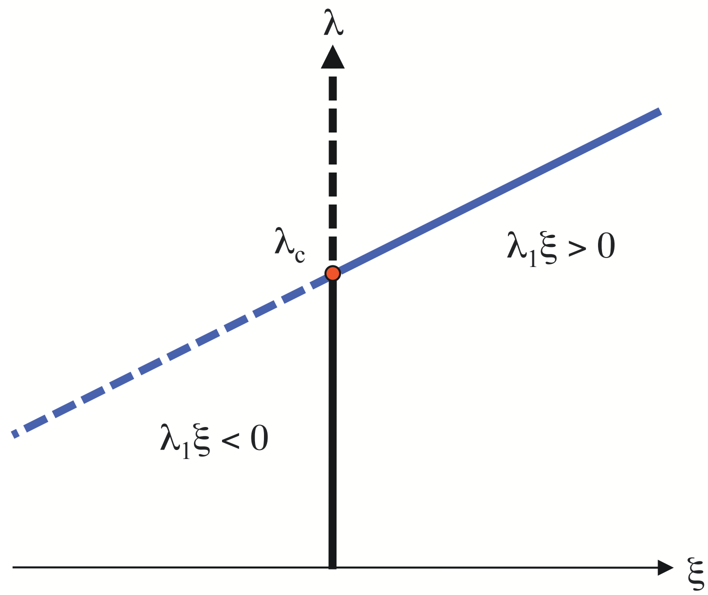

Figure AC-3.1: Case of an asymmetric bifurcation for a problem with a single eigenmode at the critical point. Stable paths are drawn in continuous lines while unstable paths are drawn in dashed lines.

while for the symmetric bifurcation case defined as $((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0$ one has:

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{1}{3}[(((\mathcal{E},^c_{uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E},^c_{uuu}v_{\xi\xi})\overset{1}{u})\overset{1}{u}]/((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}. \tag{AC-3.14}$$

where $v_{\xi\xi}$ is the unique solution of Eq. (AC-3.8). It should be noted at this point that in the above derivations it is tacitly assumed that the denominator in the expressions for λ_1 and λ_2 is nonzero [7]. A physical interpretation of this assumption will be given immediately below.

The next question of interest, pertains to the stability of the equilibrium branches through the critical point. To this end one has to investigate the sign of the minimum eigenvalue β of the stability operator $\mathcal{E},_{uu}$ evaluated on the equilibrium path in question. For the principal branch it is reasonable to assume in view of Eq. (AC-2.4), that $\overset{0}{\beta}(\lambda)$ the minimum eigenvalue of $\mathcal{E},_{uu}(\overset{0}{u}(\lambda), \lambda)$ has a strict crossing of zero at λ_c, i.e.:

$$\overset{0}{\beta}(\lambda_c) = 0 \quad , \quad (d\overset{0}{\beta}/d\lambda)_c < 0, \tag{AC-3.15}$$

with the negative sign of $(d\overset{0}{\beta}/d\lambda)_c$ explained by the fact that $\overset{0}{\beta} > 0$ for $\lambda < \lambda_c$ according to the definition of λ_c (see the discussion following Eq. (AC-2.8)). Assuming that the eigenvector corresponding to $\overset{0}{\beta}(\lambda)$ is $\overset{0}{x}(\lambda)$ one has from the definition of the eigenvalue:

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda), \lambda)\overset{0}{x}(\lambda))\delta u = \overset{0}{\beta}(\lambda)(\overset{0}{x}(\lambda), \delta u), \tag{AC-3.16}$$

[7]NOTE: From here and subsequently $(\ ),_\lambda$ denotes partial differentiation of the quantity in question with respect to the load parameter, while $d(\ )/d\lambda$ denotes differentiation of the quantity in question evaluated on the principal equilibrium branch $\overset{0}{u}(\lambda)$ with respect to the load parameter.

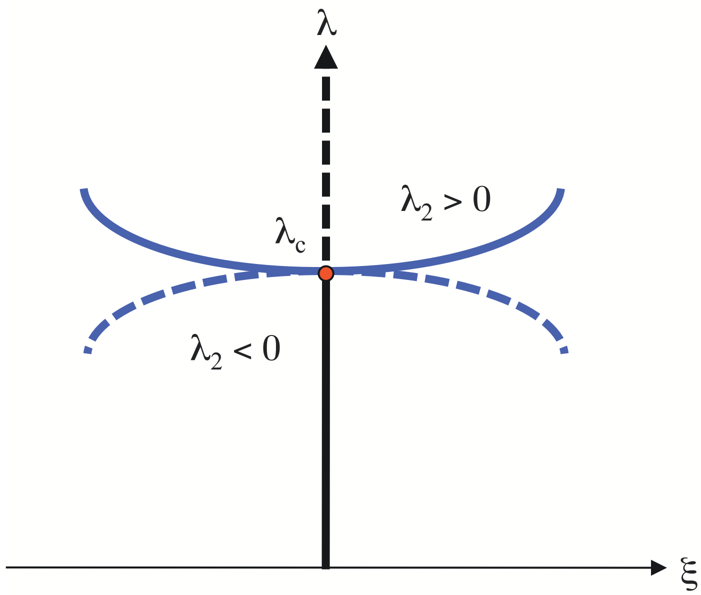

Figure AC-3.2: Case of a symmetric bifurcation for a problem with a single eigenmode at the critical point. Stable paths are drawn in continuous lines while unstable paths are drawn in dashed lines.

while to ensure uniqueness (up to sign) of the eigenvector $\overset{0}{x}$ one also requires it to have a unit norm:

$$(\overset{0}{x}(\lambda), \overset{0}{x}(\lambda)) = 1. \tag{AC-3.17}$$

Evaluating Eq. (AC-3.16), Eq. (AC-3.17) at λ_c and recalling from Eq. (AC-2.7) the uniqueness of the eigenvector of $\mathcal{E},^c_{uu}$:

$$(\mathcal{E},^c_{uu}\overset{0}{x}(\lambda_c))\delta u = 0, \quad (\overset{0}{x}(\lambda_c), \overset{0}{x}(\lambda_c)) = 1 \implies \overset{0}{x}(\lambda_c) = \overset{1}{u}. \tag{AC-3.18}$$

By differentiating Eq. (AC-3.16) with respect to λ and evaluating the resulting expression at λ_c, one obtains with the help of Eq. (AC-3.18):

$$((\mathcal{E},^c_{uuu}(d\overset{0}{u}/d\lambda)_c + \mathcal{E},^c_{uu\lambda})\overset{1}{u} + \mathcal{E},^c_{uu}(d\overset{0}{x}/d\lambda)_c)\delta u = (d\overset{0}{\beta}/d\lambda)_c(\overset{1}{u}, \delta u). \tag{AC-3.19}$$

By choosing $\delta u = \overset{1}{u}$ and recalling Eq. (AC-2.7) as well as Eq. (AC-3.15)$_1$ one finally has:

$$((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = (d\overset{0}{\beta}/d\lambda)_c. \tag{AC-3.20}$$

The above result, in conjunction with Eq. (AC-3.15)$_2$ ensures a non-zero denominator in Eq. (AC-3.13), Eq. (AC-3.14) as well in all the subsequent terms λ_n in the Taylor series expansion of $\Delta\lambda(\xi)$, and hence guarantees the existence of a bifurcated path through the critical load.

For the thus constructed bifurcated solution, a parameterization with respect to ξ is the most convenient one to study the path's stability. The starting point for the corresponding stability calculation is the definition of the minimum eigenvalue $\beta(\xi)$ of the stability operator evaluated on the bifurcated path, namely:

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda_c+\Delta\lambda(\xi))+\xi\overset{1}{u}+v(\xi,\Delta\lambda(\xi)),\lambda_c+\Delta\lambda(\xi))x(\xi))\delta u=\beta(\xi)(x(\xi),\delta u), \tag{AC-3.21}$$

which to ensure uniqueness of the eigenmode $x(\xi)$ has to be complemented by the normalization requirement:

$$(x(\xi),x(\xi))=1. \tag{AC-3.22}$$

By assuming a regular Taylor series expansion of β and x about $\xi=0$, namely:

$$\begin{aligned} \beta(\xi) &= \xi\beta_1+\frac{\xi^2}{2}\beta_2+... \\ x(\xi) &= x_0+\xi x_1+\frac{\xi^2}{2}x_2+... \end{aligned} \tag{AC-3.23}$$

and employing them together with Eq. (AC-3.13), Eq. (AC-3.14) and Eq. (AC-3.22) into Eq. (AC-3.21) one obtains by expanding about $\xi=0$ the following results:

For the $O(1)$ term, in view of Eq. (AC-3.22) and the assumed uniqueness of the eigenmode of $\mathcal{E},^c_{uu}$ (see Eq. (AC-2.7)) one has:

$$O(1)\ :\ (\mathcal{E},^c_{uu}x_0)\delta u=0,\quad (x_0,x_0)=1,\ \Longrightarrow\ x_0=\overset{1}{u}. \tag{AC-3.24}$$

Continuing with the $O(\xi)$ term of the expansion of Eq. (AC-3.21):

$$O(\xi)\ :\ ((\mathcal{E},^c_{uuu}(\lambda_1(d\overset{0}{u}/d\lambda)_c+\overset{1}{u}))\overset{1}{u}+\lambda_1\mathcal{E},^c_{uu\lambda}\overset{1}{u}+\mathcal{E},^c_{uu}x_1)\delta u=\beta_1(\overset{1}{u},\delta u). \tag{AC-3.25}$$

By taking $\delta u=\overset{1}{u}$ and recalling Eq. (AC-2.7) as well as Eq. (AC-3.13), one obtains for the asymmetric bifurcation case, i.e. for $((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}\neq 0$, the following result for β_1:

$$\beta_1=\lambda_1((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}+((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}=\lambda_1[-((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}]. \tag{AC-3.26}$$

For the symmetric bifurcation case, i.e. for $((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}=0$, the $O(\xi)$ term in the expansion of Eq. (AC-3.21) gives according to Eq. (AC-3.13), Eq. (AC-3.24) and Eq. (AC-3.26) that $\beta_1=\lambda_1=0$. For this case, Eq. (AC-3.24), Eq. (AC-3.25) and the $O(\xi)$ term in Eq. (AC-3.22) give in view of Eq. (AC-3.8):

$$O(\xi)\ :\ ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u}+\mathcal{E},^c_{uu}x_1)\delta v=0,\quad (x_1,\overset{1}{u})=0,\ \Longrightarrow\ x_1=v_{\xi\xi}. \tag{AC-3.27}$$

Continuing with the $O(\xi^2)$ term in the expansion of Eq. (AC-3.21) and making use of Eq. (AC-3.23), Eq. (AC-3.24), Eq. (AC-3.25) as well as Eq. (AC-3.14) one obtains:

$$\begin{aligned} O(\xi^2)\ :\ &((\frac{1}{2}((\mathcal{E},^c_{uuuu}\overset{1}{u})\overset{1}{u}+\mathcal{E},^c_{uuu}(\lambda_2(d\overset{0}{u}/d\lambda)_c+v_{\xi\xi})+\lambda_2\mathcal{E},^c_{uu\lambda})\overset{1}{u})x_1 \\ &(\mathcal{E},^c_{uuu}\overset{1}{u})x_1+\frac{1}{2}\mathcal{E},^c_{uu}x_2)\delta u=\frac{1}{2}\beta_2(\overset{1}{u},\delta u). \end{aligned} \tag{AC-3.28}$$

Once again by taking $\delta u = \overset{1}{u}$ and recalling Eq. (AC-2.7), Eq. (AC-3.14), Eq. (AC-3.24) as well as Eq. (AC-3.27) one finds β_2 to be:

$$\begin{aligned}\beta_2 = &\quad (((\mathcal{E},^c_{uuuu} \overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + \lambda_2((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} + ((\mathcal{E},^c_{uuu} v_{\xi\xi})\overset{1}{u})\overset{1}{u} + 2((\mathcal{E},^c_{uuu} x_1)\overset{1}{u})\overset{1}{u} \\ = &\quad 2\lambda_2(-((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}).\end{aligned} \tag{AC-3.29}$$

Consequently the stability of the bifurcated equilibrium path in the neighborhood of the critical point (u_c, λ_c) is determined by the sign of the minimum eigenvalue $\beta(\xi)$ of the corresponding stability operator which from Eq. (AC-3.20), Eq. (AC-3.26), Eq. (AC-3.29) takes the form:

$$\beta(\xi) = \begin{cases} \lambda_1\xi[-(d\overset{0}{\beta}/d\lambda)_c] + O(\xi^2) & \text{for asymmetric bifurcation } ((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0, \\ \\ \lambda_2\xi^2[-(d\overset{0}{\beta}/d\lambda)_c] + O(\xi^3) & \text{for symmetric bifurcation } ((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} = 0. \end{cases} \tag{AC-3.30}$$

From the hypothesis $(d\overset{0}{\beta}/d\lambda)_c < 0$ one concludes the following: For the transcritical asymmetric bifurcation $\lambda_1 \neq 0$, the $\lambda > \lambda_c$ branch of the solution satisfies $\lambda_1\xi > 0$ and hence the corresponding part of the bifurcated branch is stable, while for the $\lambda < \lambda_c$ part of the bifurcated branch $\lambda_1\xi < 0$ and hence the corresponding part of the bifurcated branch is unstable. For the symmetric bifurcation, the supercritical $\lambda > \lambda_c$ branch corresponds to $\lambda_2 > 0$ and it is stable while the subcritical $\lambda < \lambda_c$ branch corresponds to $\lambda_2 < 0$ and is unstable. Hence the general stability analysis for any elastic system exhibiting a simple bifurcation gives the same results with the simple rigid T model analyzed in subection AB-1.

One can also compute $\Delta\mathcal{E}$, the difference between the potential energies on the bifurcated and principal branches of the system, for a fixed value of the load parameter λ. Since by definition:

$$\Delta\mathcal{E} = \mathcal{E}(\overset{0}{u}(\lambda_c + \Delta\lambda) + \xi\overset{1}{u} + v(\xi, \Delta\lambda), \lambda_c + \Delta\lambda) - \mathcal{E}(\overset{0}{u}(\lambda_c + \Delta\lambda), \lambda_c + \Delta\lambda), \tag{AC-3.31}$$

one can expand $\Delta\mathcal{E}$ about the critical point (u_c, λ_c) and obtain a power series expansion in ξ. For the asymmetric $(\lambda_1 \neq 0)$ bifurcation case, Eq. (AC-3.31) with the help of Eq. (AC-2.7), Eq. (AC-2.27) and Eq. (AC-3.13) yields:

$$\Delta\mathcal{E} = \frac{\xi^3}{6}\lambda_1((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} + O(\xi^4), \tag{AC-3.32}$$

while for the symmetric $(\lambda_1 = 0,\ \lambda_2 \neq 0)$ bifurcation case, Eq. (AC-3.31) with the help of Eq. (AC-2.27) and Eq. (AC-3.14) yields:

$$\Delta\mathcal{E} = \frac{\xi^4}{8}\lambda_2((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} + O(\xi^5). \tag{AC-3.33}$$

These results are similar to the the one obtained for the rigid T model (see Eq. (AB-1.6), Eq. (AB-1.7).

Example

As a simple first application of the general theory developed in this section, the perfect rigid T example will be revisited. In this case the space of admissible displacement functions $u = (v, \theta)$ is $U = \mathbb{R}^2$, whose Cartesian basis is $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. From Eq. (AB-1.1) the stability operator evaluated on the principal branch $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda)$ is the rank two tensor:

$$\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda) = 2E\mathbf{e}_1\mathbf{e}_1 + (\lambda_c - \lambda)L\mathbf{e}_2\mathbf{e}_2. \tag{AC-3.34}$$

It is easy to see that $\lambda = \lambda_c$ is a singular point for $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda)$ with the corresponding unique eigenvector:

$$\overset{1}{u} = \mathbf{e}_2. \tag{AC-3.35}$$

The critical point (u_c, λ_c) is not a limit point for Eq. (AC-2.27) is satisfied, as one can easily see from Eq. (AC-3.34) and Eq. (AC-3.35):

$$\mathcal{E}^{c}_{,u\lambda}\overset{1}{u} = (-\mathbf{e}_1) \bullet \mathbf{e}_2 = 0, \tag{AC-3.36}$$

where $(\bullet)$ is the standard dyadic notation for the single dot (inner) product in finite dimension vector and tensor calculus.

Note that the rigid T model satisfies the strict crossing at zero of the minimum eigenvalue condition Eq. (AC-3.15) (see also Eq. (AC-3.20)) namely:

$$((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = ((-L\mathbf{e}_2\mathbf{e}_2) \bullet \mathbf{e}) \bullet \mathbf{e}_2 = -L < 0. \tag{AC-3.37}$$

Also note that the bifurcation amplitude parameter ξ from (AC-3.1) is given by:

$$\xi = (u - \overset{0}{u}(\lambda)) \bullet \overset{1}{u} = [(v - (\lambda/2E))\mathbf{e}_1 + \theta\mathbf{e}_2] \bullet \mathbf{e}_2 = \theta, \tag{AC-3.38}$$

which identifies θ as the bifurcation amplitude parameter ξ in this example.

For the asymmetric rigid T model $(m \neq 0, n = 0)$ the first nontrivial term λ_1 in the $\lambda - \xi$ expansion for the bifurcation branch is calculated from Eq. (AC-3.13). Noticing that in this example from Eq. (AB-1.1)

$$\mathcal{E}^{c}_{,uuu} = 2mL^3\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2, \tag{AC-3.39}$$

the value of λ_1 is found from Eq. (AC-3.13) with the help of Eq. (AC-3.35), Eq. (AC-3.37) and Eq. (AC-3.38), to be:

$$\lambda_1 = -\frac{1}{2}(2mL^3)/(-L) = mL^2, \tag{AC-3.40}$$

exactly as expected from Eq. (AB-1.4).

For symmetric model $(m = 0, n \neq 0)$, the first nontrivial term in the $\lambda - \xi$ expansion is calculated from (AC-3.14). Notice that in this case from (AB-1.1):

$$\mathcal{E}^{c}_{,uuuu} = 6nL^4\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2, \tag{AC-3.41}$$

while $\mathcal{E}_{,uuu}^{c} = 0$ and hence from Eq. (AC-3.14) with the help of Eq. (AC-3.35), Eq. (AC-3.37) and Eq. (AC-3.39) one obtains:

$$\lambda_2 = -\frac{1}{2}(6nL^4)/(-L) = 2nL^3, \tag{AC-3.42}$$

exactly as expected from Eq. (AB-1.4).

As it turns out, the above found asymptotic results describe completely the bifurcated solutions of the simple rigid T example, since one can verify that all the higher order terms λ_n in the expansion of $\lambda(\xi)$ are identically zero. For the same reason, the stability discussion based on the general asymptotic analysis of this section gives exactly the same results to the ones obtained before in the analysis of the rigid T.

The preceding general theory for simple bifurcations of elastic systems although not complete from the mathematical standpoint, is sufficient for the vast majority of engineering applications of interest. Pathological cases where $(d\mathcal{E}_{,uu}/d\lambda)_c \overset{1}{u})\overset{1}{u} = (d\overset{0}{\beta}/d\lambda)_c = 0$ do exist and require a modification of the preceding general analysis. This modification is not at all difficult since the LSK decomposition in Eq. (AC-3.1) can be used once more to obtain the new equilibrium equation (counterpart to Eq. (AC-3.12)) along the null space $\mathcal{N}$ which will automatically suggest the required parameterization of the equilibrium paths near the critical point. Depending on some additional conditions, $(d\overset{0}{\beta}/d\lambda)_c = 0$ can lead to a bifurcated branch emerging tangently from the principal path (in contrast to the results of this section where the bifurcated branches cut the principal path at a finite angle in $\lambda - \xi$ space) or might even be an isolated singular point with no bifurcation branch emerging from λ_c.

AC-4 IMPERFECT SYSTEM - SIMPLE MODE

Of interest in this section is the influence of imperfections in systems whose perfect counterparts exhibit a simple bifurcation. To this end, it is assumed that the potential energy of the imperfect system is $\overline{\mathcal{E}}(u, \lambda, w)$ where w is the imperfection field of the system while u and λ denote the displacement field and the load parameter as before. Without loss of generality, it is assumed that $w \in U$, the space which also contains all the admissible displacement functions u. If the imperfection function w vanishes, the system is reduced to its perfect counterpart and hence:

$$\overline{\mathcal{E}}(u, \lambda, 0) = \mathcal{E}(u, \lambda). \tag{AC-4.1}$$

Similarly to the perfect case, the potential energy $\overline{\mathcal{E}}$ is arbitrarily set to zero for zero displacements:

$$\overline{\mathcal{E}}(0, \lambda, w) = 0. \tag{AC-4.2}$$

The system evolves from its stress and displacement-free configuration at which $\lambda = 0$ and $u = 0$ to a loaded configuration with $\lambda \neq 0,\ u \neq 0$. For a given load level, the corresponding equilibrium solutions are found by extremizing $\overline{\mathcal{E}}$ with respect to u, namely:

$$\overline{\mathcal{E}},_u (u, \lambda, w)\delta u = 0. \tag{AC-4.3}$$

For all physically realistic problems, for a given load parameter λ and a given imperfection w, one expects the equilibrium solution $u(\lambda, w)$ to be unique in a neighborhood of $\lambda = 0$ and coincide with $\overset{0}{u}(\lambda)$ in the absence of imperfections, in agreement with Eq. (AC-2.3) and Eq. (AC-4.1):

$$\overline{\mathcal{E}},_u (u(\lambda, w), \lambda, w)\delta u = 0, \quad u(0, w) = 0, \quad u(\lambda, 0) = \overset{0}{u}(\lambda). \tag{AC-4.4}$$

Unlike $\overset{0}{u}(\lambda)$ however, $u(\lambda, w)$ is not as easy to find in applications, since the presence of imperfections destroys the symmetry of the system.

For imperfect systems, it is important to distinguish between the "*imperfection amplitude*" denoted by ϵ, and the "*imperfection shape*" denoted by $\overline{w}$. The corresponding definitions are:

$$\epsilon \equiv \| w \|, \quad \overline{w} \equiv w/\epsilon, \quad (\| \overline{w} \| = 1). \tag{AC-4.5}$$

Of interest here is the behavior of the imperfect system near the critical point $(\lambda_c,\ u_c)$ of its perfect counterpart for small imperfections, i.e. for small amplitudes ϵ of the imperfection w but for arbitrary imperfection shapes $\overline{w}$. To this end, one adopts the same LSK decomposition of the displacement $u = \overset{0}{u}(\lambda) + \xi \overset{1}{u} + v$ as in the perfect case (see Eq. (AC-3.1)). Splitting as before the equilibrium equation Eq. (AC-4.3) into two components on the subspaces $\mathcal{N}^\perp$ and $\mathcal{N}$ (see Eq. (AC-2.8) for the pertaining definitions), one obtains with the help of Eq. (AC-3.1) (in analogy to Eq. (AC-3.2), Eq. (AC-3.3)), that the equilibrium equation along $\mathcal{N}^\perp$ is:

$$\overline{\mathcal{E}},_v \delta v = 0 \Longrightarrow \overline{\mathcal{E}},_u (\overset{0}{u}(\lambda_c + \Delta\lambda) + \xi \overset{1}{u} + v, \lambda_c + \Delta\lambda, \epsilon\overline{w})\delta v = 0 \quad \forall \delta v \in \mathcal{N}^\perp, \tag{AC-4.6}$$

while the equilibrium equation along the null space $\mathcal{N}$ is:

$$\overline{\mathcal{E}},_{\xi}=0 \Longrightarrow \overline{\mathcal{E}},_u(\overset{0}{u}(\lambda_c+\Delta\lambda)+\xi\overset{1}{u}+v,\lambda_c+\Delta\lambda,\epsilon\overline{w})\overset{1}{u}=0. \quad \text{(AC-4.7)}$$

As discussed in subection AC-2, the invertibility of $\mathcal{E},^c_{vv}$ on $\mathcal{N}^{\perp}$ which follows from its positive definiteness according to Eq. (AC-2.8), ensures with the help of Eq. (AC-4.1) the existence of a unique solution $v(\xi,\Delta\lambda,\epsilon)$ to Eq. (AC-4.6). By assuming that $v(\xi,\Delta\lambda,\epsilon)$ has a Taylor series expansion about $(\xi,\Delta\lambda,\epsilon)=(0,0,0)$, i.e.:

$$\begin{aligned} v(\xi,\Delta\lambda,\epsilon)= \quad & \xi v_{\xi}+\Delta\lambda v_{\lambda}+\epsilon v_{\epsilon}+ \\ & \frac{1}{2}[\xi^2 v_{\xi\xi}+2\xi\Delta\lambda v_{\xi\lambda}+2\xi\epsilon v_{\xi\epsilon}+(\Delta\lambda)^2 v_{\lambda\lambda}+2\Delta\lambda\epsilon v_{\lambda\epsilon}+\epsilon^2 v_{\epsilon\epsilon}]+... \end{aligned} \quad \text{(AC-4.8)}$$

and expanding in a Taylor series the equilibrium equation Eq. (AC-4.8) about $(\xi,\Delta\lambda,\epsilon)=(0,0,0)$ one obtains the following results: The terms involving only powers of ξ and $\Delta\lambda$ produce exactly the same results as the perfect system, since for imperfection amplitudes $\epsilon=\parallel w\parallel=0$ the imperfect system reduces to its perfect counterpart in according to Eq. (AC-4.1). Hence for the imperfect system, in addition to Eqs. (AC-3.4) – (AC-3.11) one needs following terms to complete the Taylor series expansion of v up to the second order:

$$O(\epsilon) \ : \ (\mathcal{E},^c_{uu} v_{\epsilon}+\overline{\mathcal{E}},^c_{uw}\overline{w})\delta v=0, \quad \text{(AC-4.9)}$$

$$O(\xi\epsilon) \ : \ (\mathcal{E},^c_{uu} v_{\xi\epsilon}+(\mathcal{E},^c_{uuu} v_{\epsilon}+\overline{\mathcal{E}},^c_{uuw}\overline{w})\overset{1}{u})\delta v=0, \quad \text{(AC-4.10)}$$

$$O(\epsilon\Delta\lambda) \ : \ (\mathcal{E},^c_{uu} v_{\lambda\epsilon}+(d\mathcal{E},_{uu}/d\lambda)_c v_{\epsilon}+(d\overline{\mathcal{E}},_{uw}/d\lambda)_c\overline{w})\delta v=0, \quad \text{(AC-4.11)}$$

$$O(\epsilon^2) \ : \ (\mathcal{E},^c_{uu} v_{\epsilon\epsilon}+(\mathcal{E},^c_{uuu} v_{\epsilon})v_{\epsilon}+2(\overline{\mathcal{E}},^c_{uuw}\overline{w})v_{\epsilon}+(\overline{\mathcal{E}},^c_{uww}\overline{w})\overline{w})\delta v=0. \quad \text{(AC-4.12)}$$

All the above equations admit unique solutions in terms of the unknowns $v_{\epsilon},v_{\xi\epsilon},v_{\lambda\epsilon},v_{\epsilon\epsilon}$ in view of the existence of $(\mathcal{E},^c_{vv})^{-1}$ in $\mathcal{N}^{\perp}$. By introducing the thus constructed expansion for v into the remaining equilibrium equation Eq. (AC-4.7), one obtains with the help of Eq. (AC-2.7), Eq. (AC-2.27), Eq. (AC-3.5) and Eq. (AC-3.7):

$$\begin{aligned} 0= \quad & \epsilon(\overline{\mathcal{E}},^c_{uw}\overline{w})\overset{1}{u}+\frac{1}{2}[\xi^2((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}+2\xi\Delta\lambda((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}+2\xi\epsilon((\mathcal{E}^c_{uuu}v_{\epsilon}+\overline{\mathcal{E}},_{uuw})\overline{w}\overset{1}{u})\overset{1}{u}+ \\ & 2\epsilon\Delta\lambda((d\mathcal{E},_{uu}/d\lambda)_c v_{\epsilon}+(d\overline{\mathcal{E}},_{uw}/d\lambda)_c\overline{w})\overset{1}{u}+\epsilon^2((\mathcal{E}^c_{uuu}v_{\epsilon})v_{\epsilon}+2(\overline{\mathcal{E}},^c_{uuw}\overline{w})v_{\epsilon}+(\overline{\mathcal{E}},^c_{uww}\overline{w})\overline{w})\overset{1}{u}]+ \\ & \frac{1}{6}[\xi^3(((\mathcal{E},^c_{uuuu}\overset{1}{u})\overset{1}{u}+3\mathcal{E},^c_{uu}v_{\xi\xi})\overset{1}{u})\overset{1}{u}+...]+... \end{aligned} \quad \text{(AC-4.13)}$$

The above equation, which relates the load $\Delta\lambda=\lambda-\lambda_c$, the $\overset{1}{u}$ component of the deformation ξ and the imperfection amplitude ϵ, can be solved with respect to ϵ if $(\overline{\mathcal{E}},^c_{uw}\overline{w})\overset{1}{u}\neq 0$.

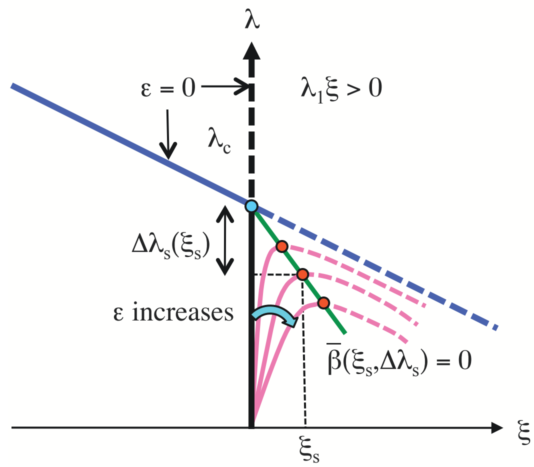

Figure AC-4.1: Case of an asymmetric bifurcation for an imperfect problem with a single eigenmode at the critical point. Stable paths are drawn in continuous lines while unstable paths are drawn in dashed lines.

Consequently, by considering the Taylor series expansion of ϵ in terms of $\Delta\lambda$ and ξ, one obtains from Eq. (AC-4.13), with the help of Eq. (AC-3.13), Eq. (AC-3.14) for the asymmetric and symmetric perfect systems respectively:

$$\epsilon(\xi,\Delta\lambda) = [((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}/(\overline{\mathcal{E}},^c_{uw}\overline{w})\overset{1}{u}]\begin{cases}(\lambda_1\xi^2 - \xi\Delta\lambda) + ... & \text{for} \quad ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0,\\ (\dfrac{\lambda_2}{2}\xi^3 - \xi\Delta\lambda) + ... & \text{for} \quad ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0.\end{cases} \tag{AC-4.14}$$

The above relation gives the magnitude of the imperfection amplitude ϵ for an equilibrium path of the imperfect system passing through the point $(\Delta\lambda, \xi)$ in the $\lambda - \xi$ space. Hence, for a given imperfection one can find the $\Delta\lambda - \xi$ relationship along the corresponding equilibrium path through Eq. (AC-4.14).

The next question of interest pertains to the stability of the aforedescribed equilibrium paths in the neighborhood of the perfect system's critical point (λ_c, u_c) and for small imperfection amplitudes. To this end one needs to investigate the minimum eigenvalue $\overline{\beta}$ of the corresponding stability operator $\overline{\mathcal{E}},_{uu}$ evaluated on the imperfect system's equilibrium solution. The defining equation for $\overline{\beta}$ and the corresponding eigenvector $\overline{x}$ is:

$$\begin{aligned}&(\overline{\mathcal{E}},_{uu}(\overset{0}{u}(\lambda_c+\Delta\lambda)+\xi\overset{1}{u}+v(\xi,\Delta\lambda,\epsilon(\xi,\Delta\lambda)),\lambda_c+\Delta\lambda,\epsilon(\xi,\Delta\lambda)\overline{w})\overline{x}(\xi,\Delta\lambda))\delta u =\\ &\overline{\beta}(\xi,\Delta\lambda)(\overline{x}(\xi,\delta\lambda),\delta u).\end{aligned} \tag{AC-4.15}$$

In addition, the normalization requirement for the eigenmode $\overline{x}(\xi,\Delta\lambda)$ is (compare with Eq. (AC-3.22)):

$$(\overline{x}(\xi,\Delta\lambda),\overline{x}(\xi,\Delta\lambda)) = 1. \tag{AC-4.16}$$

Notice in the above definition of $\overline{\beta}$, that the stability of all possible equilibrium paths in the neighborhood of the perfect system's critical load are examined, since $\Delta\lambda$ and ξ are considered as independent variables in this analysis.

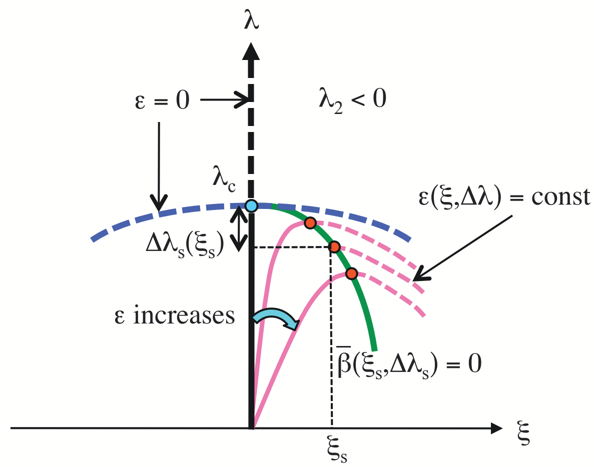

Figure AC-4.2: Case of an symmetric bifurcation for an imperfect problem with a single eigenmode at the critical point. Stable paths are drawn in continuous lines while unstable paths are drawn in dashed lines.

The minimum eigenvalue $\overline{\beta}(\xi, \Delta\lambda)$ and the corresponding eigenvector $\overline{x}(\xi, \Delta\lambda)$ are expanded in a Taylor series of their arguments in the neighborhood of $(\xi, \Delta\lambda) = (0, 0)$, i.e.:

$$\overline{\beta}(\xi, \Delta\lambda) = \xi\overline{\beta}_{\xi} + \Delta\lambda\overline{\beta}_{\lambda} + \frac{1}{2}(\xi^2\overline{\beta}_{\xi\xi} + 2\xi\Delta\lambda\overline{\beta}_{\xi\lambda} + (\Delta\lambda)^2\overline{\beta}_{\lambda\lambda}) + ... \tag{AC-4.17}$$

$$\overline{x}(\xi, \Delta\lambda) = \overline{x}_0 + \xi\overline{x}_{\xi} + \Delta\lambda\overline{x}_{\lambda} + \frac{1}{2}(\xi^2\overline{x}_{\xi\xi} + 2\xi\Delta\lambda\overline{x}_{\xi\lambda} + (\Delta\lambda)^2\overline{x}_{\lambda\lambda}) + ... \tag{AC-4.18}$$

By introducing Eq. (AC-4.17) and Eq. (AC-4.18) into Eq. (AC-4.15), expanding about $(\xi, \Delta\lambda) = (0, 0)$ and collecting the terms of the like order in ξ and $\Delta\lambda$ one obtains the following results. From the $O(1)$ term, in view of Eq. (AC-4.1), the assumed uniqueness of the eigenmode Eq. (AC-2.7) and Eq. (AC-4.16) one deduces:

$$O(1) \ : \ (\mathcal{E}^{\,c}_{,uu}\,\overline{x}_0)\delta u = 0, \quad (\overline{x}_0, \overline{x}_0) = 1 \implies \overline{x}_0 = \overset{1}{u}. \tag{AC-4.19}$$

Continuing with the $O(\xi)$ term in the Taylor series expansion of Eq. (AC-4.15) one has:

$$O(\xi) \ : \ (\mathcal{E}^{\,c}_{,uu}\,\overline{x}_{\xi} + (\mathcal{E}^{\,c}_{,uuu}\,\overset{1}{u})\overset{1}{u})\delta u = \overline{\beta}_{\xi}(\overset{1}{u}, \delta u). \tag{AC-4.20}$$

By taking $\delta u = \overset{1}{u}$, and recalling Eq. (AC-2.7) one obtains for $\overline{\beta}_{\xi}$:

$$\overline{\beta}_{\xi} = ((\mathcal{E}^{\,c}_{,uuu}\,\overset{1}{u})\overset{1}{u})\overset{1}{u}. \tag{AC-4.21}$$

while for $\delta u = \delta v$ and noticing from the $O(\xi)$ term of Eq. (AC-4.16) and Eq. (AC-4.19) that $(\overset{1}{u}, \overline{x}_{\xi}) = 0$, one obtains by comparing Eq. (AC-4.20) with Eq. (AC-3.8) that:

$$(\mathcal{E}^{\,c}_{,uu}\,\overline{x}_{\xi} + (\mathcal{E}^{\,c}_{,uuu}\,\overset{1}{u})\overset{1}{u})\delta v = 0 \implies \overline{x}_{\xi} = v_{\xi\xi}. \tag{AC-4.22}$$

Continuing with the $O(\Delta\lambda)$ term in the Taylor series expansion of Eq. (AC-4.15), one has:

$$O(\Delta\lambda) \ : \ (\mathcal{E}^{\,c}_{,uu}\,\overline{x}_{\lambda} + (d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\delta u = \overline{\beta}_{\lambda}(\overset{1}{u}, \delta u). \tag{AC-4.23}$$

By successively taking $\delta u = \overset{1}{u}$ into Eq. (AC-4.23) and recalling Eq. (AC-2.7) one finds that:

$$\overline{\beta}_\lambda = ((d\mathcal{E},_{uu}/d\lambda)_c \overset{1}{u})\overset{1}{u}. \tag{AC-4.24}$$

while by taking $\delta u = \delta v$, and noticing from the $O(\xi)$ term of Eq. (AC-4.16) and Eq. (AC-4.19) that $(\overset{1}{u}, \overline{x}_\lambda) = 0$, one obtains by comparing Eq. (AC-4.23) with Eq. (AC-3.9) that:

$$(\mathcal{E},^c_{uu} \overline{x}_\lambda + (d\mathcal{E},_{uu}/d\lambda)_c \overset{1}{u})\delta u = 0 \implies \overline{x}_\lambda = v_{\xi\lambda}. \tag{AC-4.25}$$

For the symmetric perfect system $((\mathcal{E},^c_{uuu})\overset{1}{u})\overset{1}{u} = 0$, the $O(\xi^2)$ term in the Taylor series expansion of Eq. (AC-4.15) is also required:

$$O(\xi^2) \ : \ (\frac{1}{2}(((\mathcal{E},^c_{uuuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} + \mathcal{E},^c_{uuu} v_{\xi\xi})\overset{1}{u} + (\mathcal{E},^c_{uuu} v_{\xi\xi})\overset{1}{u} + \frac{1}{2}\mathcal{E},^c_{uu} \overline{x}_{\xi\xi})\delta u = \frac{1}{2}\overline{\beta}_{\xi\xi}(\overset{1}{u}, \delta u). \tag{AC-4.26}$$

In the derivation of the above equation use was also made of Eq. (AC-4.17), Eq. (AC-4.19), Eq. (AC-4.21) and Eq. (AC-4.22). Once more by taking $\delta u = \overset{1}{u}$ and recalling Eq. (AC-2.7) one obtains:

$$\overline{\beta}_{\xi\xi} = (((\mathcal{E}^c_{uuuu}\overset{1}{u})\overset{1}{u} + 3\mathcal{E}^c_{uuu} v_{\xi\xi})\overset{1}{u})\overset{1}{u}. \tag{AC-4.27}$$

Consequently, from Eq. (AC-4.17), Eq. (AC-4.21), Eq. (AC-4.24), Eq. (AC-4.27) and Eq. (AC-3.13), Eq. (AC-3.14) the minimum eigenvalue $\overline{\beta}$ of the imperfect system's stability operator assumes the form:

$$\overline{\beta}(\xi, \Delta\lambda) = [((d\mathcal{E},_{uu}/d\lambda)_c \overset{1}{u})\overset{1}{u}] \begin{cases} (-2\lambda_1\xi + \Delta\lambda) + ... & \text{for} \quad ((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0, \\ (-\frac{3}{2}\lambda_2\xi^2 + \Delta\lambda) + ... & \text{for} \quad ((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} = 0. \end{cases} \tag{AC-4.28}$$

In the neighborhood of the perfect system's critical point $(\xi, \Delta\lambda) = (0, 0)$, one can find the load extrema of the equilibrium paths by setting $\overline{\beta} = 0$. The fact that the points with $\overline{\beta} = 0$, which are by definition the critical points of the imperfect system's stability operator $\overline{\mathcal{E}},_{uu}$, are load extrema of the corresponding equilibrium paths and not bifurcation points follows from the assumption $(\overline{\mathcal{E}}^c_{uw}\overline{w})\overset{1}{u} \neq 0$ as one can see from Eq. (AC-4.13).

Denote by $\Delta\lambda_s(\xi) = \lambda_s(\xi) - \lambda_c$ the difference between the load extremum points λ_s of the imperfect system's equilibrium paths corresponding to a given ξ and the perfect system's critical load λ_c. Assuming that $\Delta\lambda_s$ admits a Taylor series representation in terms of ξ, at least in the neighborhood of $\xi = 0$:

$$\Delta\lambda_s(\xi) = s_1\xi + \frac{s_2}{2!}\xi^2 + \frac{s_3}{3!}\xi^3 + O(\xi^4). \tag{AC-4.29}$$

Since $\overline{\beta}(\xi, \Delta\lambda_s(\xi)) = 0$, one can easily find the coefficients s_n in the Taylor series expansion of $\Delta\lambda_s$, by introducing Eq. (AC-4.29) into Eq. (AC-4.28):

$$\Delta\lambda_s(\xi) = \begin{cases} 2\lambda_1\xi + O(\xi^2) & \text{for} \quad ((\mathcal{E},^c_{uuu})\overset{1}{u})\overset{1}{u} \neq 0, \\ \frac{3}{2}\lambda_2\xi^2 + O(\xi^3) & \text{for} \quad ((\mathcal{E},^c_{uuu})\overset{1}{u})\overset{1}{u} = 0. \end{cases} \tag{AC-4.30}$$

Notice that the curve $\Delta\lambda_s(\xi)$ connects the load extrema of all the equilibrium paths of the imperfect system, each one of which corresponds to the same imperfection shape $\overline{w}$ but to a different imperfection amplitude ϵ, as shown in Fig. AC-4.1 and Fig. AC-4.2.

In applications, where imperfection w is known, i.e. for ϵ and $\overline{w}$ given, one is interested in finding $\Delta\lambda_s$ as a function of ϵ. The finding of $\Delta\lambda_s < 0$, the reduction from the critical load corresponding to the perfect system of the maximum load corresponding to the imperfect system is of particular interest, for it quantifies the critical load drop due to the presence of unavoidable imperfections in the system under investigation. Solutions for $\Delta\lambda_s > 0$, can be easily shown to correspond to equilibrium branches that do not pass through $\lambda = 0$ and hence are of no interest here.

Substitution of Eq. (AC-4.30) into Eq. (AC-4.14) and subsequent solution for $\Delta\lambda_s$ in terms of ϵ, gives to the leading order in ϵ for the asymmetric and symmetric perfect system respectively:

$$\Delta\lambda_s(\epsilon) = \begin{cases} -2[\epsilon\lambda_1(\overline{\mathcal{E}},^c_{uw}\overline{w})\overset{1}{u}/((-d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}]^{1/2} + O(\epsilon) \text{ for } ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0,\ \epsilon\lambda_1(\overline{\mathcal{E}},^c_{uw}\overline{w})\overset{1}{u} > 0, \\ \dfrac{3}{2}(\lambda_2)^{1/3}[\epsilon(\overline{\mathcal{E}},^c_{uw}\overline{w})\overset{1}{u}/((-d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}]^{2/3} + O(\epsilon) \text{ for } ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0,\ \lambda_2 < 0. \end{cases} \tag{AC-4.31}$$

Since in most practical applications the amplitude of the imperfection can be controlled but not its shape, it is important to find the imperfection shape $\overline{w}$ that maximizes the load drop ($\Delta\lambda_s$) for a given ϵ. It is not difficult to see from Eq. (AC-4.31) that ($\Delta\lambda_s$) is maximized when $\mid (\overline{\mathcal{E}},^c_{uw}\overline{w})\overset{1}{u} \mid$ is maximized, a rather straightforward problem in linear algebra.

Example

As a simple first application of the general theory developed here, the imperfect rigid T example will be revisited. In this case one can easily deduce by inspection of Eq. (AB-2.1) that $w = (0, \delta)$, $\overline{w} = (0, 1) = \mathbf{e}_2$ and $\epsilon \equiv \| w \| = \delta$. Also, from Eq. (AB-2.1) one obtains with the help of Eq. (AC-3.32):

$$\overline{\mathcal{E}},^c_{uw} = -\lambda_c L\mathbf{e}_2\mathbf{e}_2, \quad (\overline{\mathcal{E}},^c_{uw}\overline{w})\overset{1}{u} = -\lambda_c L. \tag{AC-4.32}$$

Recalling Eq. (AC-3.37), Eq. (AC-3.38), Eq. (AC-3.40) as well as Eq. (AC-3.32) one obtains from Eq. (AC-4.14) the result:

$$\epsilon = \begin{cases} (mL^2\xi^2 - \xi\Delta\lambda)/\lambda_c \implies \lambda_c\epsilon = mL^2\theta^2 - \theta(\lambda - \lambda_c) \quad \text{for} \quad m \neq 0,\ n = 0, \\ (nL^3\xi^3 - \xi\Delta\lambda)/\lambda_c \implies \lambda_c\epsilon = nL^3\theta^3 - \theta(\lambda - \lambda_c) \quad \text{for} \quad m = 0,\ n \neq 0. \end{cases} \tag{AC-4.33}$$

exactly as found in Eq. (AB-2.4).

Having independently rederived from the general theory the equilibrium equations for the rigid T model, attention is turned next in finding the minimum eigenvalue of the model's stability matrix. Recalling once more Eq. (AC-3.37), Eq. (AC-3.38) and Eq. (AC-3.40) one obtains from Eq. (AC-4.28):

$$\overline{\beta} = \begin{cases} -L(\Delta\lambda - 2mL^2\xi) = (\lambda_c - \lambda)L + 2mL^3\theta & \text{for} \quad m \neq 0,\ n = 0, \\ \\ -L(\Delta\lambda - 3nL^3\xi^2) = (\lambda_c - \lambda)L + 3nL^4\theta^2 & \text{for} \quad m = 0,\ n \neq 0. \end{cases} \tag{AC-4.34}$$

As expected, both the above expressions coincide with the results obtained in Eq. (AC-2.6) for the minimum eigenvalue of the stability matrix of the imperfect rigid T. More realistic examples from structural and solid mechanics requiring the application the general theory developed in this section will be given in the next chapters.

AC-5 PERFECT SYSTEM - SIMULTANEOUS MULTIPLE MODES

Up until now, a fundamental assumption in the bifurcation, post-bifurcation and imperfection sensitivity analyses of elastic systems was the uniqueness of the eigenmode $\overset{1}{u}$ associated with the lowest critical load λ_c (see Eq. (AC-2.7)). For the remaining sections of this section, this assumption is to be generalized as to include systems with a finite number of eigenvectors $\overset{i}{u},\ 1 \leq i \leq m$ corresponding to λ_c.

The system considered is still characterized by a potential energy $\mathcal{E}(u, \lambda)$ which obeys Eqs. (AC-2.1) – (AC-2.4). The stability of the principal branch $\overset{0}{u}(\lambda)$ is lost for the first time, as λ increases away from zero, at λ_c where:

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda_c), \lambda_c)\overset{i}{u})\delta u = 0, \quad (\overset{i}{u}, \overset{j}{u}) = \delta_{ij}; \quad i, j = 1, ..., m. \tag{AC-5.1}$$

The mode normalization condition in the above equation [8] ensures that the eigenmodes form an orthonormal set. This property will facilitate some of the subsequent calculations. Further simplification can be achieved by choosing a specific inner product, as will be described later.

In analogy to the simple bifurcation case it is also assumed that the critical point $(\overset{0}{u}(\lambda_c), \lambda_c)$ is a true multiple bifurcation point for the system, i.e. it satisfies the m-dimensional version of Eq. (AC-2.27) namely:

$$\mathcal{E},^c_{u\lambda}\overset{i}{u} = 0, \quad i = 1, ..., m. \tag{AC-5.2}$$

A criticality, the stability operator $\mathcal{E},^c_{uu}$ loses its positive definiteness only for those directions that are linear combinations of its eigenvectors $\overset{i}{u}$, i.e. for directions belonging to its null space $\mathcal{N}$. This implies the strict positive definiteness of $\mathcal{E},^c_{vv}$ on $\mathcal{N}^\perp$. Consequently Eq. (AC-2.8) continues to hold, but with the updated definition for the null space, namely $\mathcal{N} \equiv \{u \in U \mid u = \sum_{i=1}^m \xi_i \overset{i}{u},\ \xi_i \in \mathbb{R}\}$. The corresponding definition for the orthogonal complement to the null space is $\mathcal{N}^\perp \equiv \{v \in U \mid (v, \overset{i}{u}) = 0,\ i = 1, ..., m\}$.

At the neighborhood of the critical point (u_c, λ_c), the solution u to the equilibrium Eq. (AC-2.2) can be written with the help of the LSK decomposition (compare to Eq. (AC-3.1)):

$$u = \overset{0}{u}(\lambda) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v; \quad \xi_i \in \mathbb{R}, \quad v \in \mathcal{N}^\perp, \tag{AC-5.3}$$

where ξ_i is the projection of $u - \overset{0}{u}$ on $\overset{i}{u}$.

The sought displacement u is thus replaced by an equivalent set of unknowns (v, ξ_i) and the solution to the equilibrium equation $\mathcal{E},_u \delta u = 0$ proceeds in two steps: First v is determined as a function of $\Delta\lambda$ ($\Delta\lambda \equiv \lambda - \lambda_c$) and ξ_i from the equilibrium equation in $\mathcal{N}^\perp$, namely:

$$\mathcal{E},_v \delta v = 0 \implies \mathcal{E},_u(\overset{0}{u}(\lambda_c + \Delta\lambda) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v, \lambda_c + \Delta\lambda)\delta v = 0 \quad \forall \delta v \in \mathcal{N}^\perp. \tag{AC-5.4}$$

[8]Here δ_{ij} denotes the Kronecker delta symbol defined such that $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise

The resulting v is used in the remaining equilibrium equations on $\mathcal{N}$. This provides the relation between $\Delta\lambda$ and ξ_i. The m equilibrium equations that have to be solved on $\mathcal{N}$ are:

$$\mathcal{E},_{\xi_i} = 0 \implies \mathcal{E},_u (\overset{0}{u} (\lambda_c + \Delta\lambda) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v, \lambda_c + \Delta\lambda)\overset{i}{u} = 0 \tag{AC-5.5}$$

From the assumed positive definiteness of $\mathcal{E},^c_{vv}$ on $\mathcal{N}^\perp$ follows that Eq. (AC-5.4) has a unique and adequately smooth solution $v(\xi_i, \Delta\lambda)$, at least in the neighborhood of the critical point (u_c, λ_c). The Taylor series expansion of this solution is:

$$v(\xi_i, \Delta\lambda) = \sum_{i=1}^{m} \xi_i v_i + \Delta\lambda v_\lambda + \frac{1}{2}(\sum_{i=1}^{m}\sum_{j=1}^{m} \xi_i \xi_j v_{ij} + 2\Delta\lambda \sum_{i=1}^{m} \xi_i v_{i\lambda} + (\Delta\lambda)^2 v_{\lambda\lambda}) + ... \tag{AC-5.6}$$

Upon substitution of Eq. (AC-5.6) into Eq. (AC-5.4) and subsequent evaluation of its Taylor series expansion about $(\xi_i, \Delta\lambda) = (0, ..., 0)$ one obtains the following results: The $O(1)$ term of the expansion gives $\mathcal{E},^c_u \delta v = 0$ which is automatically satisfied in view of the equilibrium equation Eq. (AC-2.2). The $O(\xi_i)$ terms yield, with the help of Eq. (AC-5.2), the result: $(\mathcal{E},^c_{uu} v_i)\delta v = 0$ which in view of Eq. (AC-2.8) implies the generalization of Eq. (AC-3.5):

$$v_i = 0. \tag{AC-5.7}$$

By taking $\xi_i = 0$ into Eq. (AC-5.4), comparing the result to the equilibrium condition on the fundamental solution Eq. (AC-2.3), and invoking the uniqueness of the solution to Eq. (AC-5.4) for $v(\xi_i, \Delta\lambda)$, it is readily seen that $v(0, \Delta\lambda) = 0$. This implies once again the results already seen for the simple mode (see Eq. (AC-3.7) and Eq. (AC-3.11)):

$$v_\lambda = v_{\lambda\lambda} = ... = 0. \tag{AC-5.8}$$

This result could have also been obtained directly from the $O(\Delta\lambda)^n$ terms in the expansion of Eq. (AC-5.4).

One can similarly continue with the quadratic order terms in the expansion of the equilibrium equation (AC-5.4) to find:

$$O(\xi_i \xi_j) \ : \ (\mathcal{E},^c_{uu} v_{ij} + (\mathcal{E},^c_{uuu} \overset{i}{u})\overset{j}{u})\delta v = 0, \tag{AC-5.9}$$

$$O(\xi_i \Delta\lambda) \ : \ (\mathcal{E},^c_{uu} v_{i\lambda} + (d\mathcal{E},_{uu} / d\lambda)_c \overset{i}{u})\delta v = 0. \tag{AC-5.10}$$

The above equations have unique solutions $v_{ij}, v_{i\lambda}$ in view of Eq. (AC-2.8). In a similar way one can find the higher order terms in the expansion of $v(\xi_i, \Delta\lambda)$ and hence uniquely determine the solution to Eq. (AC-5.4). Upon substitution of the thus found $v(\xi_i, \Delta\lambda)$ into the remaining equilibrium equation Eq. (AC-5.5), and after using Eq. (AC-5.2), Eq. (AC-5.7) and Eq. (AC-5.8) one obtains the following m equations relating $\Delta\lambda$ and ξ_i:

$$\frac{1}{2}[\sum_{j=1}^{m}\sum_{k=1}^{m}\xi_j\xi_k\mathcal{E}_{ijk} + 2\Delta\lambda\sum_{j=1}^{m}\xi_j\mathcal{E}_{ij\lambda}\ ] + \frac{1}{6}[\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_j\xi_k\xi_l\mathcal{E}_{ijkl} + ...\ ]\ + ... = 0,$$

$$\mathcal{E}_{ijk} \equiv ((\mathcal{E},^c_{uuu}\overset{i}{u})\overset{j}{u})\overset{k}{u}, \tag{AC-5.11}$$

$$\mathcal{E}_{ijkl} \equiv (((\mathcal{E},^c_{uuuu}\overset{j}{u})\overset{k}{u})\overset{l}{u} + (\mathcal{E},^c_{uuu}v_{jk})\overset{l}{u} + (\mathcal{E},^c_{uuu}v_{kl})\overset{j}{u} + (\mathcal{E},^c_{uuu}v_{lj})\overset{k}{u})\overset{i}{u},$$

$$\mathcal{E}_{ij\lambda} \equiv ((d\mathcal{E},_{uu}/d\lambda)_c)\overset{i}{u})\overset{j}{u} = ((\mathcal{E},^c_{uuu}(d\overset{0}{u}/d\lambda)_c + \mathcal{E},^c_{uu\lambda})\overset{i}{u})\overset{j}{u}.$$

As expected, an obvious solution to Eq. (AC-5.11) is the principal equilibrium branch for which $\xi_i = 0$, $v = 0$ but $\Delta\lambda \neq 0$ according to Eq. (AC-5.3). The determination of the remaining equilibrium paths through the critical point, i.e. the determination of the curves $\xi_i(\Delta\lambda)$, is facilitated by introducing the "*bifurcation amplitude parameter*" ξ, defined as the projection of $u-\overset{0}{u}$ on the unit tangent of the equilibrium path at λ_c. For a neighborhood of the critical point, assuming an adequately smooth dependence of $\xi_i, \Delta\lambda$ on ξ one has:

$$\begin{aligned} \xi_i(\xi) &= \alpha_i^1\xi + \alpha_i^2\frac{\xi^2}{2} + ... \\ \Delta\lambda(\xi) &= \lambda_1\xi + \lambda_2\frac{\xi^2}{2} + ... \end{aligned} \qquad \text{where}: \quad \xi \equiv (u-\overset{0}{u}, \sum_{i=1}^{m}\alpha_i^1\overset{i}{u}). \tag{AC-5.12}$$

Two cases are to be distinguished at this point: First the asymmetric case for which $\mathcal{E}_{ijk} \neq 0$ at least for one triplet of indexes (i, j, k). By inserting Eq. (AC-5.12) into to Eq. (AC-5.11) and collecting the terms of the like order in ξ, one obtains the following system of m quadratic equations from the lowest order nontrivial term in this expansion (the $O(\xi^2)$ term):

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\alpha_j^1\alpha_k^1\mathcal{E}_{ijk} + 2\lambda_1\sum_{j=1}^{m}\alpha_j^1\mathcal{E}_{ij\lambda} = 0, \qquad \sum_{i=1}^{m}(\alpha_i^1)^2 = 1. \tag{AC-5.13}$$

where the second equation results from the unit norm of the tangent $\sum_{i=1}^{m}\alpha_i^1\overset{i}{u}$.

The above algebraic system of $m+1$ equations for the $m+1$ unknowns α_i^1, λ_1 has at most $2^m - 1$ pairs of real solutions (α_i^1, λ_1) and $(-\alpha_i^1, -\lambda_1)$, with each pair corresponding to a bifurcated equilibrium path through the critical point. Each one of these equilibrium paths can be constructed by computing its Taylor series expansion as indicated in Eq. (AC-5.12). Higher order terms in the expansion of Eq. (AC-5.11) show that the series can be continued to any desired order if the following condition is satisfied:

$$\text{Det}\ [B_{ij}] \neq 0, \qquad B_{ij} \equiv \sum_{k=1}^{m}\alpha_k^1\mathcal{E}_{ijk} + \lambda_1\mathcal{E}_{ij\lambda}. \tag{AC-5.14}$$

The second case to be investigated will be the symmetric case for which $\mathcal{E}_{ijk} = 0$ for all triplets of indexes (i, j, k). In this case Eq. (AC-5.13) implies that $\lambda_1 = 0$. By inserting Eq. (AC-5.12) into Eq. (AC-5.11) and collecting the terms of the like order in ξ, one obtains the following system of m cubic equations from the lowest order nontrivial term in this expansion (the $O(\xi^3)$ term):

$$\lambda_1 = 0, \quad \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\alpha_j^1\alpha_k^1\alpha_l^1\mathcal{E}_{ijkl} + 3\lambda_2\sum_{j=1}^{m}\alpha_j^1\mathcal{E}_{ij\lambda} = 0, \quad \sum_{i=1}^{m}(\alpha_i^1)^2 = 1. \tag{AC-5.15}$$

where again the second equation results from the unit norm of the tangent $\sum_{i=1}^{m}\alpha_i^1\overset{i}{u}$.

This algebraic system of $m+1$ equations for the $m+1$ unknowns α_i^1, λ_2 has at most $(3^m-1)/2$ pairs of real solutions (α_i^1, λ_2) and $(-\alpha_i^1, \lambda_2)$, each corresponding to a bifurcated equilibrium path through the critical point. Each one of these equilibrium paths can be constructed by computing its Taylor series expansion as indicated in Eq. (AC-5.12). The continuation of the expansion of Eq. (AC-5.11) to terms of $O(\xi^4)$ and higher for each particular branch is assured when:

$$\text{Det } [B_{ij}] \neq 0, \quad B_{ij} \equiv \sum_{k=1}^{m}\sum_{l=1}^{m}\alpha_k^1\alpha_l^1\mathcal{E}_{ijkl} + \lambda_2\mathcal{E}_{ij\lambda}. \tag{AC-5.16}$$

To complete the study of the above found bifurcated equilibrium branches, one has to investigate their stability. To this end one needs to find the sign of the minimum eigenvalue β_{min} of the stability operator $\mathcal{E}_{,uu}(u,\lambda)$ evaluated on the equilibrium path whose stability is under investigation. First the stability of the principal branch is to be investigated. To this end, assume that $\overset{0}{\beta}(\lambda)$ is any one of the m eigenvalues of $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)$ that vanish at λ_c, while $\overset{0}{x}(\lambda)$ is the corresponding normalized eigenvector. A strict crossing of zero at the critical load will again be assumed for $\overset{0}{\beta}(\lambda)$ which will again have to satisfy Eqs. (AC-3.15) – (AC-3.17). Notice that in this case each one of these equations actually represents m different equations, each one of which corresponds to one of the m different functions $\overset{0}{\beta}_i(\lambda)$ and corresponding eigenvectors $\overset{0}{x}_i(\lambda)$ 0where the subscript i has been avoided to alleviate notation.

Evaluating Eq. (AC-3.16) at the critical load one has in view of Eq. (AC-2.8) and Eq. (AC-5.1)

$$\overset{0}{x}_c = \sum_{i=1}^{m}\overset{0}{\chi}_i\overset{i}{u}. \tag{AC-5.17}$$

Assuming that each one of the $\overset{0}{\beta}(\lambda)$ and $\overset{0}{x}(\lambda)$ are smooth functions of their argument, Eq. (AC-3.16) can be differentiated with respect to λ. Evaluating the result at the critical point, and recalling from Eq. (AC-3.15) that $\overset{0}{\beta}(\lambda_c) = 0$ one has:

$$((d\mathcal{E}_{,uu}/d\lambda)_c\overset{0}{x}_c + \mathcal{E}^c_{,uu}(d\overset{0}{x}/d\lambda)_c)\delta u = (d\overset{0}{\beta}/d\lambda)_c(\overset{0}{x}_c, \delta u). \tag{AC-5.18}$$

Taking $\delta u = \overset{i}{u}$ and recalling Eq. (AC-5.1) and Eq. (AC-5.17), equation (AC-5.18) yields:

$$\sum_{j=1}^{m}\mathcal{E}_{ij\lambda}\overset{0}{\chi}_j = (d\overset{0}{\beta}/d\lambda)_c\overset{0}{\chi}_i, \tag{AC-5.19}$$

which shows that the m derivatives $(d\overset{0}{\beta}/d\lambda)_c$ are the eigenvalues of $\mathcal{E}_{ij\lambda}$ and the m vectors $\overset{0}{\chi}_i$ are the corresponding eigenvectors. Since from Eq. (AC-3.15) each such eigenvalue satisfies

$(d\overset{0}{\beta}/d\lambda)_c < 0$, one concludes that $\mathcal{E}_{ij\lambda}$ is a negative definite matrix. This condition is satisfied in the majority of applications of interest.

The stability of each bifurcated equilibrium branch trough the critical point depends on the sign of the minimum eigenvalue $\beta_{min}(\xi)$ of the corresponding stability operator $\mathcal{E}_{,uu}$. In analogy to Eq. (AC-3.21), the definitions for each one of the m lowest eigenvalues $\beta(\xi)$ and the corresponding normalized eigenvectors $x(\xi)$ are:

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c + \Delta\lambda) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v(\xi_i, \Delta\lambda), \lambda_c + \Delta\lambda)x)\delta u = \beta(x, \delta u), \tag{AC-5.20}$$

while the normalization condition for $x(\xi)$ is still given by Eq. (AC-3.22). In the above definition $\xi_i, \Delta\lambda, \beta, x$ are functions of the parameter ξ. Recall that for every bifurcated equilibrium path the m lowest eigenvalues $\beta(\xi)$ of the corresponding stability operator have to vanish at $\xi = 0$. In addition, for each bifurcated equilibrium path, $\beta(\xi), x(\xi)$ are assumed smooth functions of their argument and their Taylor series expansions as still given by Eq. (AC-3.23).

By introducing Eq. (AC-3.23) into Eq. (AC-5.20) and recalling Eq. (AC-5.1), Eqs. (AC-5.6) – (AC-5.8) and Eq. (AC-5.12), one obtains by expanding about $\xi = 0$ and collecting the terms of the like order in ξ the following results: The $O(1)$ term yields:

$$x_0 = \sum_{i=1}^{m} \chi_i \overset{i}{u}. \tag{AC-5.21}$$

Continuing with the $O(\xi)$ term in the expansion of Eq. (AC-5.20) one has:

$$O(\xi) \ : \ ((\mathcal{E}^{c}_{,uuu}(\lambda_1(d\overset{0}{u}/d\lambda)_c + \sum_{k=1}^{m} \alpha^1_k \overset{k}{u}) + \lambda_1 \mathcal{E}^{c}_{,uu\lambda})(\sum_{j=1}^{m} \chi_j \overset{j}{u}) + \mathcal{E}^{c}_{,uu} x_1)\delta u = \beta_1((\sum_{j=1}^{m} \chi_j \overset{j}{u}), \delta u). \tag{AC-5.22}$$

Taking $\delta u = \overset{i}{u}$ and recalling from Eq. $(\text{AC-5.1})_2$ the orthogonality of the eigenmodes, the above equation yields:

$$\sum_{j=1}^{m} B_{ij} \chi_j = \beta_1 \chi_i, \tag{AC-5.23}$$

where B_{ij} is defined in Eq. (AC-5.14). For the asymmetric bifurcation case, this matrix is nonsingular, which ensures that all its eigenvalues β_1 are nonzero, as well as real, in view of the symmetry of the matrix B_{ij}. It follows from Eq. (AC-5.23) that the constants χ_i introduced in Eq. (AC-5.21) are the components of the eigenvector of B_{ij} corresponding to the eigenvalue β_1.

For the symmetric bifurcation case, since $\mathcal{E}_{ijk} = \lambda_1 = 0$, $B_{ij} = 0$ as seen from Eq. (AC-5.14), which in view of Eq. (AC-5.23) also implies that $\beta_1 = 0$. Consequently, the $O(\xi)$ term in the expansion of Eq. (AC-5.20), with the help of the definition of v_{ij} given in Eq. (AC-5.9) and the normalization condition from the eigenmode $(x, x) = 1$, results in the following expression for x_1:

$$O(\xi) \ : \ (\sum_{j=1}^{m}\sum_{k=1}^{m} \chi_j \alpha^1_k (\mathcal{E}^{c}_{,uuu} \overset{j}{u})\overset{k}{u} + \mathcal{E}^{c}_{,uu} x_1)\delta u = 0, \quad (x_1, x_0) = 0, \implies x_1 = \sum_{i=1}^{m}\sum_{j=1}^{m} \chi_i \alpha^1_j v_{ij}. \tag{AC-5.24}$$

Continuing with the $O(\xi^2)$ term in the expansion of Eq. (AC-5.20) and recalling that $\lambda_1 = \beta_1 = 0$ as well as Eq. (AC-5.1), Eqs. (AC-5.6) – (AC-5.8) and Eq. (AC-5.12), one has:

$$\begin{aligned} O(\xi^2) \ : \ & (((\mathcal{E},^c_{uuuu}(\sum_{k=1}^m \alpha_k^1 \overset{k}{u}))(\sum_{l=1}^m \alpha_l^1 \overset{l}{u}) + \mathcal{E},^c_{uuu}(\lambda_2 (d\overset{0}{u}/d\lambda)_c + \sum_{k=1}^m\sum_{l=1}^m \alpha_k^1\alpha_l^1 v_{kl} + \sum_{k=1}^m \alpha_k^2 \overset{k}{u}) \\ & + \lambda_2 \mathcal{E},^c_{uu\lambda})(\sum_{j=1}^m \chi_j \overset{j}{u}) + 2(\mathcal{E},^c_{uuu}(\sum_{k=1}^m \alpha_k^1 \overset{k}{u}))x_1 + \mathcal{E},^c_{uu} x_2)\delta u = \beta_2((\sum_{j=1}^m \chi_j \overset{j}{u}), \delta u). \end{aligned} \tag{AC-5.25}$$

By subsequently taking $\delta u = \overset{i}{u}$ and using Eq. (AC-5.24) into Eq. (AC-5.25) one obtains:

$$\sum_{j=1}^m B_{ij}\chi_j = \beta_2 \chi_i, \tag{AC-5.26}$$

where the matrix B_{ij} is now given by Eq. (AC-5.16). For the symmetric bifurcation case, this matrix is nonsingular, which ensures that all its eigenvalues β_2 are nonzero (as well as real in view of the symmetry of the matrix). It also follows from Eq. (AC-5.26) that χ_i, the components of x_0 introduced in Eq. (AC-5.21), are the components of the eigenvector of B_{ij} corresponding to the eigenvalue β_2.

Hence the wanted minimum eigenvalue $\beta_{min}(\xi)$ of $\mathcal{E},_{uu}$ for the bifurcated equilibrium path in question is:

$$\beta_{min}(\xi) = \begin{cases} \xi\beta_1^{max} + O(\xi^2) \text{ if } \xi < 0, \quad \xi\beta_1^{min} + O(\xi^2) \text{ if } \xi > 0 & \text{for asymmetric bifurcation,} \\ (\xi^2/2)\beta_2^{min} + O(\xi^3) & \text{for symmetric bifurcation.} \end{cases} \tag{AC-5.27}$$

For the asymmetric bifurcation case, if for a certain bifurcated branch the maximum and minimum eigenvalues of B_{ij} respectively β_1^{max} and β_1^{min} are of the same sign, then the bifurcated branch in question changes stability as it crosses the critical point, while if the two extremal eigenvalues are of opposite sign the bifurcated branch in question is always unstable. For the symmetric bifurcation case, if the minimum eigenvalue β_2^{min} of B_{ij} is positive, the bifurcation branch in question is stable. Any negative eigenvalues render it unstable.

One can also compute $\Delta\mathcal{E}$ the difference for a fixed value of the load parameter λ, between the potential energies on a bifurcated and on the principal branch. By definition:

$$\Delta\mathcal{E} = \mathcal{E}(\overset{0}{u}(\lambda_c + \Delta\lambda) + \sum_{i=1}^m \xi_i \overset{i}{u} + v(\xi_i, \Delta\lambda), \lambda_c + \Delta\lambda) - \mathcal{E}(\overset{0}{u}(\lambda_c + \Delta\lambda), \lambda_c + \Delta\lambda). \tag{AC-5.28}$$

Expanding $\Delta\mathcal{E}$ about $\xi = 0$, and collecting terms of the like order of ξ, one obtains with the help of Eq. (AC-2.2), Eq. (AC-5.1) and Eq. (AC-5.12), Eq. (AC-5.13) for the asymmetric bifurcation case ($\mathcal{E}_{ijk} \neq 0$):

$$\Delta\mathcal{E} = \frac{\xi^3}{6}\lambda_1 \sum_{i=1}^m\sum_{j=1}^m \alpha_i^1\alpha_j^1 \mathcal{E}_{ij\lambda} + O(\xi^4), \tag{AC-5.29}$$

while following a similar procedure for the symmetric bifurcation case ($\mathcal{E}_{ijk} = 0$) and with the additional help of Eq. (AC-5.9) and Eq. (AC-5.15) one has:

$$\Delta\mathcal{E} = \frac{\xi^4}{8}\lambda_2\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i^1\alpha_j^1\mathcal{E}_{ij\lambda} + O(\xi^5). \tag{AC-5.30}$$

Recall from the discussion of Eq. (AC-5.19) that $\mathcal{E}_{ij\lambda}$ is negative definite and hence $\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i^1\alpha_j^1\mathcal{E}_{ij\lambda} < 0$. Our conclusions about the energies of the bifurcated equilibrium branches through λ_c are the same to the ones reached for the single mode case (see Eq. (AC-3.32), Eq. (AC-3.33)). Given a load level λ, one can see from Eq. (AC-5.29), Eq. (AC-5.30) that the supercritical branches $\lambda > \lambda_c$ (which correspond to $\lambda_1\xi > 0$ for the asymmetric case or to $\lambda_2 > 0$ for the symmetric case) have lower energy than the principal branch. For the subcritical branches $\lambda < \lambda_c$ (which correspond to $\lambda_1\xi < 0$ for the asymmetric case or to $\lambda_2 < 0$ for the symmetric case) the situation is reversed and the principal branch has lower energy than the bifurcated one.

Example

As a simple first application of the general theory developed in this section, the perfect rigid plate example discussed in subection AB-3 will be revisited. In this case the space of admissible displacements $u = (v, \theta, \phi)$ is $U = \mathbb{R}^3$ whose Cartesian basis is $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$. From Eq. (AB-3.1), the stability operator evaluated on the principal branch $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda)$ is the rank two tensor:

$$\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda) = 4E\mathbf{e}_1\mathbf{e}_1 + (\lambda_c - \lambda)L(\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3). \tag{AC-5.31}$$

It is obvious from Eq. (AC-5.31) that at $\lambda = \lambda_c$ the stability operator $\mathcal{E}^c_{,uu}$ has zero as a double eigenvalue and that a corresponding pair of eigenvectors are:

$$\overset{1}{u} = \mathbf{e}_2, \quad \overset{2}{u} = \mathbf{e}_3. \tag{AC-5.32}$$

Notice that the eigenvectors constitute an orthonormal basis to the null space $\mathcal{N} = \{(\theta, \phi)|\ \theta, \phi \in \mathbb{R}\}$. Hence the space $\mathcal{N}^\perp = \{v|\ v \in \mathbb{R}\}$.

The critical point (u_c, λ_c) is a double bifurcation point since it satisfies (AC-5.2), as one can see from Eq. (AC-5.31), Eq. (AC-5.32):

$$\mathcal{E}^c_{,u\lambda}\overset{1}{u} = (-\mathbf{e}_1) \bullet \mathbf{e}_2 = 0, \quad \mathcal{E}^c_{,u\lambda}\overset{2}{u} = (-\mathbf{e}_1) \bullet \mathbf{e}_3 = 0. \tag{AC-5.33}$$

Note that the two lowest eigenvalues of the stability matrix evaluated on the principal branch $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda)$ have a strict crossing of zero at λ_c and hence the matrix $\mathcal{E}_{ij\lambda}$ is negative definite. Indeed from Eq. (AC-5.31), Eq. (AC-5.32) and recalling the definition of $\mathcal{E}_{ij\lambda}$ in Eq. (AC-5.11):

$$\mathcal{E}_{ij\lambda} \equiv ((d\mathcal{E}_{,uu}/d\lambda)_c\overset{i}{u})\overset{j}{u} = -L((\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3) \bullet \overset{i}{u}) \bullet \overset{j}{u} = -L\delta_{ij}. \tag{AC-5.34}$$

The two eigenvalues $(d\overset{0}{\beta}/d\lambda)_c$ of $\mathcal{E}_{ij\lambda}$ (see Eq. (AC-5.19)) are from Eq. (AC-5.34) found to be $(d\overset{0}{\beta}/d\lambda)_c = -L < 0$.

Also note that the bifurcation amplitude parameters ξ_1 and ξ_2 are found with the help of Eq. (AC-5.3) and Eq. (AC-5.32) to be:

$$\begin{aligned} \xi_1 &= (u - \overset{0}{u}(\lambda)) \bullet \overset{1}{u} = [(v - (\lambda/4E))\mathbf{e}_1 + \theta\mathbf{e}_2 + \phi\mathbf{e}_3] \bullet \mathbf{e}_2 = \theta, \\ \xi_2 &= (u - \overset{0}{u}(\lambda)) \bullet \overset{2}{u} = [(v - (\lambda/4E))\mathbf{e}_1 + \theta\mathbf{e}_2 + \phi\mathbf{e}_3] \bullet \mathbf{e}_3 = \phi, \end{aligned} \qquad \text{(AC-5.35)}$$

which gives a meaningful physical interpretation to the bifurcation amplitude parameters ξ_i in this example.

For the asymmetric rigid plate model $m \neq 0, n = 0$ the determination of the first order coefficients $\{\alpha_i^1\}$ in the expansion of the bifurcation amplitudes ξ_i and the first order coefficient λ_1 in the expansion of the load λ for each bifurcated equilibrium path, requires in addition to Eq. (AC-5.34) the evaluation of the rank three tensor $\mathcal{E},^c_{uuu}$. Hence from Eq. (AB-3.1) one has:

$$\mathcal{E},^c_{uuu} = 2mL^3[(\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2)\mathbf{e}_2 + (\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3)\mathbf{e}_3]. \qquad \text{(AC-5.36)}$$

Using Eq. (AC-5.32), Eq. (AC-5.34) and recalling Eq. (AC-5.36) into the definition of the coefficients $\mathcal{E}_{ijk}$ (see Eq. (AC-5.11)), Eq. (AC-5.13) gives the following system for $\{\alpha_i^1\}$ and λ_1:

$$\begin{aligned} 4mL^3\alpha_1^1\alpha_2^1 - 2\lambda_1 L\alpha_1^1 &= 0 \\ 2mL^3[(\alpha_1^1)^2 + (\alpha_2^1)^2] - 2\lambda_1 L\alpha_2^1 &= 0 \end{aligned} \qquad (\alpha_1^1)^2 + (\alpha_2^1)^2 = 1. \qquad \text{(AC-5.37)}$$

The above system admits three different solutions $M1$, $M2$, $M3$ namely:

$$\begin{aligned} M1 &: \alpha_1^1 = 0, & \alpha_2^1 &= 1, & \lambda_1 &= mL^2, \\ M2 &: \alpha_1^1 = 1/\sqrt{2}, & \alpha_2^1 &= 1/\sqrt{2}, & \lambda_1 &= \sqrt{2}mL^2, \\ M3 &: \alpha_1^1 = -1/\sqrt{2}, & \alpha_2^1 &= 1/\sqrt{2}, & \lambda_1 &= -\sqrt{2}mL^2. \end{aligned} \qquad \text{(AC-5.38)}$$

By continuing with the higher order coefficients $\{\alpha_i^n\}$, $n > 1$ in the expansion of $\xi_i(\xi)$ (see Eq. (AC-5.12) as well as with the component v of $u - \overset{0}{u}$ on $N^\perp$ ($v = (u - \overset{0}{u}) \bullet \mathbf{e}_1$) one easily finds $\alpha_i^n = 0$ for $n > 1$ and $v = 0$. Hence from Eq. (AC-5.38) the $2^m - 1 = 3$ bifurcated equilibrium branches for the asymmetric rigid plate model are:

$$\begin{aligned} M1 &: u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, & \lambda &= \lambda_c + \xi mL^2; & \theta &= 0,\ \phi = \xi, \\ M2 &: u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, & \lambda &= \lambda_c + \xi\sqrt{2}mL^2; & \theta &= \phi = \xi/\sqrt{2}, \\ M3 &: u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, & \lambda &= \lambda_c - \xi\sqrt{2}mL^2; & -\theta &= \phi = \xi/\sqrt{2} \end{aligned} \qquad \text{(AC-5.39)}$$

which coincide with the solutions found in Eq. (AC-3.4)$_2$ as expected.

For the stability of the above bifurcated equilibrium branches one has to investigate the eigenvalues of the stability matrix B_{ij} defined in Eq. (AC-5.14), which with the help of

Eq. (AC-5.34) and Eq. (AC-5.36) is found to be:

$$[B_{ij}] = \begin{bmatrix} 2mL^3\alpha_2^1 - \lambda_1 L & 2mL^3\alpha_1^1 \\ 2mL^3\alpha_1^1 & 2mL^3\alpha_2^1 - \lambda_1 L \end{bmatrix} \tag{AC-5.40}$$

For each one of the equilibrium paths in Eq. (AC-5.39), the above stability matrix takes the form:

$$\begin{aligned} &M1 \ : \ [B_{ij}] = \begin{bmatrix} mL^3 & 0 \\ 0 & mL^3 \end{bmatrix}, \qquad \beta_1^{max} = \beta_1^{min} = mL^3 \implies M1 \begin{cases} m\xi > 0 & \text{stable} \\ m\xi < 0 & \text{unstable} \end{cases} \\ &M2 \ : \ [B_{ij}] = \begin{bmatrix} 0 & \sqrt{2}mL^3 \\ \sqrt{2}mL^3 & 0 \end{bmatrix}, \ \beta_1^{max} = \sqrt{2}mL^3, \ \beta_1^{min} = -\sqrt{2}mL^3 \implies M2 \text{ unstable} \\ &M3 \ : \ [B_{ij}] = \begin{bmatrix} 0 & \sqrt{2}mL^3 \\ \sqrt{2}mL^3 & 0 \end{bmatrix}, \ \beta_1^{max} = \sqrt{2}mL^3, \ \beta_1^{min} = -\sqrt{2}mL^3 \implies M3 \text{ unstable} \end{aligned} \tag{AC-5.41}$$

which coincide with the results obtained in Section AB-3 (see the discussion of Eq. (AB-3.6)$_2$).

For the symmetric rigid plate model $m = 0, n \neq 0$ as seen from Eq. (AC-5.14) one would require the additional evaluation of the rank four tensor $\mathcal{E}^{c}_{,uuuu}$ which from Eq. (AC-3.1) is found to be:

$$\begin{aligned} \mathcal{E}^{c}_{,uuuu} = \ & 12nL^4[(\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2 - \frac{1}{2}\mathbf{e}_3\mathbf{e}_3)\mathbf{e}_2\mathbf{e}_2 + (-\frac{1}{2}\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3)\mathbf{e}_3\mathbf{e}_3 + \\ & +(\mathbf{e}_2\mathbf{e}_2 - \frac{1}{2}\mathbf{e}_2\mathbf{e}_3 - \frac{1}{2}\mathbf{e}_3\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3)(\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2)], \end{aligned} \tag{AC-5.42}$$

Substitution of Eq. (AC-5.32), Eq. (AC-5.34), Eq. (AC-5.36), Eq. (AC-5.42) into Eq. (AC-5.15) gives the following system for $\{\alpha_1^1\}$ and λ_2:

$$\begin{aligned} &nL^4[12(\alpha_1^1)^3 + 36(\alpha_1^1)^2\alpha_2^1 - 18\alpha_1^1(\alpha_2^1)^2 + 12(\alpha_2^1)^3] - 3\lambda_2 L\alpha_1^1 = 0 \\ &nL^4[12(\alpha_1^1)^3 - 18(\alpha_1^1)^2\alpha_2^1 + 36\alpha_1^1(\alpha_2^1)^2 + 12(\alpha_2^1)^3] - 3\lambda_2 L\alpha_2^1 = 0 \end{aligned} \qquad (\alpha_1^1)^2 + (\alpha_2^1)^2 = 1. \tag{AC-5.43}$$

The above system admits four different real solutions $N1$, $N2$, $N3$, $N4$ namely:

$$\begin{aligned} &N1 \ : \ \alpha_1^1 = 1/\sqrt{2}, \quad \alpha_2^1 = 1/\sqrt{2}, \quad \lambda_1 = 0, \quad \lambda_2 = 7nL^3 \\ &N2 \ : \ \alpha_1^1 = -1/\sqrt{2}, \quad \alpha_2^1 = 1/\sqrt{2}, \quad \lambda_1 = 0, \quad \lambda_2 = -9nL^3 \\ &N3 \ : \ \alpha_1^1 = 2/\sqrt{5}, \quad \alpha_2^1 = 1/\sqrt{5}, \quad \lambda_1 = 0, \quad \lambda_2 = (36/5)nL^3 \\ &N4 \ : \ \alpha_1^1 = 1/\sqrt{5}, \quad \alpha_2^1 = 2/\sqrt{5}, \quad \lambda_1 = 0, \quad \lambda_2 = (36/5)nL^3 \end{aligned} \tag{AC-5.44}$$

By continuing with higher order terms $\{\alpha_i^n\}, n > 1$ in the expansion of $\xi_i(\xi)$ as well as with the component v of $u - \overset{0}{u}$ on $N^{\perp}$ $\left(v = (u - \overset{0}{u}) \bullet \mathbf{e}_1\right)$ one easily finds $\alpha_i^n = 0$ for $n > 1$ and $v = 0$. Hence from Eq. (AC-5.44) the $(3^m - 1)/2 = 4$ bifurcated equilibrium branches for the symmetric plate model are:

$$\begin{aligned}
N1 &: u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, \quad \lambda = \lambda_c + \xi^2(7nL^3/2), \quad \theta = \phi = \xi/\sqrt{2} \\
N2 &: u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, \quad \lambda = \lambda_c - \xi^2(9nL^3/2), \quad -\theta = \phi = \xi/\sqrt{2} \\
N3 &: u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, \quad \lambda = \lambda_c + \xi^2(18nL^3/5), \quad \theta = 2\phi = 2\xi/\sqrt{5} \\
N4 &: u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, \quad \lambda = \lambda_c + \xi^2(18nL^3/5), \quad 2\theta = \phi = 2\xi/\sqrt{5}
\end{aligned} \tag{AC-5.45}$$

For the stability of the above bifurcated equilibrium branches, one has to investigate the eigenvalues of the stability matrix B_{ij} defined in Eq. (AC-5.16) which with the help of Eq. (AC-5.37), Eq. (AC-5.38), Eq. (AC-5.40) and Eq. (AC-5.42) can be written as:

$$[B_{ij}] = \begin{bmatrix} 6nL^4[2(\alpha_1^1)^2 + 4\alpha_1^1\alpha_2^1 - (\alpha_2^1)^2] - \lambda_2 L & 12nL^4[(\alpha_1^1)^2 - \alpha_1^1\alpha_2^1 + (\alpha_2^1)^2] \\ 12nL^4[(\alpha_1^1)^2 - \alpha_1^1\alpha_2^1 + (\alpha_2^1)^2] & 6nL^4[-(\alpha_1^1)^2 + 4\alpha_1^1\alpha_2^1 + 2(\alpha_2^1)^2] - \lambda_2 L \end{bmatrix} \tag{AC-5.46}$$

For each one of the bifurcated equilibrium paths in Eq. (AC-5.45), the stability matrix in Eq. (AC-5.46) takes the form:

$$\begin{aligned}
N1 &: [B_{ij}] = \begin{bmatrix} 8nL^4 & 6nL^4 \\ 6nL^4 & 8nL^4 \end{bmatrix}, \quad \beta_2^{min} = 2nL^4 \implies N1 \text{ stable} \\
N2 &: [B_{ij}] = \begin{bmatrix} 0 & 18nL^4 \\ 18nL^4 & 0 \end{bmatrix}, \quad \beta_2^{min} = -18nL^4 \implies N2 \text{ unstable} \\
N3 &: [B_{ij}] = \begin{bmatrix} (54/5)nL^4 & (36/5)nL^4 \\ (36/5)nL^4 & 0 \end{bmatrix}, \quad \beta_2^{min} = -(18/5)nL^4 \implies N3 \text{ unstable} \\
N4 &: [B_{ij}] = \begin{bmatrix} 0 & (36/5)nL^4 \\ (36/5)nL^4 & (54/5)nL^4 \end{bmatrix}, \quad \beta_2^{min} = -(18/5)nL^4 \implies N4 \text{ unstable}
\end{aligned} \tag{AC-5.47}$$

which coincide with the results obtained in Section AB-3 (see the discussion of Eq. (AB-3.6)$_2$).

Chapter B

APPLICATIONS IN ELASTIC STRUCTURES AND SOLIDS

Having developed the general theory for bifurcation, stability and imperfection sensitivity of elastic structures, attention is now turned in applying it to solve problems in mechanics. This Chapter deals with structural applications.

BA ONE-DIMENSIONAL STRUCTURES

This section deals with one-dimensional structural applications. The most classical application is the inextensible beam under axial compression, the celebrated "*elastica*" problem, which was presented by Euler in 1744.

BA-1 ELASTICA

The first application to be given for the case of one-dimensional elastic structures is the well known "*elastica*" problem due to Euler. It consists of an inextensible, slender planar elastic beam of bending stiffness EI and total length L. The beam is fully clamped at end A while its end B can move freely along the y direction with a slope that remains fixed, as shown in Fig. BA-1.1. The beam is loaded by a compressive force λ which acts along the x axis.

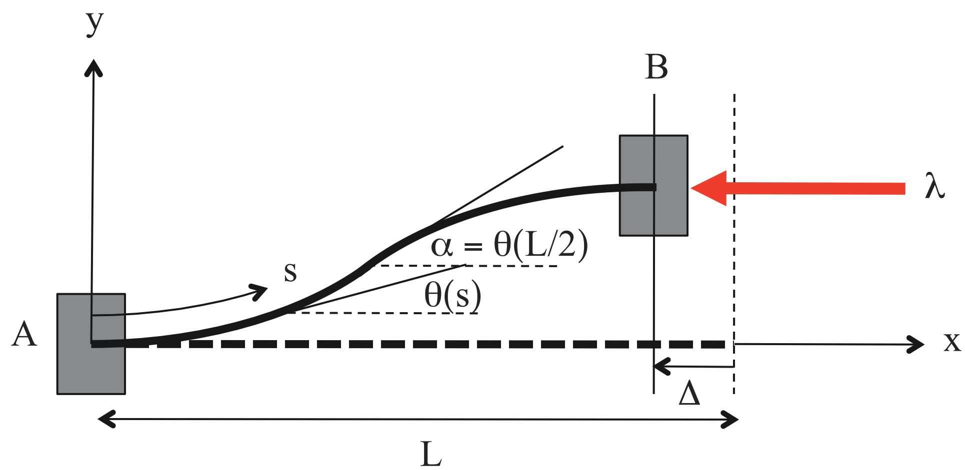

Figure BA-1.1: Undeformed and deformed configurations of Euler's elastica beam.

The role of displacement variable u for this structure is played in this problem by the scalar function $\theta(s)$ where θ denotes the rotation from its initial position of the tangent at a point with arc length coordinate s. The curvature at point s is $d\theta/ds$ and consequently the strain energy of the beam per unit length is $(1/2)EI(d\theta/ds)^2$. The point of application B of the compressive force λ displaces by $\Delta = \int_0^L (ds - dx) = \int_0^L (1-\cos\theta)ds$ and hence the beam's potential energy is given by:

$$\mathcal{E}(\theta,\lambda) = \int_0^L [\frac{1}{2}EI(\frac{d\theta}{ds})^2 - \lambda(1-\cos\theta)]ds. \qquad \text{(BA-1.1)}$$

The kinematically admissible functions $\theta(s)$ are all continuous functions in the interval $[0, L]$ which vanish at $s = 0, L$ and for which the potential energy in Eq. (BA-1.1) exists and is finite.[1] One can also define an inner product for the admissible displacement functions $\theta(s)$. The simplest possible such choice is:

$$(\theta_1, \theta_2) \equiv \frac{1}{L}\int_0^L \theta_1(s)\theta_2(s)ds. \qquad \text{(BA-1.2)}$$

[1]Note: The appropriate space U for $\theta(s)$ turns out to be the space $H_0^1[0, L]$, i.e. the set of all $\theta(s)$ such that $\{\int_0^L [(d\theta/ds)^2 + \theta^2]ds\}^{1/2} < \infty$ and which in addition satisfy $\theta(0) = \theta(L) = 0$.

From Eq. (BA-1.1) the equilibrium equation and boundary conditions of the problem are (see Eq. (AC-2.2)):

$$\frac{d^2\theta}{ds^2} + \frac{\lambda}{EI}\sin\theta = 0, \quad \theta(0) = \theta(L) = 0. \tag{BA-1.3}$$

This boundary value problem has one obvious solution, which is valid for arbitrary loads λ, namely:

$$\overset{0}{\theta}(\lambda) = 0. \tag{BA-1.4}$$

The above trivial solution is obviously the principal equilibrium solution of the beam, for it satisfies $\overset{0}{\theta}(0) = 0$ (see Eq. (AC-2.3)) and corresponds to the straight initial configuration. The principal solution is also stable, at least for adequately small values of the load λ i.e. it satisfies Eq. (AC-2.4). To prove this assertion, one has to show that for adequately small values of λ, a positive constant $\overset{0}{\beta}(\lambda)$ exists such:

$$\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u = \int_0^L [EI\left(\frac{d(\delta\theta)}{ds}\right)^2 - \lambda(\delta\theta)^2]ds \geq \overset{0}{\beta}(\lambda)\frac{1}{L}\int_0^L(\delta\theta)^2 ds, \quad \overset{0}{\beta}(\lambda) > 0, \quad \forall\delta\theta\in U. \tag{BA-1.5}$$

Any admissible $\delta\theta(s)$ can be expanded in a Fourier sine series:

$$\delta\theta = \sum_{n=1}^{\infty}[\delta\theta_n \sin(\frac{n\pi s}{L})]. \tag{BA-1.6}$$

Upon substitution of Eq. (BA-1.6) into Eq. (BA-1.5) one obtains:

$$\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u = \frac{L}{2}\sum_{n=1}^{\infty}\{(\delta\theta_n)^2[EI(\frac{n\pi}{L})^2 - \lambda]\} \geq \frac{\overset{0}{\beta}}{2}\sum_{n=1}^{\infty}(\delta\theta_n)^2. \tag{BA-1.7}$$

The above result proves that for $0 \leq \lambda < EI(\pi/L)^2$ one has $\overset{0}{\beta}(\lambda) = L(EI(\pi/L)^2 - \lambda) > 0$ and hence the straight principal configuration is stable. The $EI(\pi/L)^2$ value of the load, above which the beam loses its stability will be identified subsequently with the lowest critical load, λ_c at which the first bifurcation occurs.

Having discussed the principal solution and its stability, attention is turned next to the bifurcated solutions for the elastica, and in particular the bifurcated equilibrium solution that passes through the lowest critical load. An analytical solution for the wanted bifurcated equilibrium path does exist and is given in terms of elliptic functions. Following the general methodology developed in Section AC an asymptotic solution for the same bifurcated equilibrium path will also be derived. The results from the asymptotic method will be shown to agree with the exact solution.

i) *Exact Solution*

The exact solution for the bifurcated equilibrium path of the elastica is obtained from Eq. (BA-1.3) as follows: After multiplying the equilibrium equation by $(d\theta/ds)$ and integrating with respect to s one gets:

$$\frac{d\theta}{ds} = \mu[2(\cos\theta - \cos\alpha)]^{1/2}, \quad \mu^2 \equiv \lambda/EI, \tag{BA-1.8}$$

where α is the rotation of the beam at its inflexion point, i.e. the point for which $d\theta/ds = 0$. Introducing the change of variable $y = \sin(\theta/2)/\sin(\alpha/2)$ and the boundary condition $\theta(0) = 0$, one obtains by integration of Eq. (BA-1.8):

$$\sin\Big(\frac{\theta(s)}{2}\Big) = k \ \mathrm{sn}(\mu s), \quad k \equiv \sin(\frac{\alpha}{2}), \tag{BA-1.9}$$

where sn(x) denotes the Jacobian elliptic function "*sine-amplitude of x*" defined by:

$$x = \int_0^{\mathrm{sn}(x)} \frac{dy}{[(1-y^2)(1-k^2y^2)]^{1/2}}. \tag{BA-1.10}$$

Let $K(k)$ denote the quarter period of this periodic function, i.e. $\mathrm{sn}(x) = \mathrm{sn}(x+4nK)$ for any integer n:

$$K(k) = \int_0^1 \frac{dy}{[(1-y^2)(1-k^2y^2)]^{1/2}}, \quad \mathrm{sn}(K) = -\mathrm{sn}(3K) = 1, \quad \mathrm{sn}(2K) = \mathrm{sn}(4K) = 0. \tag{BA-1.11}$$

From the remaining boundary condition $\theta(L) = 0$, Eq. (BA-1.9) and the property $\mathrm{sn}(2K) = 0$ (although $\sin(2nK) = 0$, the value $n = 1$ is considered since of interest is the lowest corresponding value of the applied load λ) one obtains the following relation between the load λ, the material and geometric properties of the beam EI, L and the rotation α at the inflexion point of the beam:

$$\mu L = 2K(k) \quad \text{or} \quad L(\frac{\lambda}{EI})^{1/2} = 2K(\sin(\frac{\alpha}{2})). \tag{BA-1.12}$$

For small values of $k = \sin(\alpha/2)$ by expanding K about $k = 0$ one obtains up to $O(k^4)$ in accuracy:

$$\lambda = EI(\frac{\pi}{L})^2[1 + \frac{1}{2}\sin^2(\frac{\alpha}{2}) + \ldots]. \tag{BA-1.13}$$

The above relation between the applied load λ and the resulting rotation α at the middle point (note for $s = L/2$ in Eq. (BA-1.9) $\mathrm{sn}(\mu L/2) = \mathrm{sn}(K) = 1$ and so $\theta(L/2) = \alpha$) of the beam shows that a bifurcated solution is possible when the load $\lambda > \lambda_c \equiv EI(\pi/L)^2$ where λ_c is the lowest load at which instability of the principal solution occurs.

ii) *Asymptotic Solution*

The bifurcation problem for the elastica beam will now be solved using the general asymptotic methodology developed in Section AC. The starting point for these calculations is the determination of the lowest critical load λ_c which from Eq. (AC-2.7) and Eq. (BA-1.1) is found by solving the following system:

$$\frac{d^2\overset{1}{\theta}}{ds^2} + \frac{\lambda_c}{EI}\overset{1}{\theta} = 0, \quad \overset{1}{\theta}(0) = \overset{1}{\theta}(L) = 0. \tag{BA-1.14}$$

The solution to the above problem corresponding to the lowest eigenvalue is:

$$\lambda_c = EI(\pi/L)^2, \quad \overset{1}{\theta}(s) = \sqrt{2}\sin(\pi s/L), \tag{BA-1.15}$$

where $\overset{1}{\theta}(s)$ is the unique eigenmode corresponding to λ_c. The $\sqrt{2}$ term in $\overset{1}{\theta}$ comes from normalization $(\overset{1}{\theta}, \overset{1}{\theta}) = 1$.

The above value for λ_c coincides as expected with the lowest bifurcation load of the elastica found from the exact solution. The fact that λ_c is a bifurcation point can be independently verified from the general theory which gives from Eq. (AC-2.27) with the help of Eq. (BA-1.1) and Eq. (BA-1.15):

$$\mathcal{E}^c_{,u\lambda}\overset{1}{u} = -\int_0^L \sin(\overset{0}{\theta}(s))\overset{1}{\theta}(s)ds = 0. \tag{BA-1.16}$$

Proceeding with the calculation of the expansion of the load λ in terms of the bifurcation amplitude ξ, one obtains from Eq. (BA-1.1) for $((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}$:

$$((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = \lambda_c \int_0^L \sin(\overset{0}{\theta}(s))[\overset{1}{\theta}(s)]^3 ds = 0, \tag{BA-1.17}$$

which shows that the bifurcation is a symmetric one and $\lambda_1 = 0$ (see Eq. (AC-1.13)).

The calculation of λ_2, the next higher order term in the ξ expansion of the load λ_1, requires the evaluation of the various quantities appearing in Eq. (AC-3.14). Starting with $((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$ one obtains from Eq. (BA-1.1) and Eq. (BA-1.15):

$$((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = -\int_0^L \cos(\overset{0}{\theta}(s))[\overset{1}{\theta}(s)]^2 ds = -L. \tag{BA-1.18}$$

In view of Eq. (AC-3.20) the above result means that the lowest eigenvalue of the stability operator evaluated at the principal branch has a strict zero crossing at λ_c with $(d\overset{0}{\beta}/d\lambda)_c = -L$ exactly as expected from the general theory (see discussion of Eq. (AC-3.15)). The result could have also been obtained directly from Eq. (BA-1.7) which gives $\overset{0}{\beta}(\lambda) = L(EI(\pi/L)^2 - \lambda)$.

Of the two expressions appearing in the numerator of λ_2 in Eq. (AC-3.14), $((\mathcal{E}^c_{,uuu}v_{\xi\xi})\overset{1}{u})\overset{1}{u} = 0$ while $(((\mathcal{E}^c_{,uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u}$ yields from Eq. (BA-1.1) and Eq. (BA-1.15):

$$(((\mathcal{E}^c_{,uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} \quad = \lambda_c \int_0^L \cos(\overset{0}{\theta}(s))[\overset{1}{\theta}(s)]^4 ds = \frac{3}{2}L\lambda_c = \frac{3}{2}EI\frac{\pi^2}{L}. \tag{BA-1.19}$$

Hence using the results in Eq. (BA-1.18), Eq. (BA-1.19) into Eq. (AC-3.14) one has:

$$\lambda_2 = -\frac{1}{3}(\frac{3}{2}L\lambda_c)/(-L) = \frac{\lambda_c}{2}. \tag{BA-1.20}$$

Thus the asymptotic expansion for the load λ in terms of the bifurcation amplitude parameter ξ is:

$$\lambda = \lambda_c[1 + \frac{\xi^2}{4} + O(\xi^4)) = EI(\frac{\pi}{L})^2[1 + \frac{\xi^2}{4} + \cdots]. \tag{BA-1.21}$$

Similarly the expansions for $\theta(s)$ are in view of Eq. (BA-1.4) and Eq. (BA-1.15):

$$\theta(s) = \xi\sqrt{2}\sin(\frac{\pi s}{L}) + O(\xi^3). \tag{BA-1.22}$$

It is of interest to compare the above obtained asymptotic results for the bifurcated equilibrium solution with their exact counterparts in Eq. (BA-1.13) and Eq. (BA-1.9). Since the exact solution is given in terms of $k = \sin(\alpha/2)$, while the asymptotic one is given in terms of ξ, the starting point of the comparison is the relation between those two quantities. Recalling from Eq. (AC-3.1) the definition of $\xi = ((u - \overset{0}{u}), \overset{1}{u})$, one obtains up to $O(k^2)$ from Eq. (BA-1.4), Eq. (BA-1.9) and Eq. (BA-1.15):

$$\xi = \frac{1}{L}\int_0^L \theta(s)\overset{1}{\theta}(s)ds = \frac{\sqrt{2}}{L}\int_0^L 2\sin^{-1}[k\mathrm{sn}(\mu s)]\sin(\frac{\pi s}{L})ds = k\sqrt{2} + O(k^2). \tag{BA-1.23}$$

By introducing Eq. (BA-1.23) into Eq. (BA-1.9), Eq. (BA-1.13) one recovers as expected Eq. (BA-1.22) and Eq. (BA-1.21).

BB TWO-DIMENSIONAL STRUCTURES

Attention is now turned in applications involving two-dimensional problems in structural mechanics. The first application to be given is the stability of a flat plate subjected to in-plane loading. Two cases will be considered: a plate with a single eigenmode at the critical load and a plate with a double mode at the critical load.

BB-1 RECTANGULAR PLATE

The first application of the general theory presented in Section AC for two dimensional structures is the buckling of a simply supported rectangular plate which is uniformly compressed in its own plane. Depending on the geometry of the plate, the lowest critical eigenvalue can be either simple or double. Both cases will be studied.

i) *Von Karman Plate Model*

The simplest possible nonlinear plate model that can be successfully employed in the prediction of buckling of an axially compressed plate is due to T. Von Karman. This model is in essence a kinematically nonlinear (but constitutively linear) elastic model and takes into account the contributions to the strains of the squares of the rotations of the middle surface.

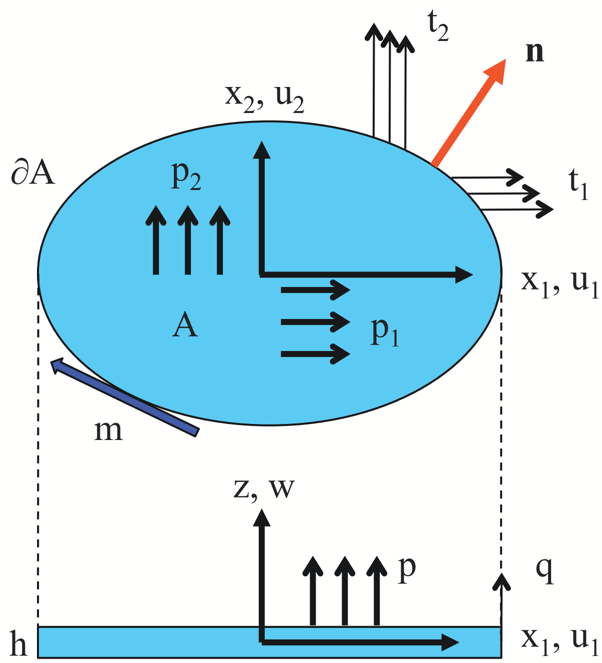

Figure BB-1.1: Von Karman plate.

Consider a thin rectangular plate whose middle surface is a region A in the x_1, x_2 plane and with thickness h, as shown in Fig. BB-1.2. Let $u_\alpha(x_1, x_2)$ $(\alpha = 1, 2)$ be the tangential displacements of an arbitrary point on the middle surface along x_α and $w(x_1, x_2)$ be the vertical displacement of the same point. The small strain - moderate rotation kinematical assumption leads to the following strain-displacement relations for the in-plane membrane strains $E_{\alpha\beta}$ and the curvature strains $K_{\alpha\beta}$

$$E_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + \frac{1}{2}w_{,\alpha}\, w_{,\beta}\,, \qquad K_{\alpha\beta} = -w_{,\,\alpha\beta}, \tag{BB-1.1}$$

where $(\ )_{,\alpha} \equiv \partial(\ )/\partial x_\alpha$. Note also that from here and subsequently in this section Greek indexes range from 1 to 2 while the presence of repeated indexes in an expression implies summation with respect to those indexes from 1 to 2 (Einstein's summation convention).

The membrane resultants $N_{\alpha\beta}$ (N_{11}, N_{22} are the axial forces while $N_{12} = N_{21}$ is the shear force) and the moment resultants $M_{\alpha\beta}$ (M_{11}, M_{22} are the bending moments while $M_{12} = M_{21}$ is the twisting moment) are related to their work conjugate strains $E_{\alpha\beta}$ and $K_{\alpha\beta}$ by

$$N_{\alpha\beta} = hL_{\alpha\beta\gamma\delta}E_{\gamma\delta}, \quad M_{\alpha\beta} = \frac{h^3}{12}L_{\alpha\beta\gamma\delta}K_{\gamma\delta}, \tag{BB-1.2}$$

where the plane stress elastic moduli $L_{\alpha\beta\gamma\delta}$ of the material are given in terms of its Young's modulus E and its Poisson's ratio ν by

$$L_{\alpha\beta\gamma\delta} = \frac{E}{1-\nu^2}\left[\frac{1-\nu}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) + \nu\delta_{\alpha\beta}\delta_{\gamma\delta}\right]. \tag{BB-1.3}$$

The plate's potential energy $\mathcal{E}$, assuming that its midsurface is a region A in the (x_1, x_2) plane with boundary ∂A is

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\int_A [L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}E_{\gamma\delta} + \frac{h^2}{12}K_{\alpha\beta}K_{\gamma\delta})h]dA \\ &- \int_A [p_\alpha u_\alpha + pw]\,dA - \int_{\partial A} [t_\alpha u_\alpha + qw + m(-w_{,n})]\,ds, \end{aligned} \tag{BB-1.4}$$

where p_α, p are the distributed forces on the surface of the plate and t_α, q are the boundary tractions acting along the x_α, z directions respectively while m is the imposed bending moment at the plate's edge and $(-w_{,n})$ is its work conjugate edge rotation ($w_{,n}$ is the directional derivative of w along the outward normal $\mathbf{n}$ to the boundary ∂A).

ia) *Simply Supported Perfect Plate Under Compression – Simple Eigenvalue Case*

Consider the thin rectangular plate shown in Fig. BB-1.2 of dimensions $a_1 \times a_2$ and of thickness h. The plate is simply supported on all its four edges. Moreover the plate is axially compressed along the x_1 and x_2 directions with the help of four rigid bars which keep the plate's edges straight. The rigid bars are considered to be lubricated at their surface of contact with the thin plate and can transmit normal stresses but not shear ones. The normal forces F_α acting on the rigid bars are assumed to increase in proportion to a scalar parameter $\lambda \geq 0$. More specifically it is assumed that $F_1 = \lambda \overset{0}{\sigma}_{11} a_2 h,\ F_2 = \lambda \overset{0}{\sigma}_{22} a_1 h$ where $\overset{0}{\sigma}_{\alpha\beta}$ are constants whose physical meaning will be made apparent in the sequel.

From the absence of distributed forces $p_\alpha = p = 0$, and in view of the simple support at the boundary, the vertical displacement $w = 0$ and the edge bending moment $m = 0$. Consequently, the external loading part of the potential energy $\mathcal{E}$ of the plate takes the form

$$\mathcal{E}_{\text{ext}} = -\int_{\partial A} [t_\alpha u_\alpha]ds. \tag{BB-1.5}$$

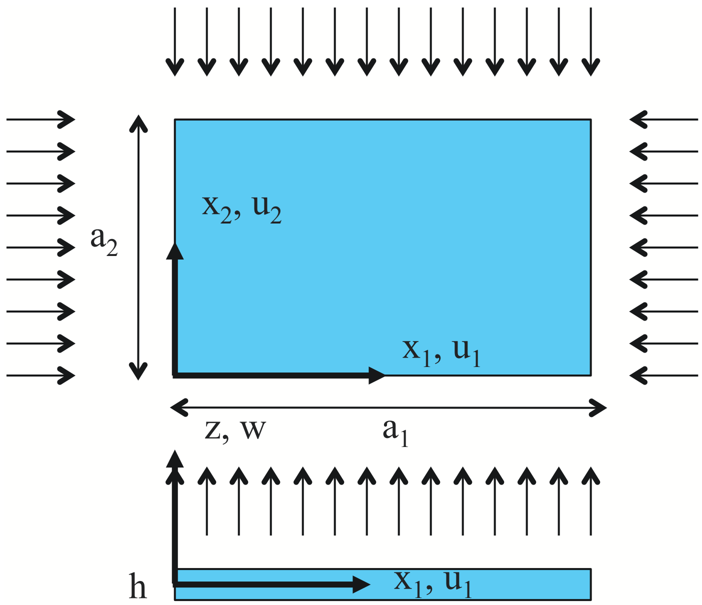

Figure BB-1.2: Rectangular Von Karman plate under in-plane loading.

The trivial principal solution to the plate problem is the one in which the forces applied on the rigid bars are uniformly distributed along the edges of the plate, in which case the traction $t_\alpha = \lambda \overset{0}{\sigma}_{\alpha\beta} n_\beta h$ ($\mathbf{n}$ the outer normal to ∂A) thus implying that $\lambda \overset{0}{\sigma}_{\alpha\beta}$ is the resulting constant stress field inside the plate. From the divergence theorem applied to the two dimensional domain A one has from Eq. (BB-1.5)

$$\mathcal{E}_{\text{ext}} = -\lambda \int_A [\overset{0}{\sigma}_{\alpha\beta} u_{\alpha,\beta}] h dA, \tag{BB-1.6}$$

and consequently from Eq. (BB-1.4) – (BB-1.6) the plate's potential energy can be written as

$$\begin{aligned} \mathcal{E} = & \int_A [\frac{1}{2} L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}E_{\gamma\delta} + \frac{h^2}{12} K_{\alpha\beta}K_{\gamma\delta}) - \lambda \overset{0}{\sigma}_{\alpha\beta} u_{\alpha,\beta}] h dA \\ w(0, x_2) = \ & w(a_1, x_2) = w(x_1, 0) = w(x_1, a_2) = 0 \\ u_1(0, x_2) = \ & u_{1,2}(a_1, x_2) = u_2(x_1, 0) = u_{2,1}(x_1, a_2) = 0, \end{aligned} \tag{BB-1.7}$$

with the membrane ($E_{\alpha\beta}$) and curvature ($K_{\alpha\beta}$) strains given by Eq. (BB-1.1). The above essential boundary conditions given in Eq. (BB-1.7)$_2$ reflect the fact that the vertical displacement at the edges of the plate has to vanish and that the ends of the plate have to remain straight during the deformation. It is also tacitly assumed that the admissible displacements u_α, w have adequate smoothness as to ensure the finiteness of the potential energy integral in Eq. (BB-1.7)$_1$.

The equilibrium equations of the plate are given by extremizing $\mathcal{E}$ with respect to the displacement Eq. (see (AC-2.2)). Hence from Eq. (BB-1.7) and by recalling Eq. (BB-1.1) one obtains

$$\mathcal{E},_u \delta u = \int_A [L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}\delta E_{\gamma\delta} + \frac{h^2}{12}K_{\alpha\beta}\delta K_{\gamma\delta}) - \lambda\overset{0}{\sigma}_{\alpha\beta}\delta u_{\alpha,\beta}]hdA = 0$$

$$\delta E_{\alpha\beta} = \frac{1}{2}(\delta u_{\alpha,\beta} + \delta u_{\beta,\alpha}) + \frac{1}{2}(w,_\alpha \delta w,_\beta + w,_\beta \delta w,_\alpha), \quad \delta K_{\alpha\beta} = -\delta w,_{\alpha\beta}, \tag{BB-1.8}$$

where the admissible displacement δu_α, δw satisfy the essential boundary conditions Eq. (BB-1.7)$_2$.

It is not difficult to see that the principal solution to Eq. (BB-1.8) is the constant membrane force, zero moment solution, i.e.,

$$\overset{0}{N}_{\alpha\beta} = \lambda h\overset{0}{\sigma}_{\alpha\beta}, \quad (\overset{0}{\sigma}_{11}, \overset{0}{\sigma}_{22} \neq 0, \quad \overset{0}{\sigma}_{12} = \overset{0}{\sigma}_{21} = 0); \quad \overset{0}{M}_{\alpha\beta} = 0$$

$$\overset{0}{u}_1 = (\lambda x_1/E)(\overset{0}{\sigma}_{11} - \nu\overset{0}{\sigma}_{22}), \quad \overset{0}{u}_2 = (\lambda x_2/E)(\overset{0}{\sigma}_{22} - \nu\overset{0}{\sigma}_{11}); \quad \overset{0}{w} = 0, \tag{BB-1.9}$$

since a simple substitution of Eq. (BB-1.9) into Eq. (BB-1.8) proves that the equilibrium equations and boundary conditions are satisfied and since the solution vanishes for $\lambda = 0$.

Of interest is the lowest critical load λ_c for the above found principal solution. To this end one has to find the lowest load λ_c that satisfies Eq. (AC-2.7), namely $(\mathcal{E},_{uu}(\overset{0}{u}(\lambda_c), \lambda_c)\overset{1}{u})\delta u = 0$. Upon taking an additional derivative of Eq. (BB-1.8) with respect to u, and after noting also from the principal solution Eq. (BB-1.9) that $\overset{0}{N}_{\alpha\beta} = hL_{\alpha\beta\gamma\delta}\overset{0}{E}_{\alpha\beta} = \lambda h\overset{0}{\sigma}_{\alpha\beta}$, one obtains at the critical load λ_c

$$(\mathcal{E}^c,_{uu}\overset{1}{u})\delta u = \int_A [L_{\alpha\beta\gamma\delta}(\overset{1}{u}_{\alpha,\beta}\delta u_{\gamma,\delta} + \frac{h^2}{12}\overset{1}{w},_{\alpha\beta}\delta w,_{\gamma\delta}) + \lambda\overset{0}{\sigma}_{\alpha\beta}\overset{1}{w},_\alpha \delta w,_\beta]hdA = 0. \tag{BB-1.10}$$

Note that in the derivation of Eq. (BB-1.10) use was also made of the symmetries of the plane stress moduli $L_{\alpha\beta\gamma\delta}$ given in Eq. (BB-1.3). It is also understood that the eigenmode $\overset{1}{u} = (\overset{1}{u}_\alpha, \overset{1}{w})$ has to satisfy the essential boundary conditions Eq. (BB-1.7)$_2$.

Integration of Eq. (BB-1.10) by parts gives the following Euler-Lagrange form of the corresponding equations plus boundary conditions:

$$\begin{aligned}
\delta_{u_\alpha}: \quad & (L_{\alpha\beta\gamma\delta}\overset{1}{u}_{\gamma,\delta}),_\beta = 0 \text{ in } A, \\
& L_{12\gamma\delta}\overset{1}{u}_{\gamma,\delta} = 0 \text{ on } \partial A, \quad \overset{1}{u}_1(0,x_2) = \overset{1}{u}_{1,2}(a_1,x_2) = \overset{1}{u}_2(x_1,0) = \overset{1}{u}_{2,1}(x_1,a_2) = 0, \\
\delta w: \quad & (h^2/12)L_{\alpha\beta\gamma\delta}\overset{1}{w},_{\alpha\beta\gamma\delta} - \lambda\overset{0}{\sigma}_{\alpha\beta}\overset{1}{w},_{\alpha\beta} = 0 \text{ in } A, \\
& \overset{1}{w} = 0 \text{ on } \partial A, \quad \overset{1}{w},_{11}(0,x_2) = \overset{1}{w},_{11}(a_1,x_2) = \overset{1}{w},_{22}(x_1,0) = \overset{1}{w},_{22}(x_1,a_2) = 0.
\end{aligned} \tag{BB-1.11}$$

Observe that Eq. (BB-1.11)$_1$ is the equilibrium equation of plane stress linear elasticity with zero distributed loads on A and zero boundary conditions on ∂A. Consequently the unique solution for $\overset{1}{u}_\alpha$ is the zero one i.e.

$$\overset{1}{u}_\alpha = 0. \tag{BB-1.12}$$

On the other hand, one observes that Eq. (BB-1.11)$_2$ admits the following solutions $\overset{1}{w}$

$$\overset{1}{w} = h \sin(m\pi x_1/a_1) \sin(n\pi x_2/a_2), \tag{BB-1.13}$$

where m and n are arbitrary positive integers ($m, n \in \mathbb{N}$). A substitution of Eq. (BB-1.13) into Eq. (BB-1.11)$_2$ gives for the value of $\lambda = \lambda(m,n)$ corresponding to the $h \sin(m\pi x_1/a_1) \sin(n\pi x_2/a_2)$ eigenmode

$$\lambda(m,n) = -\frac{h^2}{12}\,\frac{E}{1-\nu^2}\,\frac{\left[(m\pi/a_1)^2 + (n\pi/a_2)^2\right]^2}{\overset{0}{\sigma}_{11}(m\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n\pi/a_2)^2}. \tag{BB-1.14}$$

Consequently there exists an infinity of critical points for the stability operator of the simply supported plate. Moreover all these points are bifurcation points since according to the general theory from Eq. (AC-2.27) $\mathcal{E}^c,_{u\lambda}\,\overset{1}{u} = 0$. Indeed from Eq. (BB-1.8) and Eq. (BB-1.12) we deduce

$$\mathcal{E}^c,_{u\lambda}\,\overset{1}{u} = -\int_A [\overset{0}{\sigma}_{\alpha\beta}\overset{1}{u}_{\alpha,\beta}] h dA = 0. \tag{BB-1.15}$$

Of all these bifurcation points of interest of course is the one corresponding to the lowest critical load. Assume for the time being that the minimum of $\lambda(m,n)$ is achieved for a unique pair of real integers (m_c, n_c)

$$\lambda_c = \lambda(m_c, n_c) = \min_{m,n\in\mathbb{N}}[\lambda(m,n)], \tag{BB-1.16}$$

then the $h \sin(m_c\pi x_1/a_1) \sin(n_c\pi x_2/a_2)$ is the unique eigenmode corresponding to λ_c at which point the principal equilibrium branch encounters a simple bifurcation.

One expects the principal branch of the solution to be stable for $0 \le \lambda < \lambda_c$. Indeed it will be shown that the stability functional of the principal solution $(\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u$ is positive definite in the aforementioned range of loads λ. From Eq. (BB-1.8) and Eq. (BB-1.9) we have

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u = \int_A [L_{\alpha\beta\gamma\delta}(\delta u_{\alpha,\beta}\delta u_{\gamma,\delta} + \frac{h^2}{12}\delta w,_{\alpha\beta}\delta w,_{\gamma\delta}) + \lambda\overset{0}{\sigma}_{\alpha\beta}\delta w,_{\alpha}\,\delta w,_{\beta}] h dA. \tag{BB-1.17}$$

A straightforward calculation using Eq. (BB-1.3) shows that for $1 > \nu > 0$, $L_{\alpha\beta\gamma\delta}\delta u_{\alpha,\beta}\delta u_{\gamma,\delta} > 0$ if $[\delta u_{\alpha,\beta}]_s \neq 0$ and hence one has to concentrate on the δw dependent part of the integrand in Eq. (BB-1.17). Noting that any admissible δw satisfying the essential boundary conditions $\delta w = 0$ or ∂A (see Eq. (BB-1.7)$_2$) can be put in the form

$$\delta w = h \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \delta W_{mn} \sin(m\pi x_1/a_1) \sin(n\pi x_2/a_2), \tag{BB-1.18}$$

where the δw dependent part of the quadratic in δu form in Eq. (BB-1.17) takes the form:

$$\begin{gathered}\int_A [\frac{h^2}{12} L_{\alpha\beta\gamma\delta}\delta w,_{\alpha\beta}\delta w,_{\gamma\delta} + \lambda\overset{0}{\sigma}_{\alpha\beta}\delta w,_{\alpha}\,\delta w,_{\beta}] h dA = \\ \frac{a_1 a_2 h^3}{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\delta W_{mn})^2\,[\lambda(m,n) - \lambda]\,[-\overset{0}{\sigma}_{11}(m\pi/a_1)^2 - \overset{0}{\sigma}_{22}(n\pi/a_2)^2].\end{gathered} \tag{BB-1.19}$$

Since $\overset{0}{\sigma}_{11}$, $\overset{0}{\sigma}_{22} < 0$ (the plate is compressed), it follows from Eq. (BB-1.7) and Eq. (BB-1.19) that for $0 \leq \lambda < \lambda_c$ (with λ_c given by Eq. (BB-1.16)) that the δw part of the stability functional Eq. (BB-1.17) is positive definite and hence, given that the δu part of the stability functional is always positive irrespective of the value of λ, the stability of the principal solution for loads up to the lowest bifurcation load λ_c follows from the just proved positive definiteness of Eq. (BB-1.17).

One should note at this point that with some additional effort, and after defining an inner product in the space of all admissible displacements, one could have shown the ellipticity of the stability operator in the sense of Eq. (AC-2.4), i.e., that a $\overset{0}{\beta}(\lambda) > 0$ exists $(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda)\delta u)\delta u \geq \overset{0}{\beta}(\lambda)(\delta u, \delta u)$ for $0 \leq \lambda < \lambda_c$, exactly as shown for the relatively simpler case of the elastica (see Eq. (BA-1.5)).

Having found the lowest bifurcation load $\lambda_c = \lambda(m_c, n_c)$, attention is focused on the post-bifurcated expansion of the bifurcated equilibrium path. To this end notice that $((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$ can be found from Eq. (BB-1.7), Eq. (BB-1.9) and Eq. (BB-1.13)

$$((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = \int_A [\overset{0}{\sigma}_{\alpha\beta}\overset{1}{w}_{,\alpha}\overset{1}{w}_{,\beta}]hdA = \frac{h^3}{4}a_1a_2[\overset{0}{\sigma}_{11}(m_c\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n_c\pi/a_2)^2]. \tag{BB-1.20}$$

Notice that since $\overset{0}{\sigma}_{\alpha\beta} < 0$, $((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} < 0$, as expected from the general theory in Section AC-3.

According to the general theory, the first term λ_1 in the asymptotic expansion of the bifurcated equilibrium path through λ_c is given by Eq. (AC-3.13). The denominator in that expression has just been calculated above in Eq. (BB-1.20) while the numerator, with the help of Eq. (BB-1.7), Eq. (BB-1.9) and Eq. (BB-1.12) yields

$$((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = \int_A 3[L_{\alpha\beta\gamma\delta}(\overset{1}{w}_{,\alpha}\overset{1}{w}_{,\beta}\overset{1}{u}_{\gamma,\delta})]hdA = 0, \tag{BB-1.21}$$

which implies from Eq. (AC-3.13) that $\lambda_1 = 0$ and hence that the structure will undergo a symmetric bifurcation.

The calculation of the next term λ_2 in the expansion of the load requires according to Eq. (AC-3.14) the determination of the next order term $v_{\xi\xi}$ in the expansion of the displacement which is found according to the general theory from Eq. (AC-3.8). With the help of Eq. (BB-1.7), Eq. (BB-1.9), one obtains from Eq. (AC-3.8) the following variational equation for $v_{\xi\xi} = (\overset{2}{u}_\alpha, \overset{2}{w})$

$$\begin{aligned} 0 = ((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u} + \mathcal{E}^c_{,uu}v_{\xi\xi})\ \delta v = \int_A [L_{\alpha\beta\gamma\delta}(\overset{1}{w}_{,\gamma}\overset{1}{w}_{,\delta}\delta u_{\alpha,\beta} + 2\overset{1}{u}_{\gamma,\delta}\overset{1}{w}_{,\alpha}\delta w_{,\beta})]hdA \\ + \int_A L_{\alpha\beta\gamma\delta}(\overset{2}{u}_{\gamma,\delta}\delta u_{\alpha,\beta} + \frac{h^2}{12}\overset{2}{w}_{,\gamma\delta}\delta w_{,\alpha\beta}) + \lambda_c\overset{0}{\sigma}_{\alpha\beta}\overset{2}{w}_{,\alpha}\delta w_{,\beta}]hdA. \end{aligned} \tag{BB-1.22}$$

Integration of Eq. (BB-1.22) by parts and recalling the admissibility conditions Eq. (BB-

1.7)$_2$ gives the following point-wise equations and boundary conditions for $\overset{2}{u}_\alpha, \overset{2}{w}$

$$\delta u_\alpha : \qquad (L_{\alpha\beta\gamma\delta}(\overset{2}{u}_{\gamma,\delta} + \overset{1}{w}_{,\gamma}\,\overset{1}{w}_{,\delta}))_{,\beta} = 0 \text{ in } A;$$

$$L_{12\gamma\delta}(\overset{2}{u}_{\gamma,\delta} + \overset{1}{w}_{,\gamma}\,\overset{1}{w}_{,\delta}) = 0 \text{ on } \partial A, \;\; \overset{2}{u}_1(0,x_2) = \overset{2}{u}_{1,2}(a_1,x_2) = \overset{2}{u}_2(x_1,0) = \overset{2}{u}_{2,1}(x_1,a_2) = 0$$

$$\delta w : \qquad (h^2/12)L_{\alpha\beta\gamma\delta}\overset{2}{w}_{,\alpha\beta\gamma\delta} - \lambda_c \overset{0}{\sigma}_{\alpha\beta}\overset{2}{w}_{,\alpha\beta} = 0 \text{ in } A;$$

$$\overset{2}{w} = 0 \text{ on } \partial A, \quad \overset{2}{w}_{,11}(0,x_2) = \overset{2}{w}_{,11}(a_1,x_2) = \overset{2}{w}_{,22}(x_1,0) = \overset{2}{w}_{,22}(x_1,a_2) = 0. \tag{BB-1.23}$$

Notice that the equation governing $\overset{2}{w}$ is identical to the one for $\overset{1}{w}$ (compare Eq. (BB-1.23)$_1$ and Eq. (BB-1.11)$_2$ for $\lambda = \lambda_c$) and hence $\overset{2}{w} = c\overset{1}{w}$. However, in view of the orthogonality $(v_{\xi\xi}, \overset{1}{u}) = 0$ (recall that $v_{\xi\xi} \in \mathcal{N}^\perp$ according to the general theory in subsection AC-3) and the fact that $\overset{1}{u}_\alpha = 0$ from Eq. (BB-1.12), one concludes $0 = (v_{\xi\xi}, \overset{1}{u}) = (\overset{2}{w}, \overset{1}{w}) = c(\overset{1}{w}, \overset{1}{w})$ which implies that the constant $c = 0$ and hence

$$\overset{2}{w} = 0. \tag{BB-1.24}$$

The determination of $\overset{2}{u}_\alpha$ is a somewhat more tricky task and is based on the observation that Eq. (BB-1.23)$_1$ are the equations of plane stress linear elasticity with a distributed body force $(L_{\alpha\beta\gamma\delta}\overset{1}{w}_{,\gamma}\,\overset{1}{w}_{,\delta})_{,\beta}$. Consequently, the Airy stress function technique of linear elasticity will be employed in this case. To this end, define the following quantities

$$s_{\alpha\beta} \equiv L_{\alpha\beta\gamma\delta}(\overset{2}{u}_{\gamma,\delta} + \overset{1}{w}_{,\gamma}\,\overset{1}{w}_{,\delta}) = L_{\alpha\beta\gamma\delta}\mathrm{e}_{\gamma\delta}, \quad e_{\alpha\beta} \equiv \frac{1}{2}(\overset{2}{u}_{\alpha,\beta} + \overset{2}{u}_{\beta,\alpha}) + \overset{1}{w}_{,\alpha}\,\overset{1}{w}_{,\beta}\,. \tag{BB-1.25}$$

Moreover define the Airy stress function $f(x_1, x_2)$ to be

$$s_{11} \equiv f_{,22}\,, \quad s_{22} \equiv f_{,11}\,, \quad s_{12} = s_{21} \equiv -f_{,12}\,. \tag{BB-1.26}$$

It is not difficult to see that with this definition the in-plane equilibrium equations in Eq. (BB-1.23)$_1$ take the form $s_{\alpha\beta,\beta} = 0$ i.e. they are identically satisfied. On the other hand, by inverting the relation between $s_{\alpha\beta}$ and $e_{\alpha\beta}$ from Eq. (BB-1.25) by using the elastic moduli expressions in Eq. (BB-1.3), one has

$$e_{11} = \frac{1}{E}(s_{11} - \nu s_{22}), \quad e_{22} = \frac{1}{E}(s_{22} - \nu s_{11}), \quad e_{12} = e_{21} = \frac{1+\nu}{E}s_{12}. \tag{BB-1.27}$$

In addition, by exploiting the relation between $e_{\alpha\beta}$, $\overset{2}{u}_\alpha$ and $\overset{1}{w}$ one obtains the compatibility condition

$$e_{11,22} + e_{22,11} - 2e_{12,12} = 2[(\overset{1}{w}_{,12})^2 - \overset{1}{w}_{,11}\,\overset{1}{w}_{,22}]. \tag{BB-1.28}$$

Upon substitution of Eq. (BB-1.27), Eq. (BB-1.26) and Eq. (BB-1.13) into Eq. (BB-1.28) one obtains the following governing equation for f

$$\frac{1}{E}\nabla^4 f = h^2(m_c\pi/a_1)^2(n_c\pi/a_2)^2[\cos(2m_c\pi x_1/a_1) + \cos(2n_c\pi x_2/a_2)], \tag{BB-1.29}$$

where ∇^4 denotes the biharmonic operator i.e. $\nabla^4 f \equiv f_{,1111} + 2f_{,1122} + f_{,2222}$.

The solution of Eq. (BB-1.29) is found to be

$$f = \frac{Eh^2}{16}\left[\frac{(n_c/a_2)^2}{(m_c/a_1)^2}\cos(2m_c\pi x_1/a_1) + \frac{(m_c/a_1)^2}{(n_c/a_2)^2}\cos(2n_c\pi x_2/a_2)\right], \tag{BB-1.30}$$

and thus from Eq. (BB-1.26), Eq. (BB-1.27) upon substitution of Eq. (BB-1.30) one has for $s_{\alpha\beta}$

$$s_{11} = -\frac{Eh^2}{4}(m_c\pi/a_1)^2\cos(2n_c\pi x_2/a_2), \quad s_{22} = -\frac{Eh^2}{4}(n_c\pi/a_2)^2\cos(2m_c\pi x_1/a_1), \quad s_{12} = s_{21} = 0, \tag{BB-1.31}$$

and for $e_{\alpha\beta}$

$$\begin{aligned} e_{11} = & -\frac{h^2}{4}[(m_c\pi/a_1)^2\cos(2n_c\pi x_2/a_2) - \nu(n_c\pi/a_2)^2\cos(2m_c\pi x_1/a_1)], \quad e_{12} = 0. \\ e_{22} = & -\frac{h^2}{4}[(n_c\pi/a_2)^2\cos(2m_c\pi x_1/a_1) - \nu(m_c\pi/a_1)^2\cos(2n_c\pi x_2/a_2)], \quad e_{12} = 0. \end{aligned} \tag{BB-1.32}$$

Finally, from Eq. (BB-1.25)$_2$ and Eq. (BB-1.32) one obtains for $\overset{2}{u}_\alpha$

$$\begin{aligned} \overset{2}{u}_1 &= -\frac{h^2}{4}\{[(m_c\pi/a_1)^2x_1 + [\frac{(m_c\pi/a_1)^2 - \nu(n_c\pi/a_2)^2}{2(m_c\pi/a_1)} - \frac{1}{2}(m_c\pi/a_1)\cos(2n_c\pi x_2/a_2)]\sin(2m_c\pi x_1/a_1)\} \\ \overset{2}{u}_2 &= -\frac{h^2}{4}\{[(n_c\pi/a_2)^2x_2 + [\frac{(n_c\pi/a_2)^2 - \nu(m_c\pi/a_1)^2}{2(n_c\pi/a_1)} - \frac{1}{2}(n_c\pi/a_2)\cos(2m_c\pi x_1/a_1)]\sin(2n_c\pi x_2/a_2)\} \end{aligned} \tag{BB-1.33}$$

It is not difficult to verify that the $\overset{2}{u}_\alpha$ found above, also verify all the boundary conditions in Eq. (BB-1.23)$_1$.

The last ingredient according to the general theory (see (AC-3.14)) required for the calculation of λ_2 is the numerator $(((\mathcal{E}^c,_{uuuu}\overset{1}{u})\overset{1}{u} + 3\,\mathcal{E}^c,_{uuu}v_{\xi\xi})\overset{1}{u})\overset{1}{u}$. To this end by using Eq. (BB-1.7), the wanted numerator in the expression for λ_2 according to Eq. (AC-3.14) becomes

$$\begin{aligned} (((\mathcal{E}^c,_{uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E}^c,_{uuu}v_{\xi\xi})\overset{1}{u})\overset{1}{u} &= 3\int_A[L_{\alpha\beta\gamma\delta}(\overset{2}{u}_{\gamma,\delta} + \overset{1}{w},_\gamma\overset{1}{w},_\delta)\overset{1}{w},_\alpha\overset{1}{w},_\beta]hdA = \\ 3\int_A[s_{\alpha\beta}\overset{1}{w},_\alpha\overset{1}{w},_\beta]hdA &= \frac{3}{32}Eh^5a_1a_2[(m_c\pi/a_1)^4 + (n_c\pi/a_2)^4], \end{aligned} \tag{BB-1.34}$$

where in the derivation of Eq. (BB-1.34) use was also made of Eq. (BB-1.12), Eq. (BB-1.13), Eq. (BB-1.24), Eq. (BB-1.25), Eq. (BB-1.31). The denominator of λ_2 has already been found in Eq. (BB-1.20) and consequently from Eq. (AC-3.14) λ_2 is finally given by

$$\lambda_2 = -\frac{Eh^2}{8}\,\frac{(m_c\pi/a_1)^4 + (n_c\pi/a_2)^4}{\overset{0}{\sigma}_{11}(m_c\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n_c\pi/a_2)^2} = \frac{3}{2}\lambda_c(1-\nu^2)\frac{(m_c\pi/a_1)^4 + (n_c\pi/a_2)^4}{[(m_c\pi/a_1)^2 + (n_c\pi/a_2)^2]^2}. \tag{BB-1.35}$$

Recalling from the discussion of Eq. (BB-1.20) that $\overset{0}{\sigma}_{11}(m_c\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n_c\pi/a_2)^2 < 0$ (or equivalently since $\lambda_c > 0$) we conclude that $\lambda_2 > 0$ which implies that the simple symmetric bifurcation of the simply supported plate gives a stable bifurcation equilibrium branch.

Example

As an application to the above analysis consider the case of a square plate $a_1 = a_2 = a$ under equibiaxial compression $\overset{0}{\sigma}_{11} = \overset{0}{\sigma}_{22} = -1$. Then from Eq. (BB-1.14), Eq. (BB-1.35)

$$\lambda_c = \frac{\pi^2}{6}\,\frac{E}{1-\nu^2}\left(\frac{h}{a}\right)^2, \quad m_c = n_c = 1; \quad \lambda_2 = \frac{3}{4}\,\frac{\lambda_c}{1-\nu^2}. \tag{BB-1.36}$$

ib) *Simply Supported Perfect Plate Under Compression – Double Eigenvalue Case*

The analysis in the previous subsection is valid when the lowest bifurcation load λ_c corresponds to a unique eigenmode i.e. when the pair of integers m_c, n_c that minimizes $\lambda(m,n)$ (see Eq. (BB-1.14)) is unique. Suppose now that $a_1 = \sqrt{2}a$, $a_2 = a$ and that the plate is compressed only along the x_1 direction, i.e. $\overset{0}{\sigma}_{11} = -1$, $\overset{0}{\sigma}_{22} = 0$. Under those conditions it is not difficult to see that the lowest bifurcation load λ_c corresponds to a double eigenmode since from Eq. (BB-1.16)

$$\lambda_c = \frac{3\pi^2}{8}\frac{E}{1-\nu^2}\left(\frac{h}{a}\right)^2, \quad (m_c, n_c) = (1,1) \text{ or } (2,1), \tag{BB-1.37}$$

while the corresponding bifurcation eigenmodes are (see Eq. (BB-1.13))

$$\overset{1}{w} = h\sin(\pi x_1/a_1)\sin(\pi x_2/a_2), \quad \overset{2}{w} = h\sin(2\pi x_1/a_1)\sin(\pi x_2/a_2); \quad a_1 = \sqrt{2}a_2. \tag{BB-1.38}$$

The corresponding in-plane displacements $\overset{i}{u}_\alpha$ to each mode with $\overset{i}{w}$ out-of-plane displacement are from Eq. (BB-1.12)

$$\overset{1}{u}_\alpha = 0, \quad \overset{2}{u}_\alpha = 0. \tag{BB-1.39}$$

The bifurcation is still a symmetric one i.e. $\mathcal{E}_{ijk} = 0$ since according to Eq. (AC-5.11) and with the help of Eq. (BB-1.7), Eq. (BB-1.9) and Eq. (BB-1.39) one obtains

$$((\mathcal{E}^c{}_{,uuu}\overset{i}{u})\overset{j}{u})\overset{k}{u} = \int_A [L_{\alpha\beta\gamma\delta}(\overset{i}{w}_{,\alpha}\overset{j}{w}_{,\beta}\overset{k}{u}_{\gamma,\delta} + \overset{j}{w}_{,\alpha}\overset{k}{w}_{,\beta}\overset{i}{u}_{\gamma,\delta} + \overset{k}{w}_{,\alpha}\overset{i}{w}_{,\beta}\overset{j}{u}_{\gamma,\delta})]hdA = 0 \tag{BB-1.40}$$

Following the general theory developed in Section AC-5, the calculation of the bifurcated equilibrium paths through λ_c requires first the evaluation of $v_{ij} = (\overset{ij}{u}_\alpha, \overset{ij}{w})$ satisfying (see Eq. (AC-5.9) for the corresponding definition)

$$\begin{aligned} 0 = [(\mathcal{E}^c{}_{,uuu}\overset{i}{u})\overset{j}{u} + \mathcal{E}^c_{,uu}v_{ij})\delta v = &\int_A [L_{\alpha\beta\gamma\delta}(\overset{i}{w}_{,\gamma}\overset{j}{w}_{,\delta}\,\delta u_{\alpha,\beta} + \overset{i}{u}_{\gamma,\delta}\overset{j}{w}_{,\alpha}\,\delta w_{,\beta} + \overset{j}{u}_{\gamma,\delta}\overset{i}{w}_{,\alpha}\,\delta w_{,\beta}]hdA \\ + &\int_A [L_{\alpha\beta\gamma\delta}(\overset{ij}{u}_{\gamma,\delta}\delta u_{\alpha,\beta} + \frac{h^2}{12}\overset{ij}{w}_{,\gamma\delta}\,\delta w_{,\alpha\beta}) + \lambda_c\overset{0}{\sigma}_{\alpha\beta}\overset{ij}{w}_{,\alpha}\,\delta w_{,\beta}]hdA \end{aligned} \tag{BB-1.41}$$

In the above derivations use was made of Eq. (BB-1.7), Eq. (BB-1.9). Integrating Eq. (BB-1.41) by parts and recalling once more the admissibility conditions Eq. $(BB\text{-}1.7)_2$ as well as

Eq. (BB-1.39) we obtain the following Euler - Lagrange equations for $\overset{ij}{u}$, $\overset{ij}{w}$

$$\begin{aligned}
\delta u_\alpha \ : \quad & L_{\alpha\beta\gamma\delta}(\overset{ij}{u}_{\gamma,\delta} + \overset{i}{w}_{,\gamma}\overset{i}{w}_{,\delta})_{,\beta} = 0 \text{ in } A, \\
& L_{12\gamma\delta}(\overset{ij}{u}_{\gamma,\delta} + \overset{i}{w}_{,\gamma}\overset{j}{w}_{,\delta}) = 0 \text{ on } \partial A, \quad \overset{ij}{u}_1(0,x_2) = \overset{ij}{u}_{1,2}(a_1,x_2) = \overset{ij}{u}_2(x_1,0) = \overset{ij}{u}_{2,1}(x_1,a_2) = 0 \\
\delta w \ : \quad & \frac{h^2}{12} L_{\alpha\beta\gamma\delta}\overset{ij}{w}_{,\alpha\beta\gamma\delta} - \lambda_c \overset{0}{\sigma}_{\alpha\beta}\overset{ij}{w}_{,\alpha\beta} = 0 \text{ in } A, \\
& \overset{ij}{w} = 0 \text{ on } \partial A, \quad \overset{ij}{w}_{,11}(0,x_2) = \overset{ij}{w}_{,11}(a_1,x_2) = \overset{ij}{w}_{,22}(x_1,0) = \overset{ij}{w}_{,22}(x_1,a_2) = 0.
\end{aligned} \tag{BB-1.42}$$

Notice that similarly to the simple eigenmode case, the equation governing $\overset{ij}{w}$ is identical to Eq. (BB-1.11)$_2$ governing each $\overset{k}{w}$ and thus $\overset{ij}{w} = c_{ij}\overset{1}{w} + d_{ij}\overset{2}{w}$ where c_{ij} and d_{ij} are constants. Since $v_{ij} = (\overset{ij}{u}_\alpha, \overset{ij}{w})$ is orthogonal to the eigenmodes $\overset{k}{u} = (0, \overset{k}{w})$, $k = 1,2$ at λ_c, we have $0 = (v_{ij}, \overset{k}{u}) = (\overset{ij}{w}, \overset{k}{w})$ which implies $c_{ij} = d_{ij} = 0$ and hence

$$\overset{ij}{w} = 0. \tag{BB-1.43}$$

The determination of $\overset{ij}{u}_\alpha$ is done in a similar way as for the simple buckling mode case. To this end define, in analogy to Eq. (BB-1.25) the stress $\overset{ij}{s}_{\alpha\beta}$ and strain $\overset{ij}{e}_{\alpha\beta}$ quantities

$$\overset{ij}{s}_{\alpha\beta} \equiv L_{\alpha\beta\gamma\delta}(\overset{ij}{u}_{\gamma,\delta} + \overset{i}{w}_{,\gamma}\overset{j}{w}_{,\delta}) = L_{\alpha\beta\gamma\delta}\overset{ij}{e}_{\gamma\delta}, \quad \overset{ij}{e}_{\alpha\beta} \equiv \frac{1}{2}(\overset{ij}{u}_{\alpha,\beta} + \overset{ij}{u}_{\beta,\alpha}) + \frac{1}{2}(\overset{i}{w}_{,\alpha}\overset{j}{w}_{,\beta} + \overset{j}{w}_{,\alpha}\overset{i}{w}_{,\beta}). \tag{BB-1.44}$$

In analogy to Eq. (BB-1.26) one also defines the Airy stress functions $\overset{ij}{f}$ to be

$$\overset{ij}{s}_{11} \equiv \overset{ij}{f}_{,22}, \quad \overset{ij}{s}_{22} \equiv \overset{ij}{f}_{,11}, \quad \overset{ij}{s}_{12} = \overset{ij}{s}_{21} \equiv -\overset{ij}{f}_{,12}, \tag{BB-1.45}$$

which satisfy automatically the equilibrium equations Eq. (BB-1.42)$_1$ in A, i.e. $\overset{ij}{s}_{\alpha\beta,\beta} = 0$. By inverting the relation Eq. (BB-1.44) between $\overset{ij}{s}_{\alpha\beta}$ and $\overset{ij}{e}_{\alpha\beta}$ one has, in analogy to Eq. (BB-1.27)

$$\overset{ij}{e}_{11} = \frac{1}{E}(\overset{ij}{s}_{11} - \nu\overset{ij}{s}_{22}), \quad \overset{ij}{e}_{22} = \frac{1}{E}(\overset{ij}{s}_{22} - \nu\overset{ij}{s}_{11}), \quad \overset{ij}{e}_{12} = \overset{ij}{e}_{21} = \frac{1+\nu}{E}\overset{ij}{s}_{12}. \tag{BB-1.46}$$

The compatibility condition, analogous to Eq. (BB-1.28) is found by exploiting the definition of $\overset{ij}{e}_{\alpha\beta}$ in terms of $\overset{ij}{u}_\alpha$ and $\overset{i}{w}$ in Eq. (BB-1.44) to be

$$\overset{ij}{e}_{11,22} + \overset{ij}{e}_{22,11} - 2\overset{ij}{e}_{12,12} = 2\overset{i}{w}_{,12}\overset{j}{w}_{,12} - \overset{i}{w}_{,11}\overset{j}{w}_{22} - \overset{j}{w}_{,11}\overset{i}{w}_{,22}. \tag{BB-1.47}$$

Upon substitution of Eq. (BB-1.38) and Eq. (BB-1.45), Eq. (BB-1.46) into (BB-1.47) one obtains the following equations for $\overset{ij}{f}$

$$\frac{1}{E}\nabla^4(\overset{ij}{f}) = 2\overset{i}{w}_{,12}\overset{j}{w}_{,12} - \overset{i}{w}_{,11}\overset{j}{w}_{22} - \overset{j}{w}_{,11}\overset{i}{w}_{,22} \tag{BB-1.48}$$

The solution of Eq. (BB-1.48) gives the following results for $\overset{ij}{f}$

$$\overset{11}{f} = \frac{Eh^2}{16}\left[\left(\frac{a_1}{a_2}\right)^2 \cos(2\pi x_1/a_1) + \left(\frac{a_2}{a_1}\right)^2 \cos(2\pi x_2/a_2)\right]$$

$$\overset{22}{f} = \frac{Eh^2}{16}\left[\left(\frac{a_1}{2a_2}\right)^2 \cos(4\pi x_1/a_1) + \left(\frac{2a_2}{a_1}\right)^2 \cos(2\pi x_2/a_2)\right]$$

$$\overset{12}{f} = \frac{Eh^2}{4}\left\{\frac{\cos(3\pi x_1/a_1)}{(3a_2/a_1)^2} - \frac{\cos(\pi x_1/a_1)}{(a_2/a_1)^2} + \left[\frac{9\cos(\pi x_1/a_1)}{[(a_2/a_1)+(4a_1/a_2)]^2} - \frac{\cos(3\pi x_1/a_1)}{[(9a_2/a_1)+(4a_1/a_2)]^2}\right]\cos(2\pi x_2/a_2)\right\} \tag{BB-1.49}$$

Consequently, using Eq. (BB-1.49) into Eq. (BB-1.45) one obtains for $\overset{ij}{s}_{\alpha\beta}$. Likewise one can also calculate $\overset{ij}{e}_{\alpha\beta}$ from Eq. (BB-1.46) and Eq. (BB-1.49) and subsequently from Eq. (BB-1.44) with the help of Eq. (BB-1.39) one can find $\overset{ij}{u}_{\alpha}$. It turns out only $\overset{ij}{s}_{\alpha\beta}$ are required in the subsequent calculations for λ_2.

As discussed in the general theory in Section AC-5, the coefficients $\mathcal{E}_{ijkl}$ and $\mathcal{E}_{ij\lambda}$ are required in order to calculate λ_2 and the initial tangents $\{\alpha_i^1\}$ to the different equilibrium paths through λ_c for the symmetric multiple bifurcation (see Eq. (AC-5.15) and the definitions for $\mathcal{E}_{ijkl}$ and $\mathcal{E}_{ij\lambda}$ in Eq. (AC-5.11). By using Eq. (BB-1.3), Eq. (BB-1.7) and Eq. (BB-1.44) one obtains for $\mathcal{E}_{ijkl}$

$$\begin{aligned}\mathcal{E}_{ijkl} &= \int_A [L_{\alpha\beta\gamma\delta}[\overset{i}{w}_{,\alpha}\overset{j}{w}_{,\beta}(\overset{kl}{u}_{\gamma,\delta} + \overset{k}{w}_{,\gamma}\overset{l}{w}_{,\delta}) + \overset{i}{w}_{,\alpha}\overset{l}{w}_{,\beta}(\overset{jk}{u}_{\gamma,\delta} + \overset{j}{w}_{,\gamma}\overset{k}{w}_{,\delta}) + \overset{i}{w}_{,\alpha}\overset{k}{w}_{,\beta}(\overset{jl}{u}_{\gamma,\delta} + \overset{j}{w}_{,\gamma}\overset{l}{w}_{,\delta})]hdA \\ &= \int_A [\overset{kl}{s}_{\alpha\beta}\overset{i}{w}_{,\alpha}\overset{j}{w}_{,\beta} + \overset{kj}{s}_{\alpha\beta}\overset{i}{w}_{,\alpha}\overset{l}{w}_{,\beta} + \overset{jl}{s}_{\alpha\beta}\overset{i}{w}_{,\alpha}\overset{k}{w}_{,\beta}]hdA\end{aligned} \tag{BB-1.50}$$

and for $\mathcal{E}_{ij\lambda}$

$$\mathcal{E}_{ij\lambda} = \int_A [\overset{0}{\sigma}_{\alpha\beta}\overset{i}{w}_{,\alpha}\overset{j}{w}_{,\beta}]hdA \tag{BB-1.51}$$

It is interesting to note that $\mathcal{E}_{ijkl}$ and $\mathcal{E}_{ij\lambda}$ are completely symmetric with respect to any permutation of indexes. By using Eq. (BB-1.49), as well as Eq. (BB-1.38), into Eq. (BB-1.50) one obtains for the nonzero coefficients $\mathcal{E}_{ijkl}$

$$\mathcal{E}_{1111} = \frac{15\sqrt{2}}{128}Eh^5\frac{\pi^4}{a^2}, \quad \mathcal{E}_{2222} = \frac{15\sqrt{2}}{32}Eh^5\frac{\pi^4}{a^2}, \quad \mathcal{E}_{1122} = \frac{14135\sqrt{2}}{83232}Eh^5\frac{\pi^4}{a^2} \tag{BB-1.52}$$

The remaining nonzero $\mathcal{E}_{ijkl}$ are given by all the possible index permutations in view of the symmetries discussed. The $\mathcal{E}_{ij\lambda}$ coefficients are found by substituting Eq. (BB-1.38) into Eq. (BB-1.51) (and recalling also that the only nonzero $\overset{0}{\sigma}_{\alpha\beta}$ is $\overset{0}{\sigma}_{11} = -1$. Hence one obtains for the nonzero coefficients $\mathcal{E}_{ij\lambda}$

$$\mathcal{E}_{11\lambda} = -\frac{\sqrt{2}}{8}h^3\pi^2, \quad \mathcal{E}_{22\lambda} = -\frac{\sqrt{2}}{2}h^3\pi^2 \tag{BB-1.53}$$

A more convenient way to proceed with the algebraic calculations is to rewrite the nonzero components of the coefficients $\mathcal{E}_{ijkl}$ and $\mathcal{E}_{ij\lambda}$ in terms of three positive constants $\alpha,\ \delta,\ \gamma$

as follows:

$$\mathcal{E}_{1111} = \alpha,\ \mathcal{E}_{2222} = 4\alpha,\ \mathcal{E}_{1122} = \mathcal{E}_{2211} = (4\alpha/3)(1+\delta); \quad \mathcal{E}_{11\lambda} = -\gamma,\ \mathcal{E}_{22\lambda} = -4\gamma,$$
$$\alpha \equiv (15\sqrt{2}/128)Eh^5\pi^4/a^2, \quad \gamma \equiv (\sqrt{2}/8)h^3\pi^2, \quad \delta \equiv 226/2601. \tag{BB-1.54}$$

Consequently, from Eq. (AC-5.15) of the general theory and by substituting in Eq. (BB-1.54), the initial direction α_i^1 of the bifurcated equilibrium solutions at λ_c satisfy the following system

$$\begin{aligned} \alpha(\alpha_1^1)\left[(\alpha_1^1)^2 + 4(1+\delta)(\alpha_2^1)^2 - 3\lambda_2(\gamma/\alpha)\right] &= 0 \\ 4\alpha(\alpha_2^1)\left[(1+\delta)(\alpha_1^1)^2 + (\alpha_2^1)^2 - 3\lambda_2(\gamma/\alpha)\right] &= 0 \\ (\alpha_1^1)^2 + (\alpha_2^1)^2 &= 1 \end{aligned} \tag{BB-1.55}$$

The above system admits four real solutions

$$\begin{aligned} &N1: \ \alpha_1^1 = 1, \quad \alpha_2^1 = 0; \quad \lambda_2 = \alpha/3\gamma \\ &N2: \ \alpha_1^1 = 0, \quad \alpha_2^1 = 1; \quad \lambda_2 = \alpha/3\gamma \\ &N3: \ \alpha_2^1/\alpha_1^1 = [\delta/(3+4\delta)]^{1/2}; \quad \lambda_2 = (\alpha/3\gamma)[1+\delta(3+4\delta)/(3+5\delta)] \\ &N4: \ \alpha_2^1/\alpha_1^1 = -[\delta/(3+4\delta)]^{1/2}; \quad \lambda_2 = (\alpha/3\gamma)[1+\delta(3+4\delta)/(3+5\delta)] \end{aligned} \tag{BB-1.56}$$

Notice that since all $\lambda_2 > 0$, all the bifurcated paths are supercritical at λ_c, i.e. occur under increasing load.

Next the stability of the above bifurcated branches is to be examined, at least in the neighborhood of λ_c. To this end one has according to the general theory presented in Section AC-5 to examine the positive definiteness of the matrix B_{ij} given by Eq. (AC-5.16)

$$B_{ij} = \begin{bmatrix} \mathcal{E}_{1111}(\alpha_1^1)^2 + \mathcal{E}_{1122}(\alpha_2^1)^2 + \lambda_2\mathcal{E}_{11\lambda} & 2\mathcal{E}_{1122}\alpha_1^1\alpha_2^1 \\ 2\mathcal{E}_{1122}\alpha_1^1\alpha_2^1 & \mathcal{E}_{2211}(\alpha_1^1)^2 + \mathcal{E}_{2222}(\alpha_2^1)^2 + \lambda_2\mathcal{E}_{22\lambda} \end{bmatrix} \tag{BB-1.57}$$

where in the derivation of (BB-1.57) from (AC-5.16) use was made of the fact that the only nonzero coefficients $\mathcal{E}_{ijkl}$ and $\mathcal{E}_{ij\lambda}$ are given by (BB-1.52), (BB-1.53). The evaluation of the stability matrix (BB-1.57) for the four bifurcated branches found in (BB-1.56) gives the following results

$$\begin{aligned} &N1: \quad B_{ij} = \frac{2\alpha}{3}\begin{bmatrix} 1 & 0 \\ 0 & 2\delta \end{bmatrix} \Longrightarrow \text{Stable} \\ &N2: \quad B_{ij} = \frac{\alpha}{3}\begin{bmatrix} 3+4\delta & 0 \\ 0 & 8 \end{bmatrix} \Longrightarrow \text{Stable} \\ &N3: \quad B_{ij} = \frac{2\alpha}{3(3+5\delta)}\begin{bmatrix} 3+4\delta & 4(1+\delta)\sqrt{\delta(3+4\delta)} \\ 4(1+\delta)\sqrt{\delta(3+4\delta)} & 4\delta \end{bmatrix} \Longrightarrow \text{Unstable} \\ &N4: \quad B_{ij} = \frac{2\alpha}{3(3+5\delta)}\begin{bmatrix} 3+4\delta & -4(1+\delta)\sqrt{\delta(3+4\delta)} \\ -4(1+\delta)\sqrt{\delta(3+4\delta)} & 4\delta \end{bmatrix} \Longrightarrow \text{Unstable} \end{aligned} \tag{BB-1.58}$$

It is interesting to note that unlike the simple mode case where the post-bifurcation equilibrium branch of the plate was stable, in the double eigenmode buckling case one finds that two of the four bifurcated equilibrium branches, N3 and N4, are unstable. However, since for all the post-bifurcated branches $\lambda_2 > 0$, no imperfection sensitivity is expected and the corresponding imperfect plate will not exhibit any snap-through instability at a load lower than λ_c.

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