STABILITY AT DIFFERENT SCALES - B

IMPORTANCE OF CELLULAR SOLIDS

- Cellular solids are extremely important in technological applications as either natural or man-made materials.

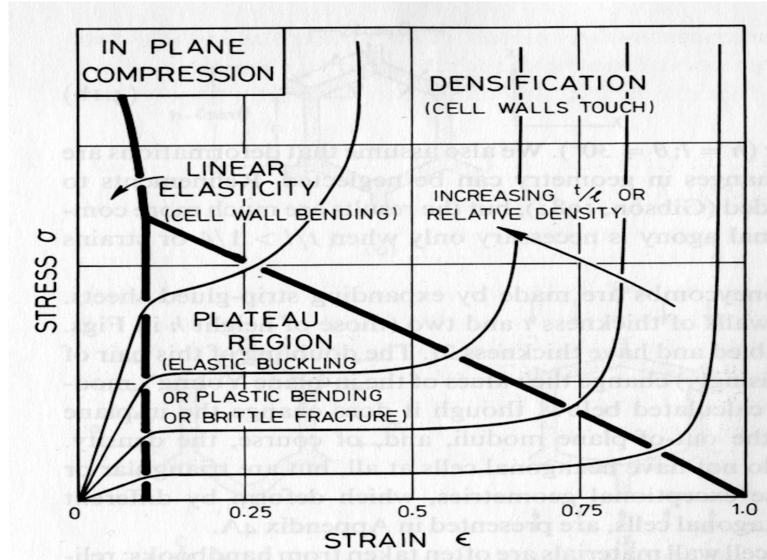
- Their microstructures are either regular (honeycomb) or random (foams).

- They have many advantages, due to their high-stiffness/weight ratios.

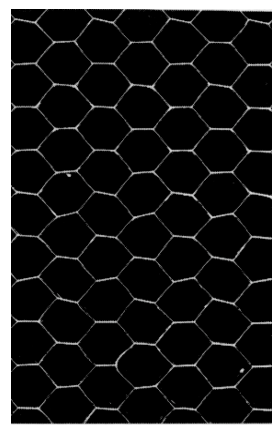
- They enjoy a wide variety of applications (aerospace, packaging, shock absorption).

- They are mainly designed to work in compression. An useful feature of cellular solids is the existence of a large plateau in their macroscopic stress-strain response under compression which is beneficial in absorbing shocks under constant forces.

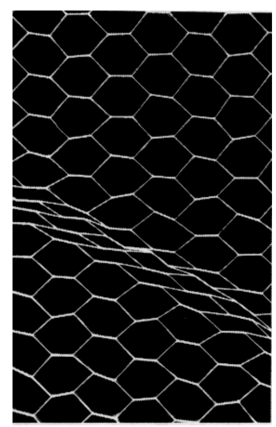
- The explanation of this plateau lies in micro-structural instabilities.



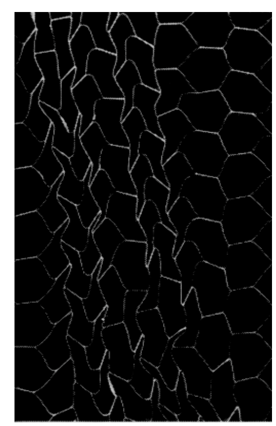
Typical behavior of cellular solids under compression
FROM: Gibson & Ashby, Cellular Solids, Cambridge, 1988



Initial Configuration

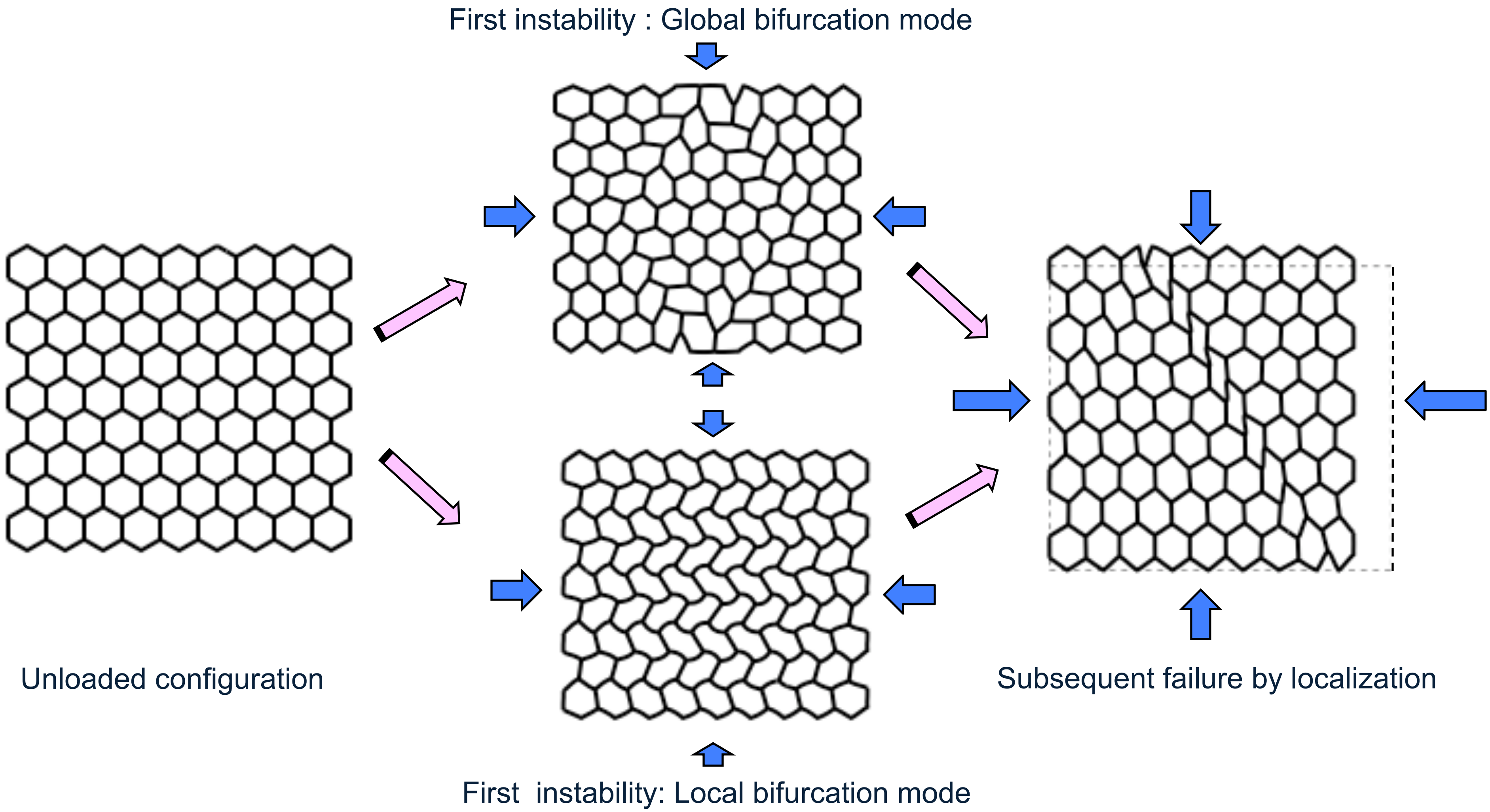


One band localization



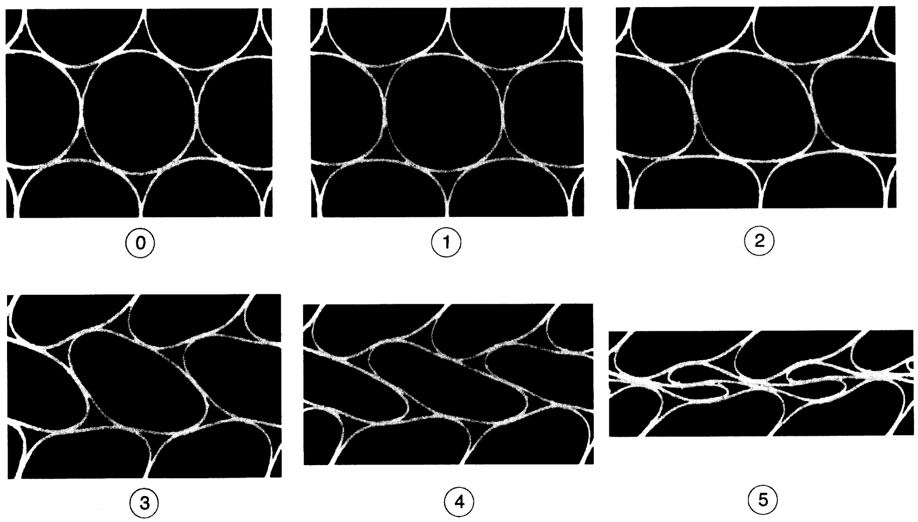
Multi-band localization

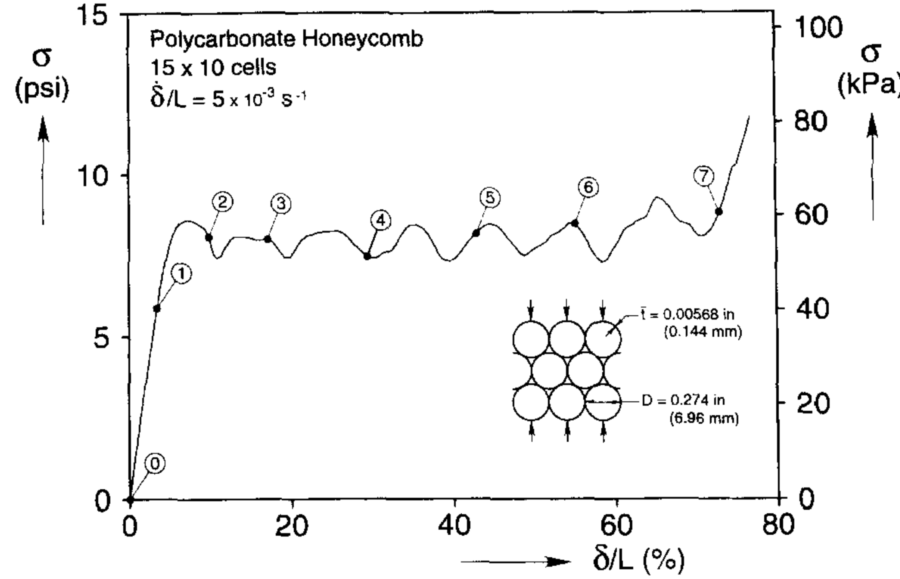
COMPRESSION-INDUCED FAILURE IN CELLULAR SOLIDS

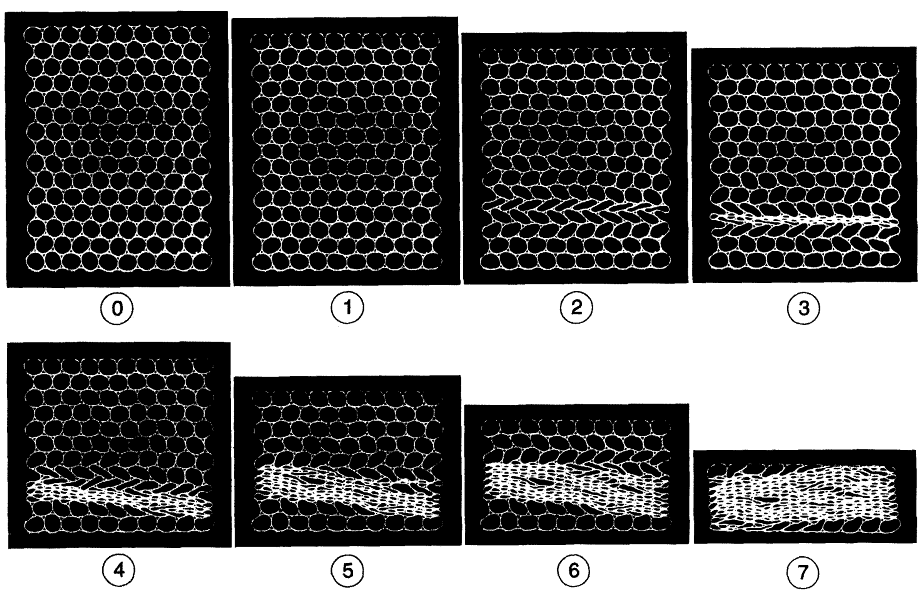


Typically, upon loading, an initial bifurcation evolves to an ultimate failure by localization.

CELLULAR SOLIDS – 2D LOADING





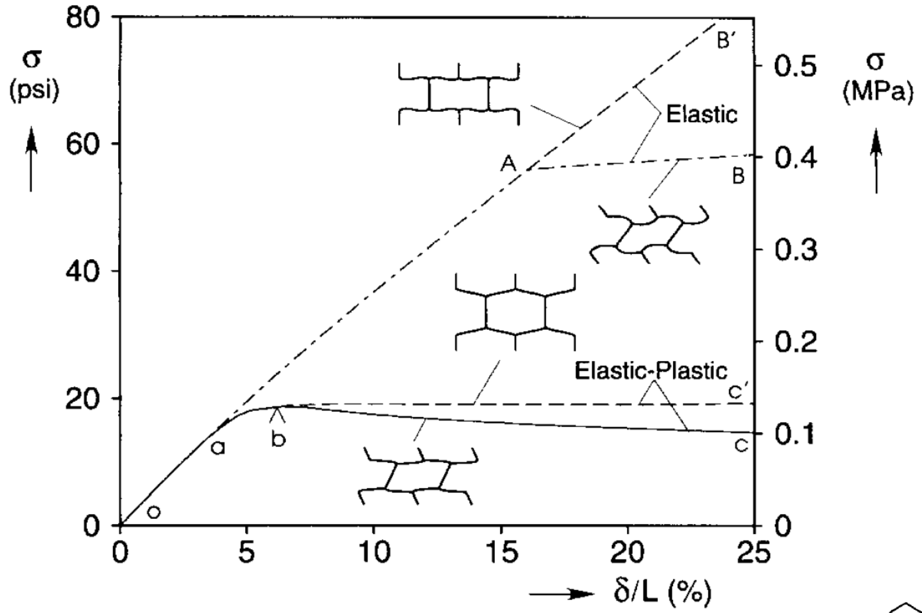


Typical sequence of events in 2D loading of a cellular (circular cells) solid:

- **Initial bifurcation (local mode, involving one row) occurs under reduced loads.**
- **Deformation localizes in that row until entire row collapses (contact).**
- **Mechanism restarts in another row.**

FROM: Papka & Kyriakides IJSS, 1998, **35**, pp. 239-267

CELLULAR SOLIDS – 2D LOADING

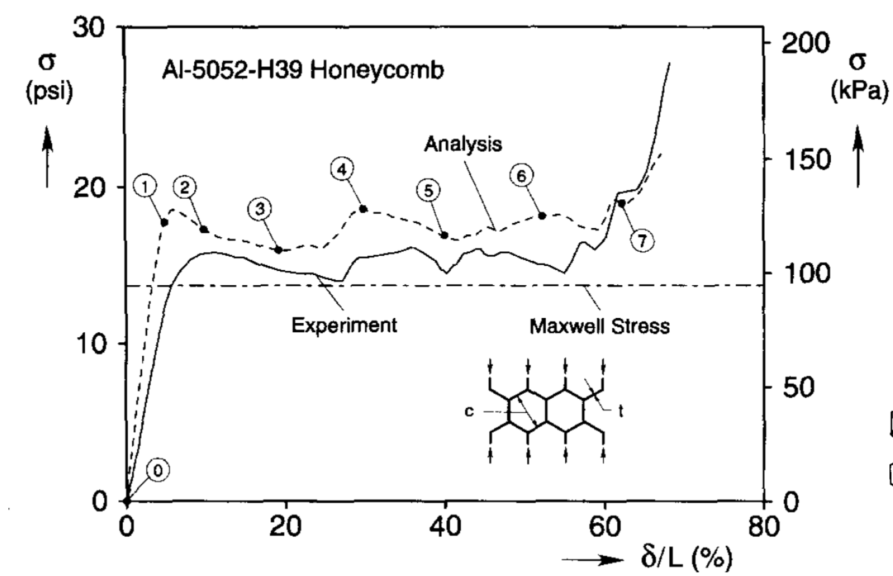


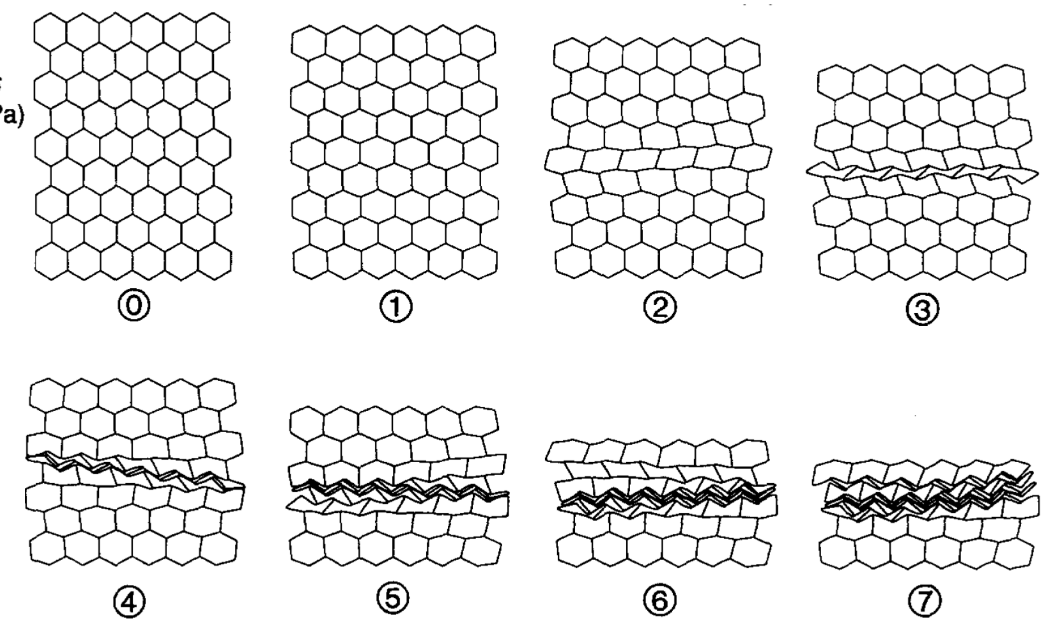
Sequence of events in 2D loading of a cellular (hexagonal cells) solid:

• **Initial bifurcation** (local mode, involving one row) occurs under **reduced loads**.

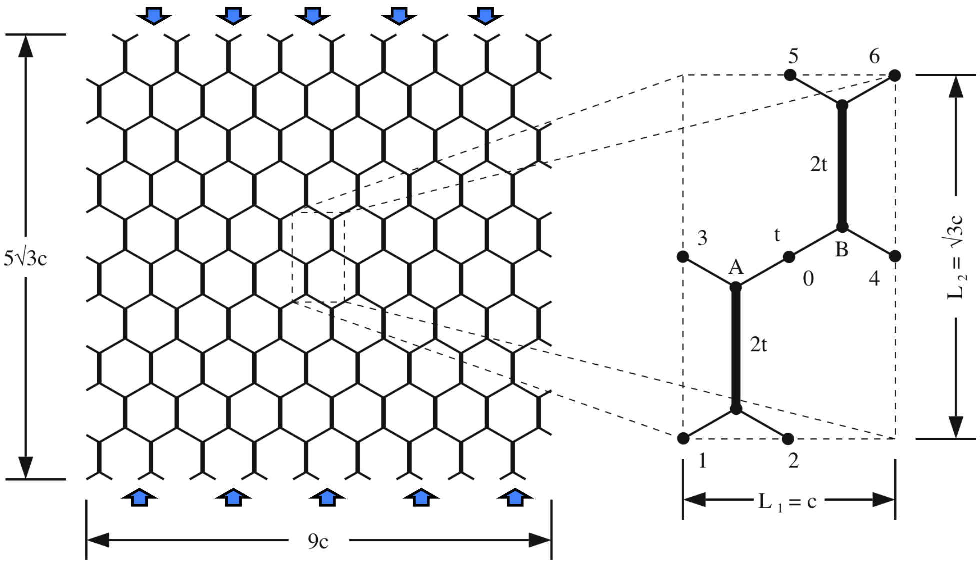
• Deformation **localizes in that row** until entire row collapses (contact) and the process restarts. Notice strong **boundary effects** in this experiment.

FROM: Papka & Kyriakides JMPS, 1994, **42**, pp. 1499-1532

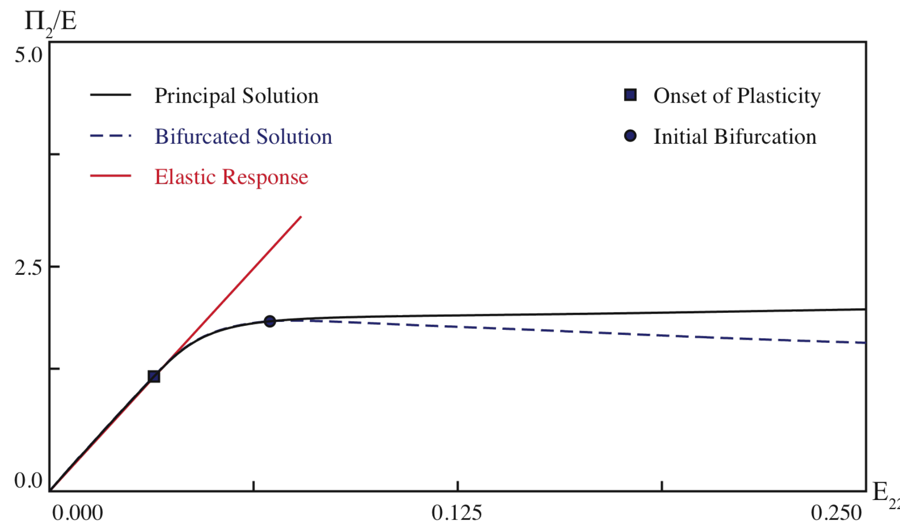




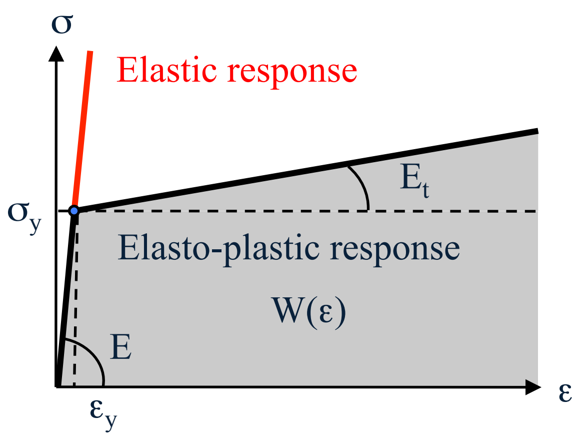
CELLULAR SOLIDS – 2D LOADING



Periodic, perfect honeycomb and unit cell chosen



Onset of first bifurcation in infinite, periodic solid



Stress-strain response of cell walls

- **Initially all cells of perfect structure deform identically (principal solution).**
- **Upon increase of compressive loading shown, principal solution of honeycomb enters plastic range (governed by modulus E_t), thus considerably softening response.**
- **A bifurcation is found to occur under reduced load, thus causing collapse of a row and starting crushing mode.**

COMPRESSION-INDUCED FAILURE IN CELLULAR SOLIDS

• To model the entire deformation history of a cellular solid under compression, one has to solve for a finite-size structure (boundary effects are important) with many unit cells, using an elastoplastic constitutive law (unloading is important). In addition, initial imperfections have to be used, with different imperfection shapes often giving very different results.

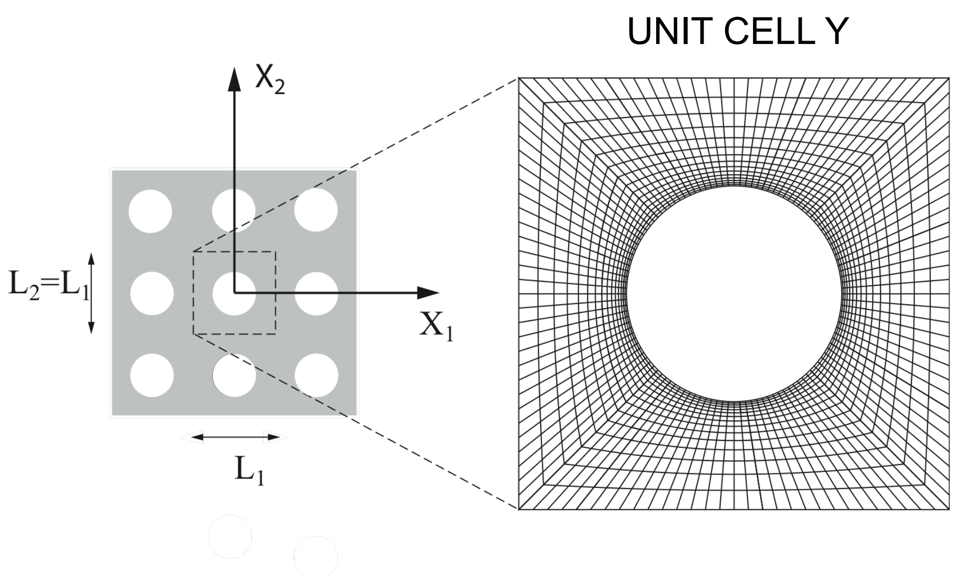
• However, the onset of the first instability during the load increase in an infinite, perfect structure can be found accurately from the principal solution, by using one unit cell.

• The use of only a unit cell in finding the first instability occurring in a loading process is guaranteed from Bloch wave representation theorem for the solution of a system of linear differential equations with periodic coefficients. The Bloch wave representation theorem is the generalization to PDE's of Floquet's theorem for ODE's with periodic coefficients.

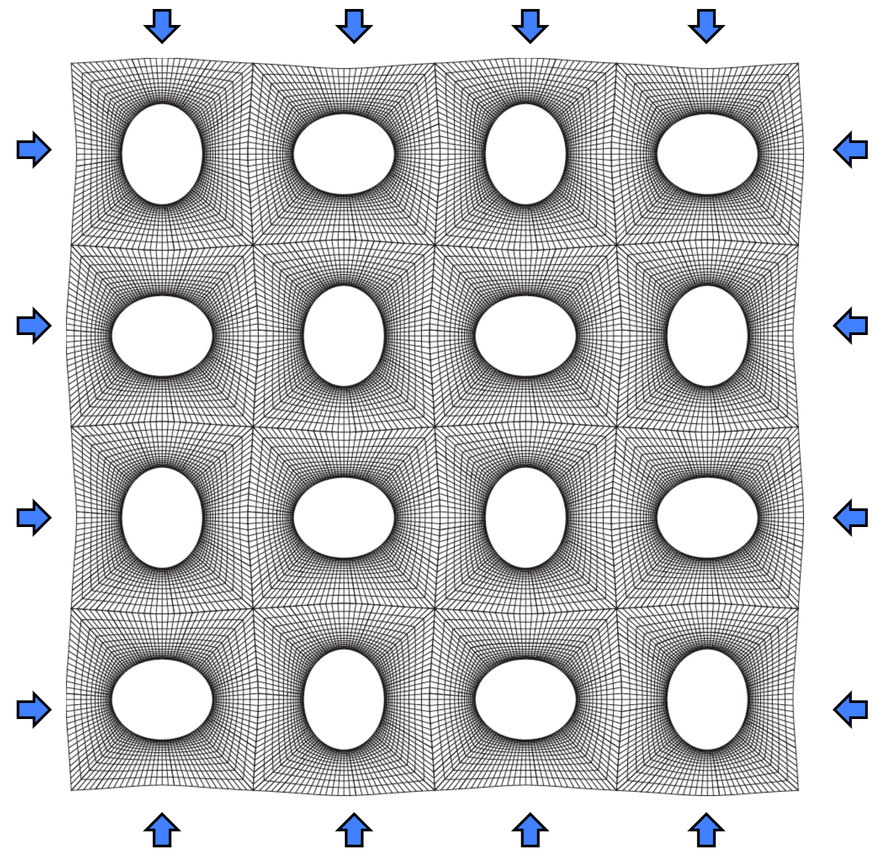
• The Bloch wave representation theorem is easily adapted to cellular geometry.

• The use of an elastic model in calculations for elastoplastic solids is based on the assumption that in the principal solution, all cell walls satisfy loading condition as load increases, thus using a deformation theory of plasticity (which has a stored energy).

STABILITY OF INFINITE PERIODIC SOLIDS: BLOCH WAVE



- The **Y-periodic, principal solution** of **infinite**, perfect solid is **initially stable**.
- Upon load increase, the system **bifurcates** with modes that are **no longer Y-periodic**.
- **Critical load** and corresponding **eigenmode** can be found based on calculations on **one unit cell Y** with the help of **Bloch wave representation** theorem.



First bifurcation mode of a porous, compressible neo-Hookean solid ($W = 0.5[\mu(I_2\text{-}2\text{-}lnI_2)+(\kappa\text{-}\mu)(\sqrt{I_2}\text{-}1)^2]$) loaded under compressive, equi-biaxial plane strain ($E_{11} = E_{22} = -\lambda$). The critical load is $\lambda_c = 8\%$

CELLULAR SOLIDS – 2D LOADING

STABILITY OF INFINITE PERIODIC SOLIDS: BLOCH WAVE

$$\overset{0}{\beta}(\lambda) = \min_{\delta u \in U} \left[(\mathcal{E},_{uu}(\overset{0}{u}(\lambda), \lambda)\delta u)\delta u \right], \; ||\delta u|| = 1; \quad \text{stability of } \overset{0}{u}(\lambda)$$

$$\overset{0}{u}(\lambda) = Y\text{– periodic (unit cell } Y) : \; \overset{0}{u}_{i,j}(X_k + n_k L_k) = \overset{0}{u}_{i,j}(X_k) \; (!); \; n_k \in \mathbb{N}.$$

$$(\mathcal{E},_{\overset{0}{uu}}\delta u)\delta u = \int_V \left\{ \left[\frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}} \right]_{\overset{0}{u}} \delta u_{i,j} \delta u_{k,l} \right\} dV; \quad \delta u_{i,j} \equiv \partial \delta u_i / \partial X_j, \; V = \mathbb{R}^2 \text{ or } \mathbb{R}^3$$

$$\forall \; \delta u \in U : \quad \delta u_j(\boldsymbol{X}) = \delta p_j(\boldsymbol{X}) \exp(i \omega_k X_k) : \; \text{BLOCH WAVE THEOREM.}$$

$$\text{where} : \quad \omega_k L_k \; (!) \in (0, 2\pi); \quad k = 1, 2 \text{ if } V = \mathbb{R}^2 \text{ or } k = 1, 2, 3 \text{ if } V = \mathbb{R}^3$$

$$\delta \mathbf{p} = Y\text{– periodic} : \quad \delta p_j(X_k + n_k L_k) = \delta p_j(X_k) \; (!); \quad n_k \in \mathbb{N}.$$

$$\overset{0}{\beta}(\lambda) = \inf_{\boldsymbol{\omega}} \left[\min_{\delta \mathbf{p}} \int_Y \left\{ \left[\frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}} \right]_{\overset{0}{u}} \overline{\delta u}_{i,j} \delta u_{k,l} \right\} dV \right]; \; \text{calculations need only } Y$$

HEXAGONAL HONEYCOMB: MODELING

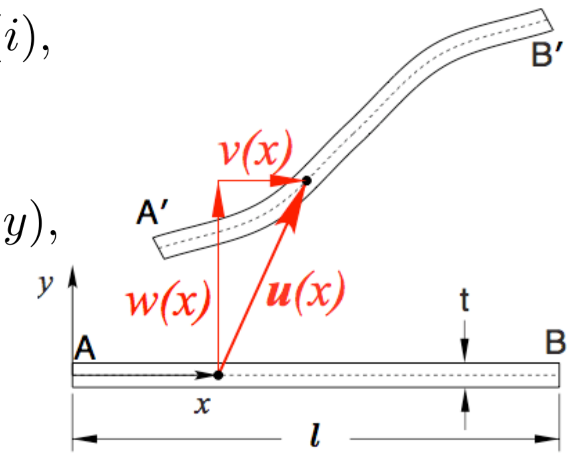
$$\mathcal{E}_{wall}^{(i)} = \int_0^{l_i} \left[\int_{-t/2}^{t/2} W(e) dy \right] dx; \text{ energy stored in cell wall } (i),$$

$$W(e) = \int_0^e \sigma(\epsilon) d\epsilon; \text{ energy density of fiber with strain } e(x,y),$$

$$e(x,y) = \varepsilon(x) + y\kappa(x); \text{ axial strain at point } (x,y),$$

$$\boxed{\varepsilon(x) = \left[(1 + v,_x)^2 + (w,_x)^2\right]^{1/2} - 1;} \text{ mid} - \text{fiber axial strain},$$

$$\boxed{\kappa(x) = [w,_x v,_{xx} - (1 + v,_x) w,_{xx}] / [(1 + v,_x)^2 + (w,_x)^2];} \text{ mid} - \text{fiber curvature}.$$



NOTE A : Finite rotation kinematics, (Euler – Lagrange : correct equilibrium equs.).

NOTE B : General nonlinear (no unloading) uniaxial response : $\sigma(\epsilon)$ is considered.

HEXAGONAL HONEYCOMB: EQUILIBRIUM

$$\mathcal{E} \;=\; \sum_{(i)} \mathcal{E}^{(i)}_{wall} = \sum_{(i)} \int_0^{l_i} \left[\int_{-t/2}^{t/2} W(e)dy \right] dx; \quad \text{energy of structure.}$$

$$\mathcal{E},_u \delta u \;=\; \sum_{(i)} \int_0^{l_i} \left[\int_{-t/2}^{t/2} \left[\frac{dW(e)}{de} (\delta\varepsilon(x) + y\delta\kappa(x)) \right] dy \right] dx =$$

$$= \sum_{(i)} \int_0^{l_i} \left[\left(\int_{-t/2}^{t/2} \left[\frac{dW(e)}{de} \right] dy \right) \delta\varepsilon(x) + \left(\int_{-t/2}^{t/2} \left[\frac{dW(e)}{de} y \right] dy \right) \delta\kappa(x) \right] dx =$$

$$= \sum_{(i)} \int_0^{l_i} [N(x)\delta\varepsilon(x) + M(x)\delta\kappa(x)]\, dx = 0; \quad \text{equilibrium of structure.}$$

$$N(x) = \int_{-t/2}^{t/2} \sigma(e)dy, \;\text{axial force;}\; M(x) = \int_{-t/2}^{t/2} \sigma(e)ydy, \;\text{bending moment;}\; \sigma(e) = \frac{dW(e)}{de}$$

HEXAGONAL HONEYCOMB: STABILITY

$$(\mathcal{E},_{uu}^{0} \Delta u)\delta u \;=\; \sum_{(i)} \int_0^{l_i} \left[\int_{-t/2}^{t/2} [\Delta(\delta W(e)\delta e)\Delta e]\, dy \right] dx, \text{ stability of path : } \overset{0}{u}(\lambda).$$

$$(\mathcal{E},_{uu}^{0} \Delta u)\delta u \;=\; \sum_{(i)} \int_0^{l_i} \left[\int_{-t/2}^{t/2} \left[\frac{d^2W(e)}{de^2} (\Delta\varepsilon(x) + y\Delta\kappa(x))\,(\delta\varepsilon(x) + y\delta\kappa(x)) \; + \right.\right.$$

$$\left.\left. + \;\; \frac{dW(e)}{de}(\Delta(\delta\varepsilon(x)) + y\Delta(\delta\kappa(x))) \right] dy \right] dx \implies \text{Bloch representation :}$$

$$\overset{0}{\beta}(\lambda) \;=\; \min_{\delta u \in U} \left[(\mathcal{E},_{uu}^{structure}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u \right] = \min_{\delta \boldsymbol{p},\ \boldsymbol{\omega}} \left[(\mathcal{E},_{uu}^{unit\ cell}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u \right].$$

$$\text{Bloch thm. :} \quad \delta u_j(\boldsymbol{X}) = \delta p_j(\boldsymbol{X}) \exp(i\omega_k X_k),\ \forall\ \delta u \in U,\ ||\delta u|| = 1; \quad (u \equiv (u_j(\boldsymbol{X})),$$

$$\text{where :} \quad \omega_k L_k \in (0, 2\pi); \quad \delta p_j(X_k + n_k L_k) = \delta p_j(X_k)\ (!); \quad n_k \in \mathbb{N}.$$

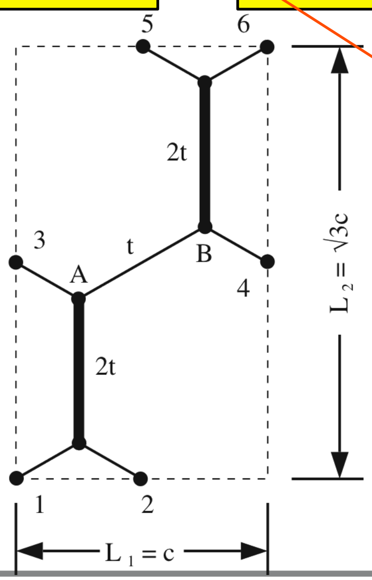
HEXAGONAL HONEYCOMB: STABILITY

$$\overset{0}{\beta}(\lambda) = \min_{\delta \boldsymbol{p},\ \boldsymbol{\omega}} \left[(\mathcal{E},^{unit\ cell}_{uu} (\overset{0}{u}(\lambda), \lambda)\delta u)\delta u \right]; \quad \Longrightarrow \quad \text{F.E.M. discretization :}$$

$$\overset{0}{\beta}(\lambda) = \min_{\delta \boldsymbol{p},\ \boldsymbol{\omega}} \left\{ [\delta \boldsymbol{u}]^* \left[\sum_{(i)}^{unit\ cell} \boldsymbol{K}_{(i)}(\lambda) \right] [\delta \boldsymbol{u}] \right\} = \min_{\delta \boldsymbol{p},\ \boldsymbol{\omega}} \left\{ \sum_{i,j=1}^{6} [\delta \boldsymbol{u}_j]^* \boldsymbol{K}_{ij}(\lambda) [\delta \boldsymbol{u}_i] \right\}$$

$$\begin{bmatrix} \delta \boldsymbol{u}_4 \\ \delta \boldsymbol{u}_5 \\ \delta \boldsymbol{u}_6 \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \exp(i\omega_1 L_1)\boldsymbol{I} \\ \boldsymbol{0} & \exp(i\omega_2 L_2)\boldsymbol{I} & \boldsymbol{0} \\ \exp(i\omega_1 L_1 + i\omega_2 L_2)\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{u}_1 \\ \delta \boldsymbol{u}_2 \\ \delta \boldsymbol{u}_3 \end{bmatrix}$$

$$\delta \boldsymbol{u}_\beta = \boldsymbol{A}(\omega_1 L_1, \omega_2 L_2) \delta \boldsymbol{u}_\alpha ,$$



$$\overset{0}{\beta}(\lambda) = \min \begin{bmatrix} \delta \boldsymbol{u}^*_\alpha & \delta \boldsymbol{u}^*_\beta \end{bmatrix} \begin{bmatrix} \boldsymbol{K}_{\alpha\alpha}(\lambda) & \boldsymbol{K}_{\alpha\beta}(\lambda) \\ \boldsymbol{K}_{\beta\alpha}(\lambda) & \boldsymbol{K}_{\beta\beta}(\lambda) \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{u}_\alpha \\ \delta \boldsymbol{u}_\beta \end{bmatrix},$$

$$\overset{0}{\beta}(\lambda) = \min\{\delta \boldsymbol{u}^*_\alpha \boldsymbol{K}(\lambda; \omega_1 L_1, \omega_2 L_2) \delta \boldsymbol{u}_\alpha\} = \min \text{eigenvalue of :}\ \ \boldsymbol{K},$$

$$\boldsymbol{K} \equiv \boldsymbol{K}_{\alpha\alpha} + \boldsymbol{K}_{\alpha\beta}\boldsymbol{A} + \boldsymbol{A}^*\boldsymbol{K}_{\beta\alpha} + \boldsymbol{A}^*\boldsymbol{K}_{\beta\beta}\boldsymbol{A}, \quad \text{where :}\ \ \boldsymbol{K}^* = \boldsymbol{K}.$$

HEXAGONAL HONEYCOMB: STABILITY

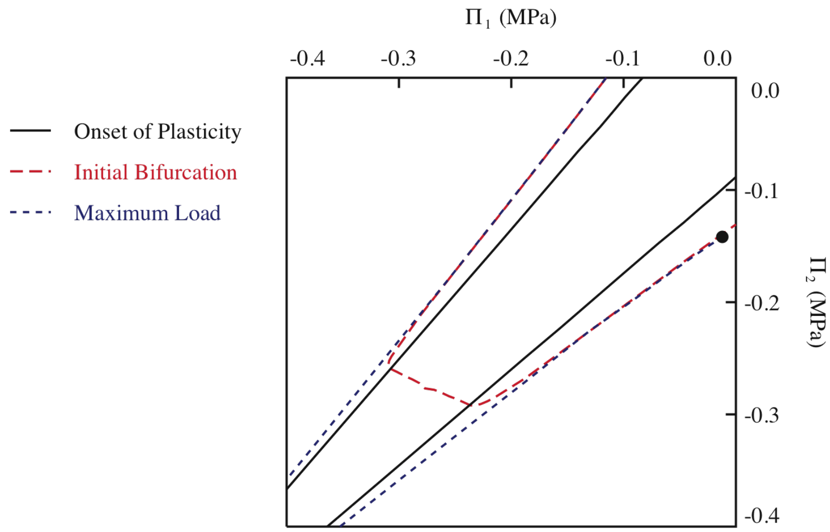
• **First (as load increases) bifurcation instability of the principal (unit-cell periodic) solution of the infinite structure is captured, through Bloch wave analysis. One finds the loss of positive definiteness of a low-dimension (half the # of unit cell's boundary nodes) Hermitian matrix $\mathbf{K}(\lambda;\omega_1 L_1, \omega_2 L_2)$. Reason it works: modes of all possible wavelengths (with respect to unit-cell size) are considered.**

• **Calculations work as follows: for fixed $(\omega_1 L_1, \omega_2 L_2)$ the lowest load $\lambda_m(\omega_1 L_1, \omega_2 L_2)$ for vanishing of the minimum eigenvalue of $\mathbf{K}(\lambda;\omega_1 L_1, \omega_2 L_2)$ is found. The sought critical load corresponding to the first (as load increases) bifurcation instability is given by the minimum of $\lambda_m(\omega_1 L_1, \omega_2 L_2)$ over all $0 < \omega_1 L_1, \omega_2 L_2 < 2\pi$.**

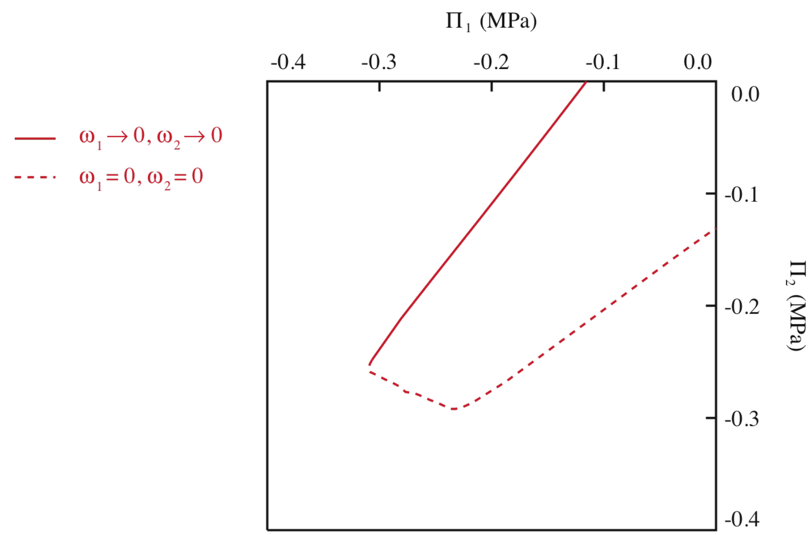
CAUTION: Surface $\lambda_m(\omega_1 L_1, \omega_2 L_2)$ can be singular near the origin $(\omega_1 L_1, \omega_2 L_2) = (0, 0)$ since two different types of modes coexist in that neighborhood: long wavelength modes for which $(\omega_1 L_1, \omega_2 L_2) \rightarrow (0, 0)$ and unit-cell periodic modes for which $(\omega_1 L_1, \omega_2 L_2) = (0, 0)$. If the minimum of $\lambda_m(\omega_1 L_1, \omega_2 L_2)$ occurs on a $(0, 0)$ singularity, then a global (with a wavelength much larger than the unit cell dimensions) mode is the critical one. If not, a local (commensurate with the unit cell dimensions) mode is the critical one.

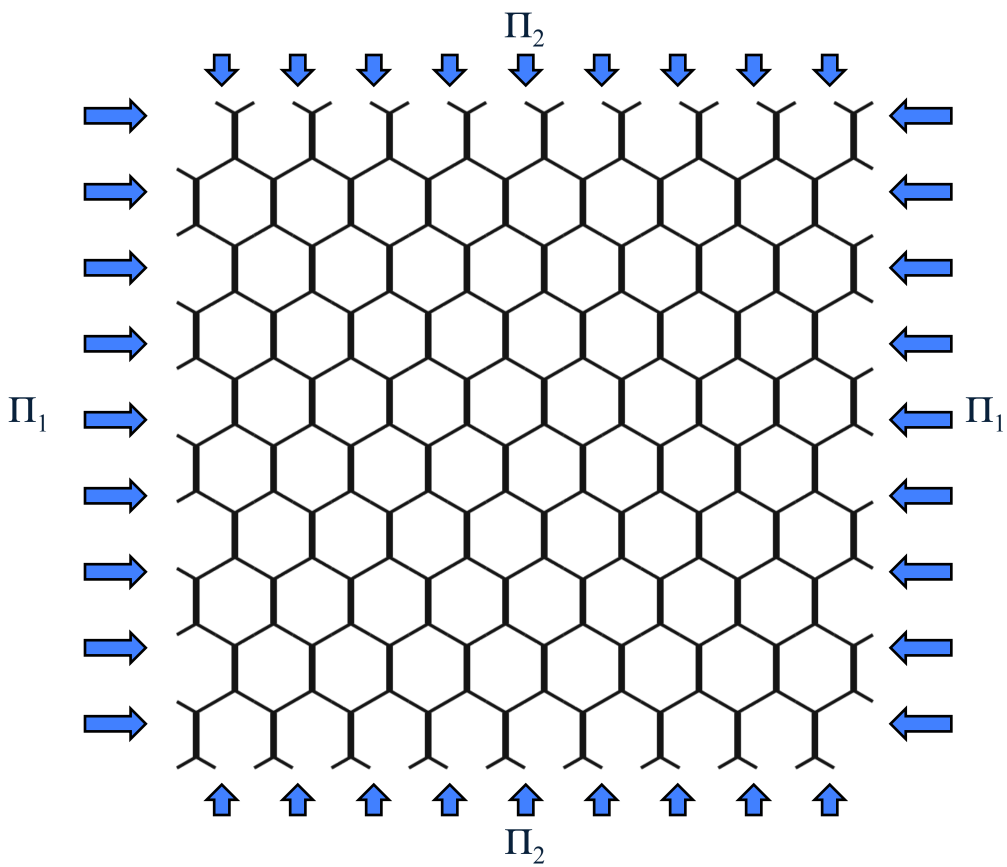
NOTE: Bloch wave technique described here is useful for the stability analysis of a wide range of periodic solids, discrete (cellular solids, lattices) as well as continua.

HEXAGONAL HONEYCOMB: INFLUENCE OF LOAD PATH



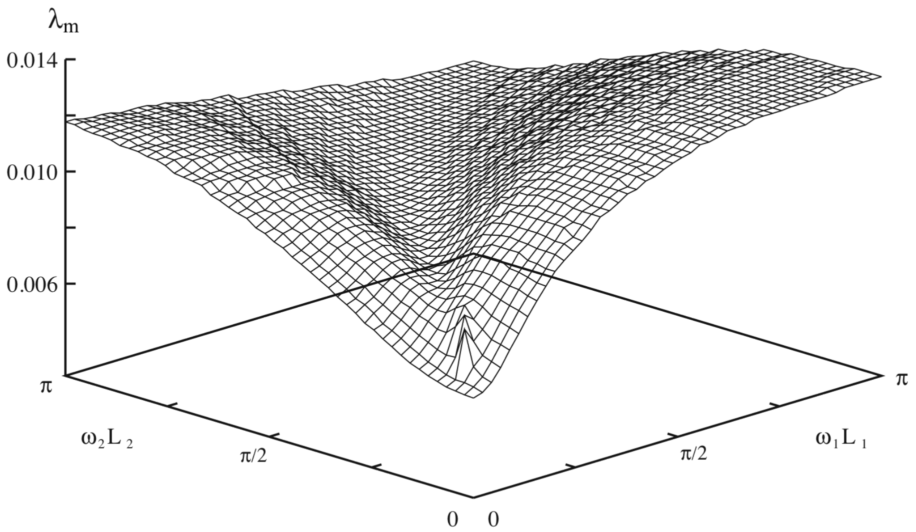
Loading paths: $\Pi_2 / \Pi_1 = \tan(\phi)$

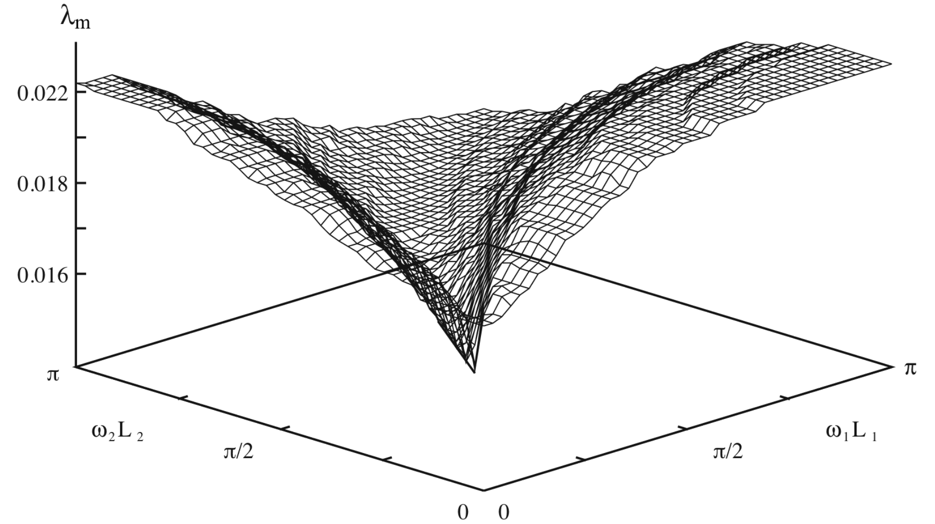




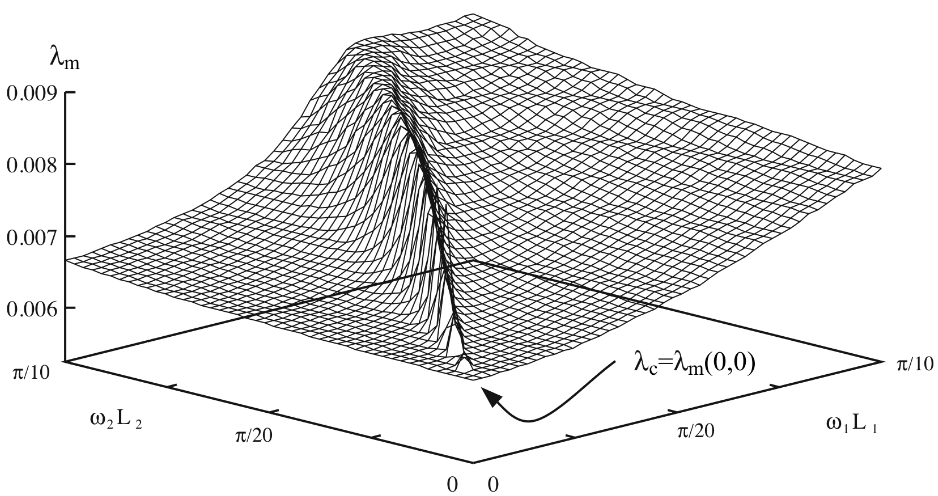
Onset-of-bifurcation for various load paths in stress space ($0 < \phi < 90°$). Calculations for the infinite perfect structure using Bloch waves.

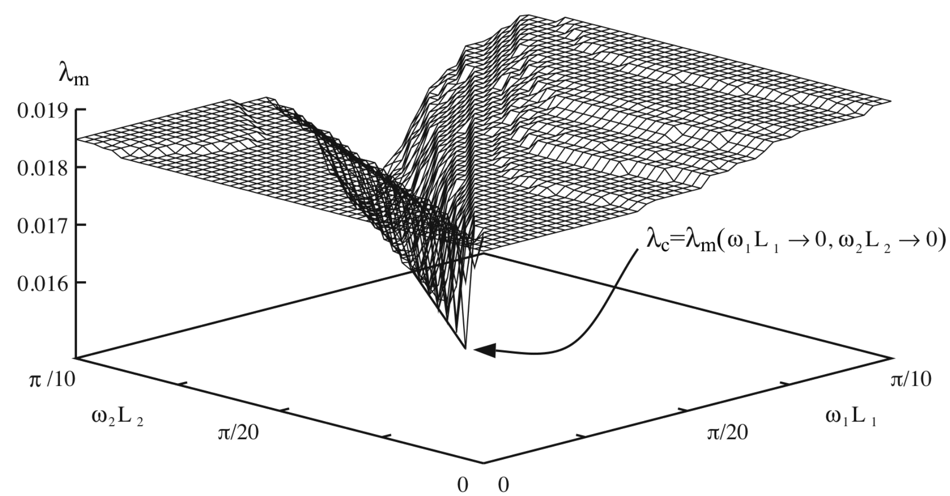
HEXAGONAL HONEYCOMB: INFLUENCE OF LOAD PATH



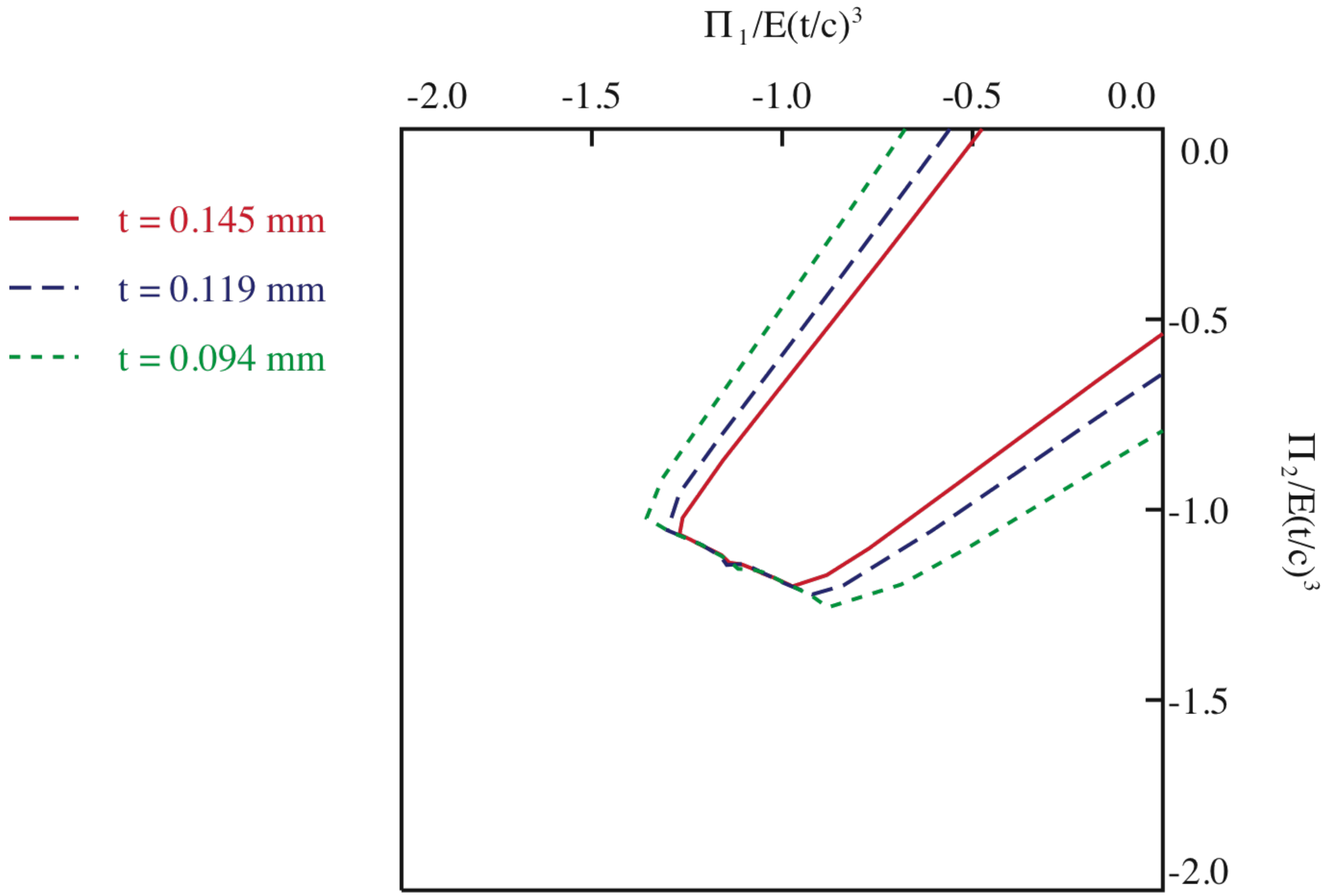


Local (unit-cell periodic) mode is critical (ϕ=35°) Global (long wavelength) mode is critical (ϕ=40°)



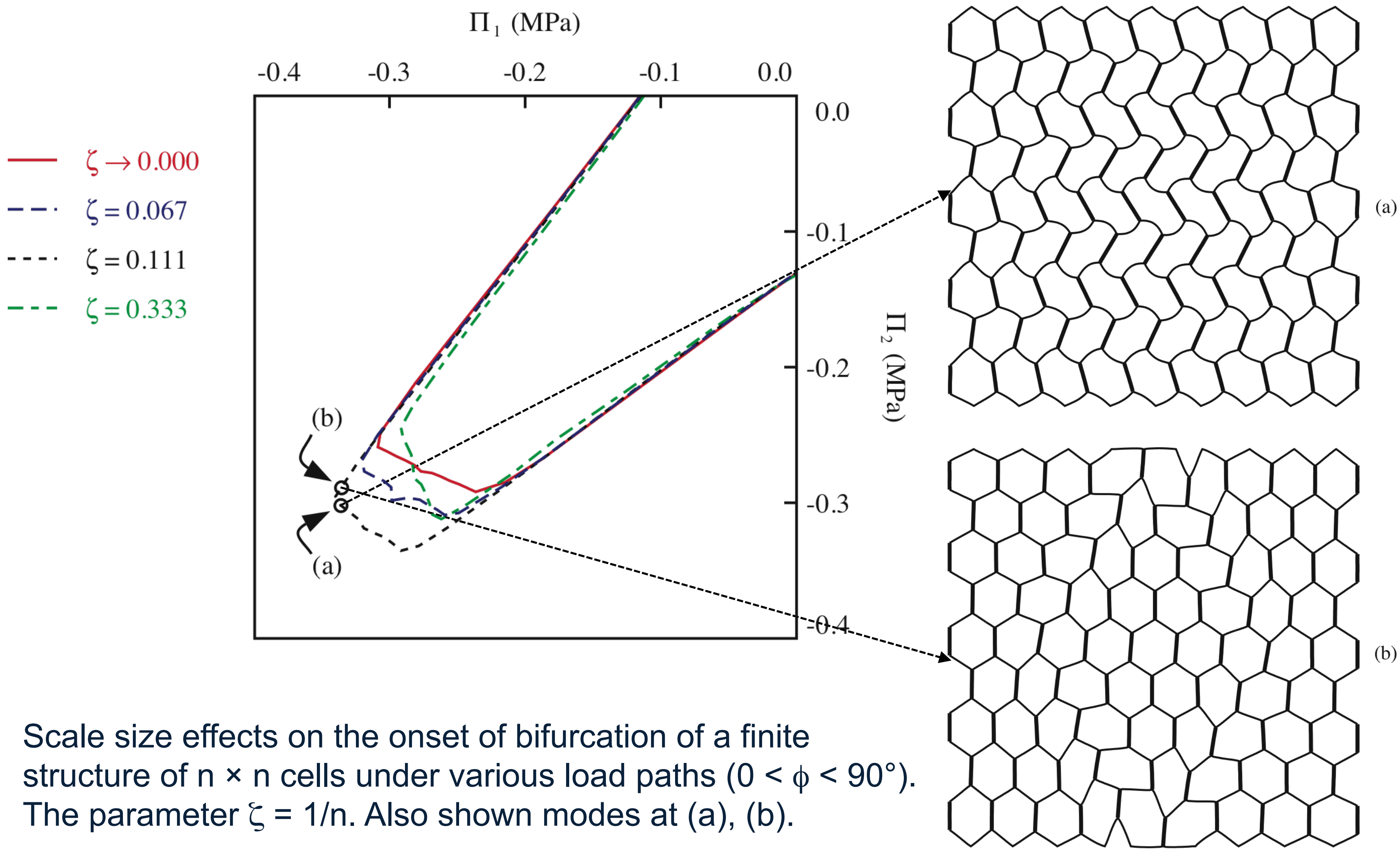


HEXAGONAL HONEYCOMB: INFLUENCE OF WALL THICK.



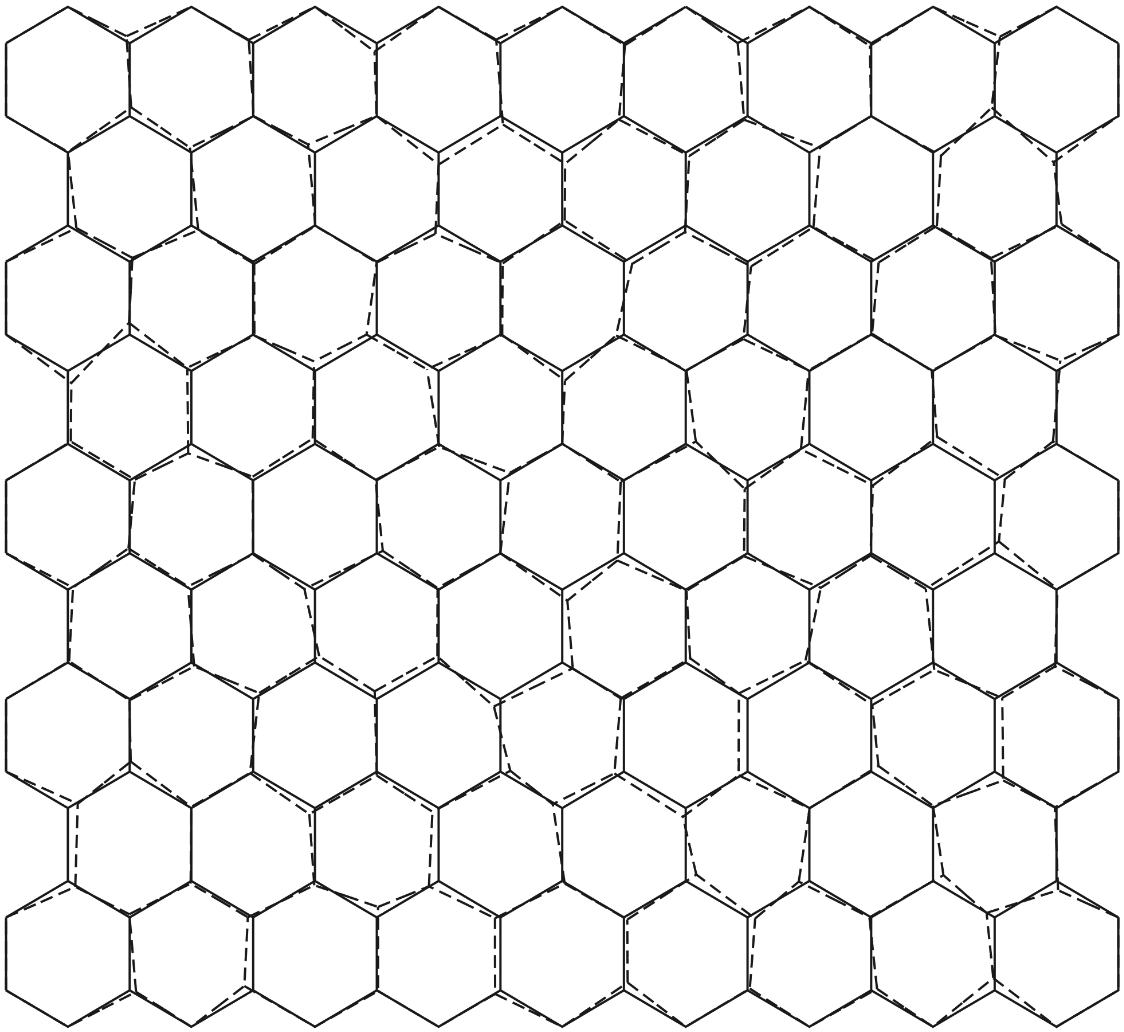
Influence of cell wall thickness on the onset-of-bifurcation for various load paths (0 < ϕ < 90°).
Calculations for the infinite perfect structure using Bloch waves.

HEXAGONAL HONEYCOMB: INFLUENCE OF SIZE



Scale size effects on the onset of bifurcation of a finite structure of n × n cells under various load paths ($0 < \phi < 90°$). The parameter $\zeta = 1/n$. Also shown modes at (a), (b).

HEXAGONAL HONEYCOMB: INFLUENCE OF IMPERFECTIONS



— **PERFECT**

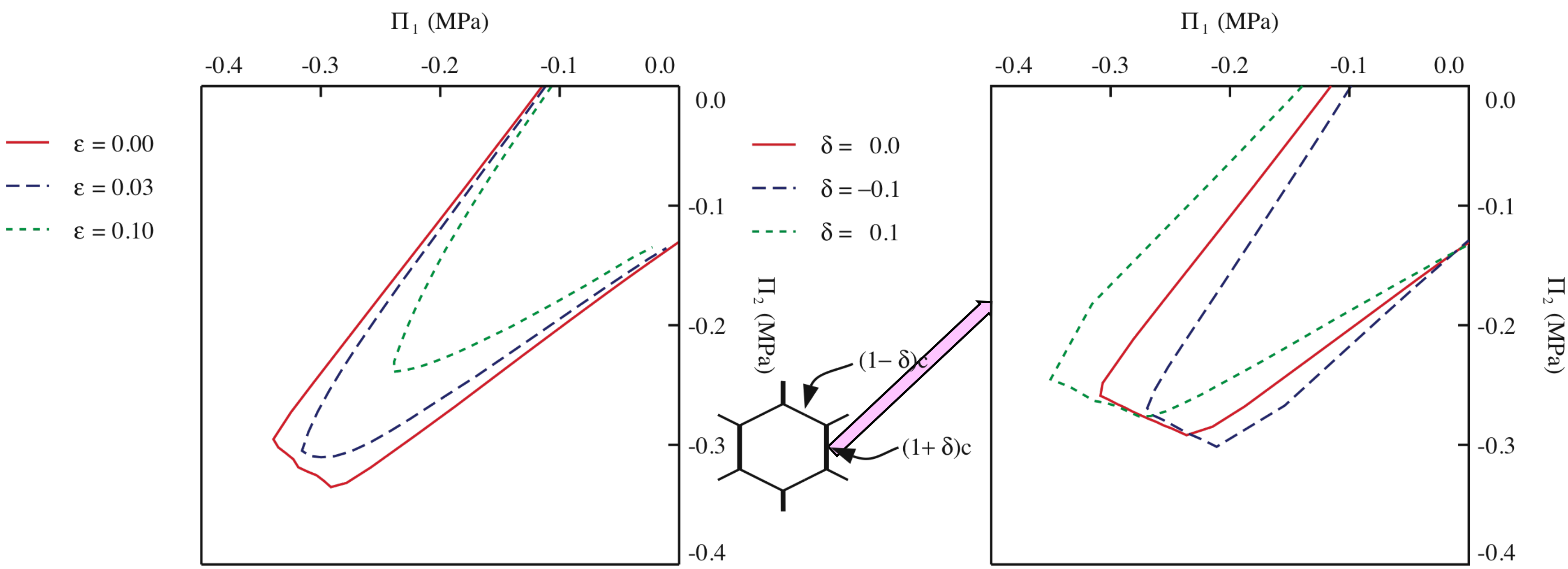
--- **IMPERFECT**

The nodes of the perfect structure are randomly displaced within a disc by:

$\Delta x_1 = \varepsilon\, cr \cos(2\pi q)$
$\Delta x_2 = \varepsilon\, cr \sin(2\pi q)$

ε: imperfection amplitude
c: cell wall length
r: random number 0 < r < 1
q: random number 0 < q < 1

HEXAGONAL HONEYCOMB: INFLUENCE OF IMPERFECTIONS



Influence of the imperfection amplitude ε on the onset of instability (Det **K** = 0) for various load paths ($0 < \phi < 90°$). The nodes of the structure are randomly displaced within a disc of maximum radius εc (where c is the wall cell size for the perfect case).

Influence of the systematic manufacturing error parameter δ on the onset of bifurcation for the infinite, perfect structure for various load paths ($0 < \phi < 90°$). The systematic manufacturing error results in a periodic structure but with non-hexagonal cells.

WHAT HAVE WE LEARNED:

- Onset-of-failure in cellular solids is due to a local bifurcation (for perfect case) instability mechanism and is the precursor of the ultimate failure mechanism.

- Onset-of-failure is very sensitive to constitutive properties, microstructure geometry and macroscopic loading.

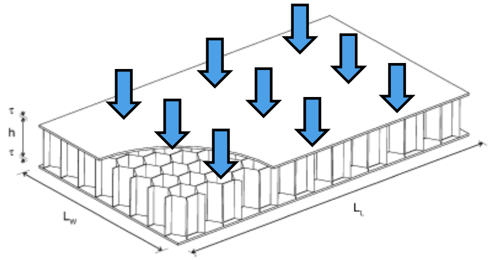
- One can construct onset-of-failure surfaces in macroscopic stress or strain space using powerful concept of Bloch wave representation for the eigenmode, which requires investigation of the smallest unit cell of the perfect, periodic structure.

- Concept of onset-of-failure surfaces in macroscopic stress or strain space is a general method, applicable to arbitrary loadings and geometry in rate-independent, ductile materials and gives a consistent method for predicting the beginning of the failure sequence in real, imperfect cellular solids.

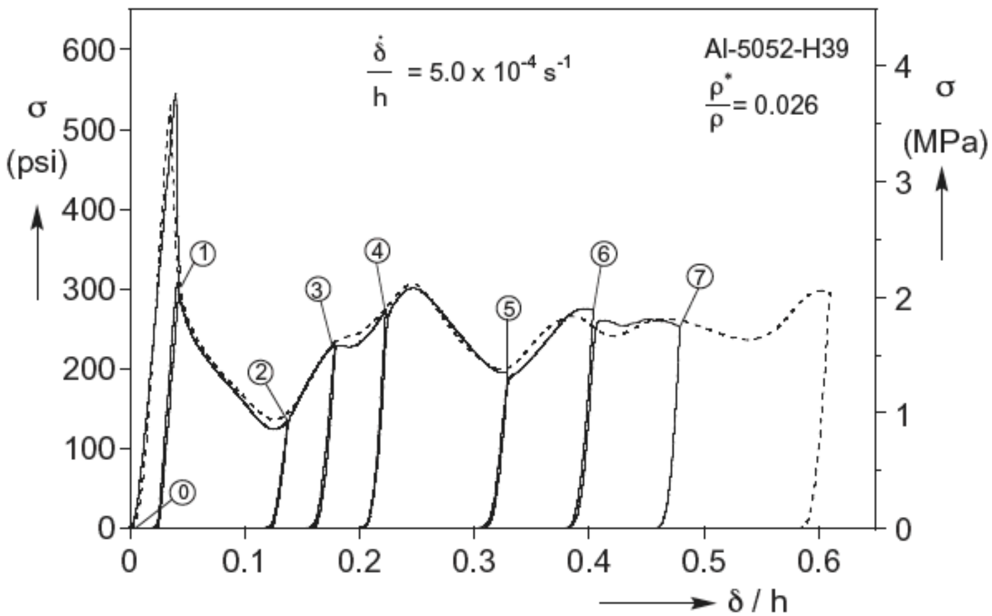
- Predicting actual failure does require solution of large (multi-cell) boundary value problems of the full structure with actual boundary conditions. Since such solutions are strongly dependent on imperfection shape (huge number of possibilities), onset-of-failure surfaces provide excellent guide to analyze such complicated problems.

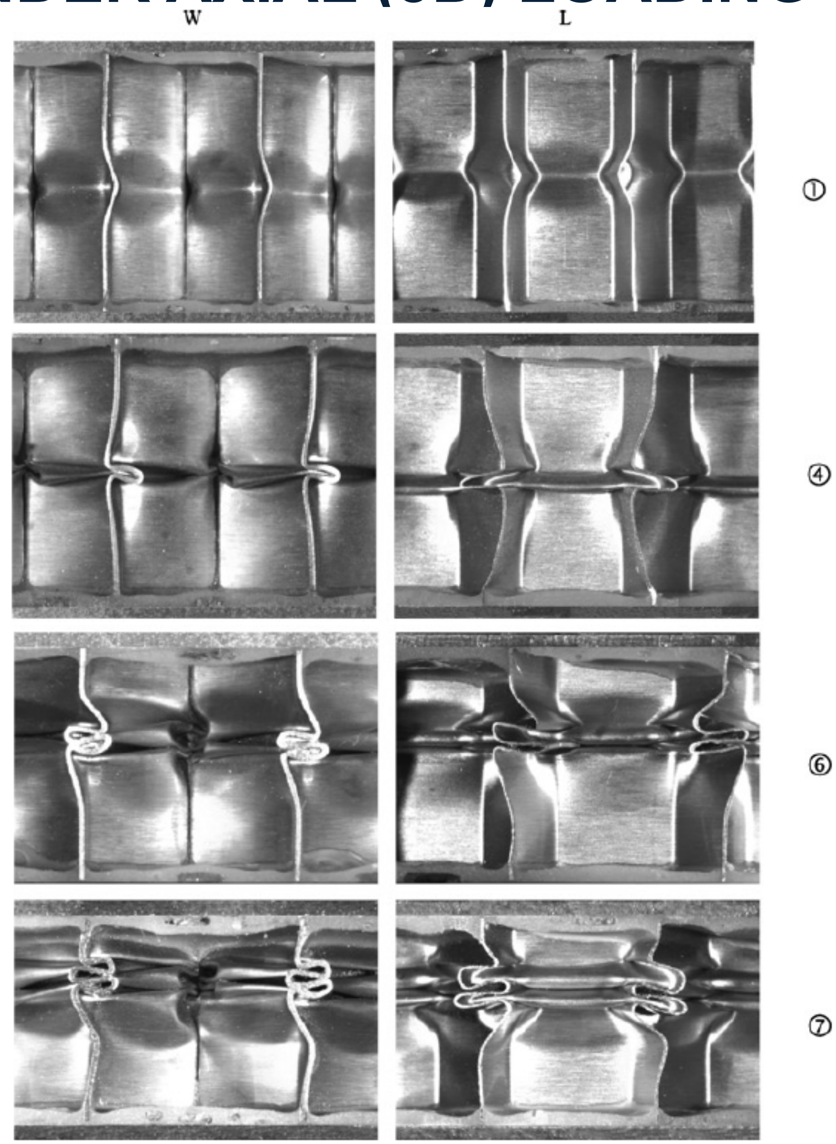
CELLULAR SOLIDS – 3D LOADING

HEXAGONAL HONEYCOMB UNDER AXIAL (3D) LOADING



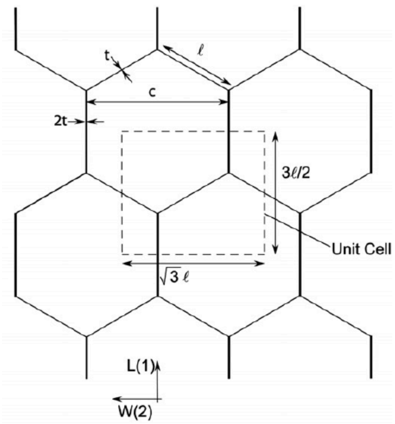
Honeycomb usually crushed
along axial direction (h)

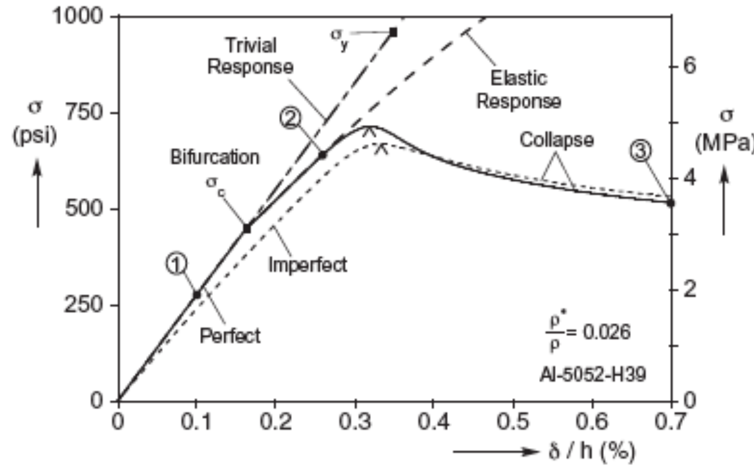




From: Wilbert, Yang, Kyriakides & Floccari IJSS, 2011, **48**, pp. 803-816

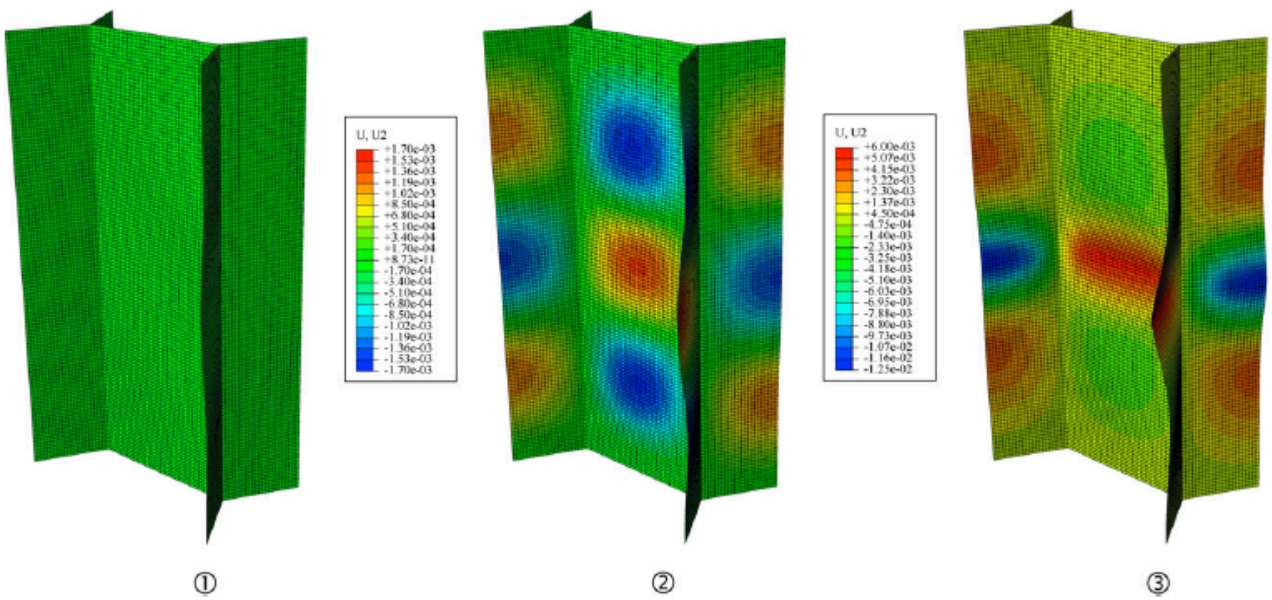
HEXAGONAL HONEYCOMB UNDER AXIAL (3D) LOADING

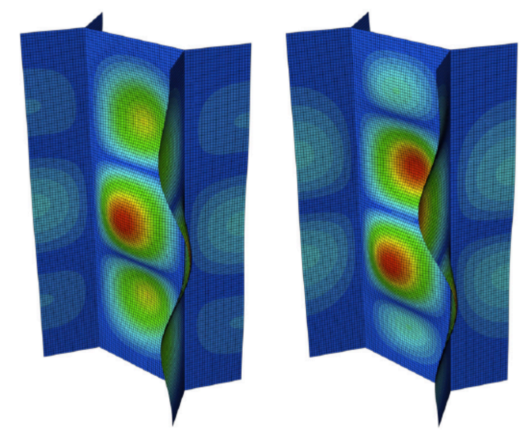




SEQUENCE OF EVENTS:

1. **Bifurcation (elastic)**
2. **Onset of placticity**
3. **Maximum load**
4. **Localization**





First two buckling modes are close at 2.95 MPa & 2.99 Mpa

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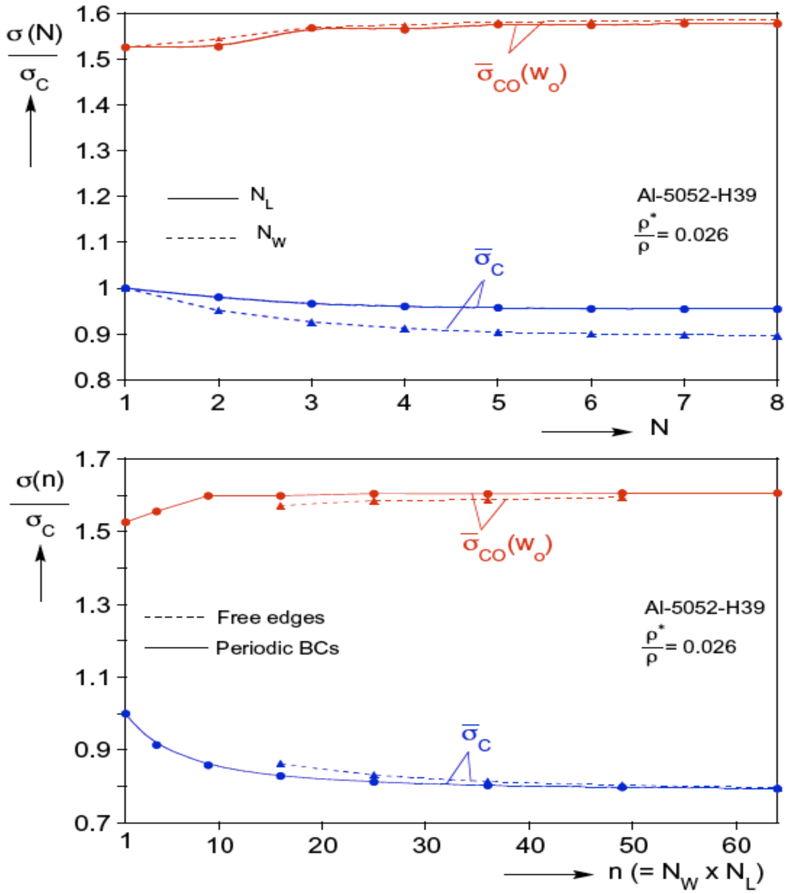
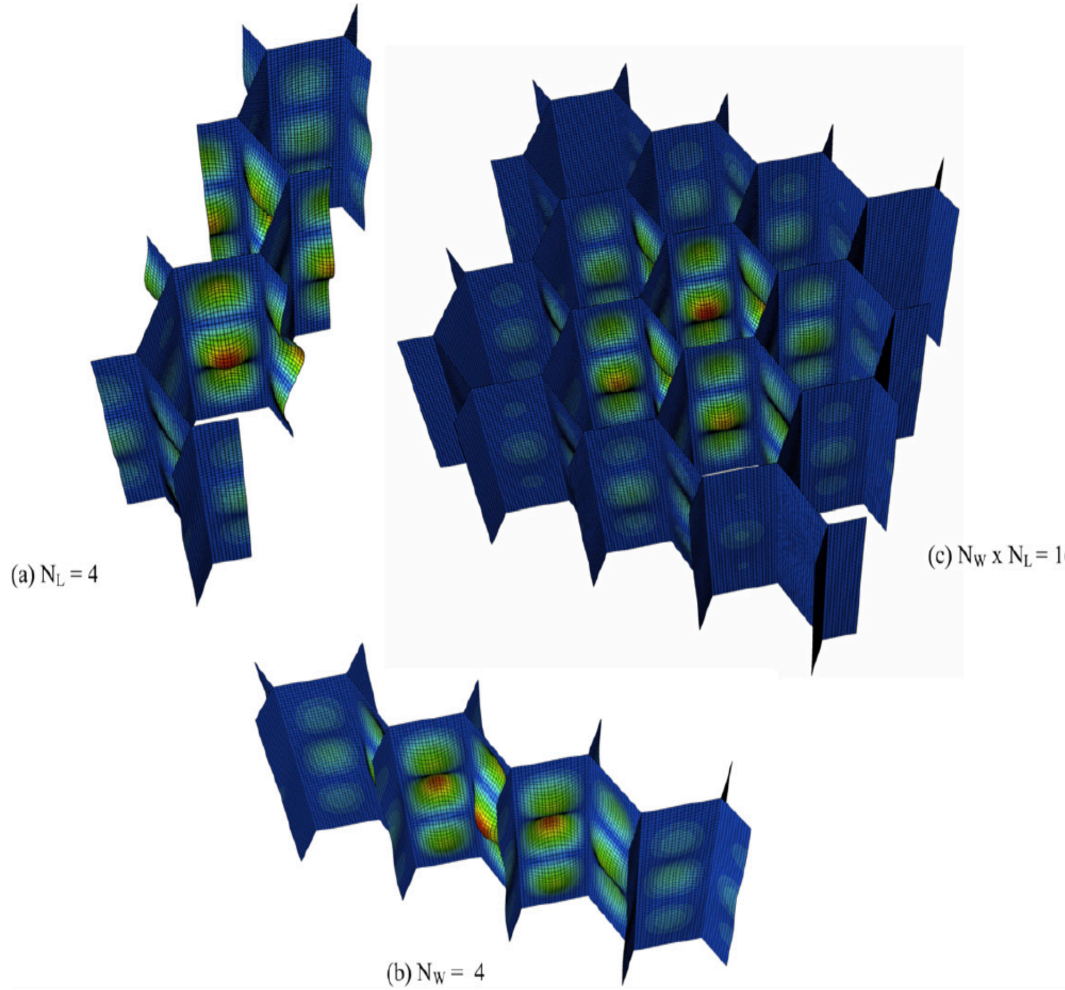
CELLULAR SOLIDS – 3D LOADING

HEXAGONAL HONEYCOMB UNDER AXIAL (3D) LOADING



First buckling and collapse stresses as a function of FEM domain size



First buckling modes depends on size of specimen used

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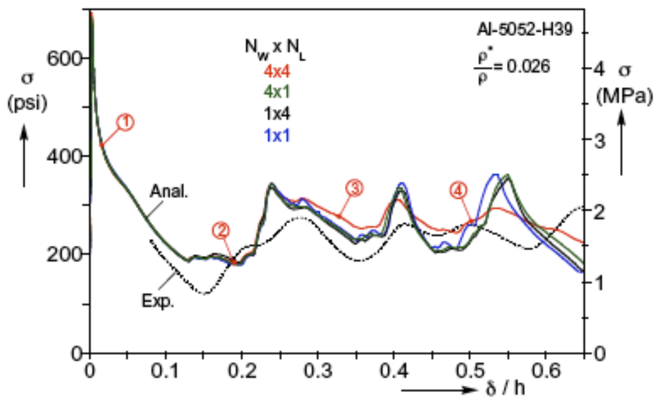
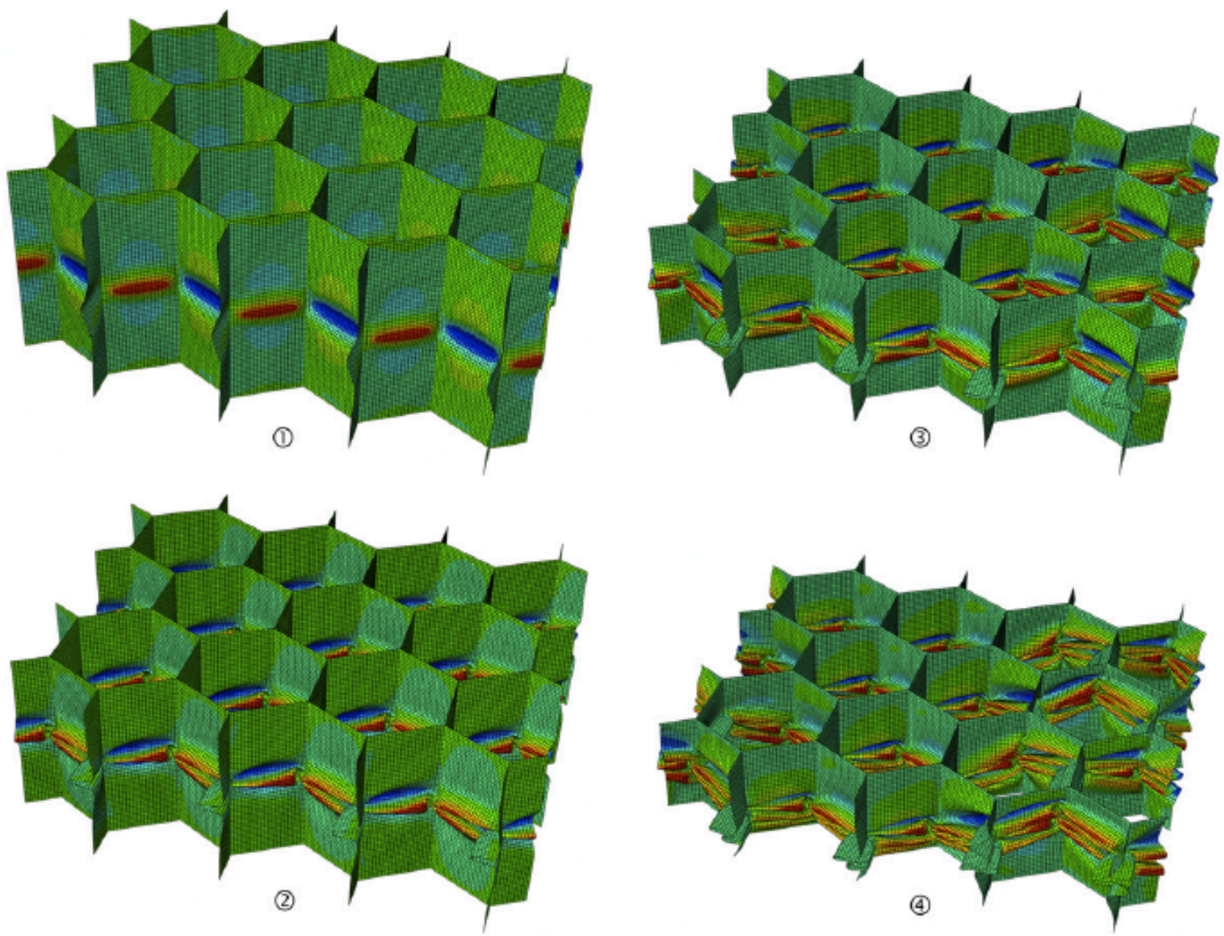
HEXAGONAL HONEYCOMB UNDER AXIAL (3D) LOADING

FEM calculations of loading along axial direction (h)



FEM calculations results depend on specimen size



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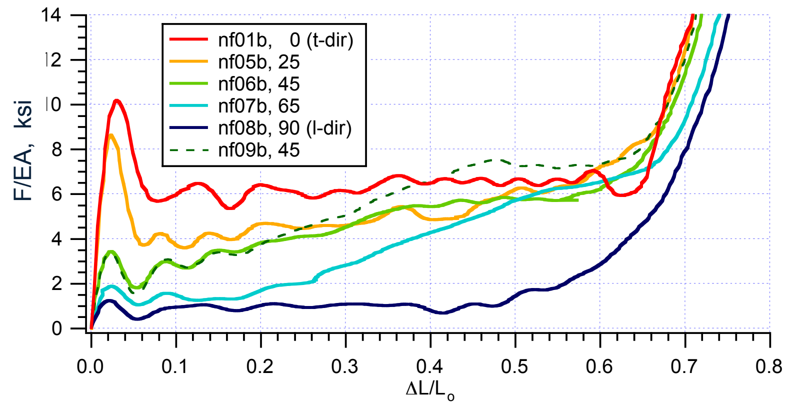
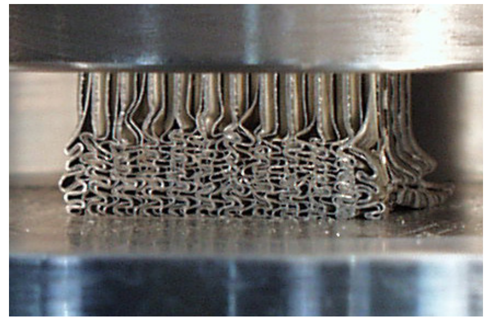
CELLULAR SOLIDS – 3D LOADING

INFLUENCE OF LOAD ORIENTATION, TEMPERATURE:



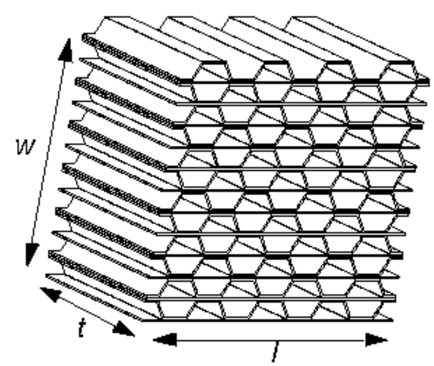


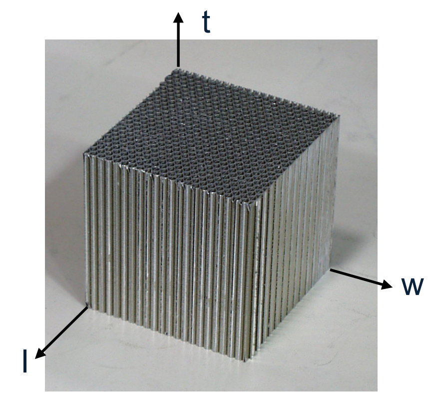
Failure mechanism at room temperature

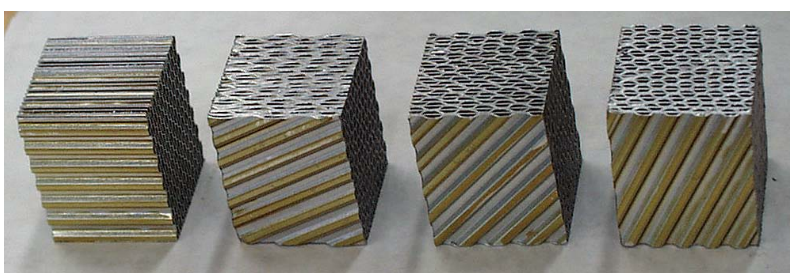
Failure mechanism at 165 ^{o}F

Strong dependence of first instability (and thus maximum force) on loading orientation – from confined compression experiments.

Courtesy: W. Y. Lu SANDIA National Labs CA







90° 65° 45° 25°

Failure mechanisms of reinforced aluminum honeycomb depend strongly on macroscopic load orientation and temperature (glue softens)