LSK ASYMPTOTICS – MULTIPLE MODE CASE

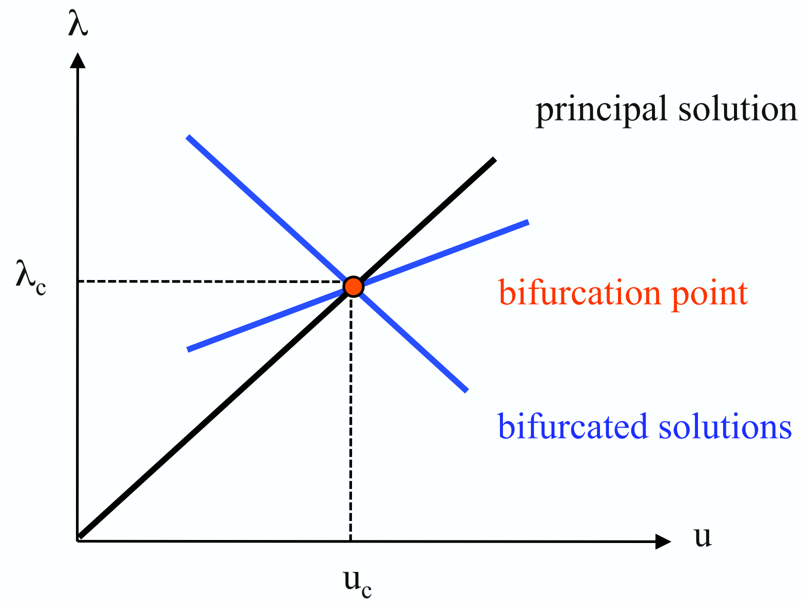
• **Interest** of multiple mode case for applications: systems with an **initially high symmetry**: e.g. cylinder buckling, stability of cubic crystals.

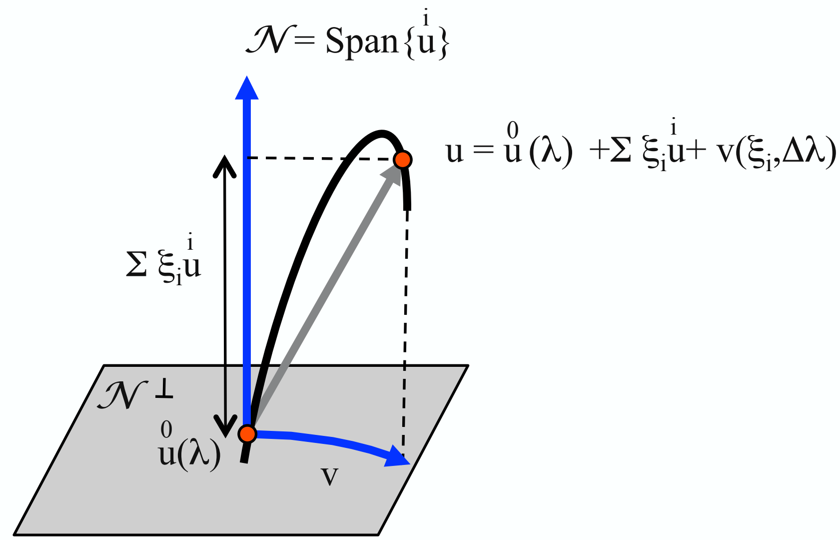
• **IDEA**: Study the **projection** of equilibrium equations along the **finite dimensional null space** of the system's stability operator at critical point. This way the study of a large problem is **reduced to the study of a nonlinear system of m equations** where m is the multiplicity of the stability operator's eigenvalue at the critical point.

• Method **follows asymptotically all the equilibrium paths** emerging from the bifurcation point of the perfect system and **determines their stability**.

• Method also investigates the **equilibrium** and **stability** of **imperfect** systems, near critical points of their perfect counterparts, for **small imperfection amplitudes**

•**NOTE**: Method is useful in determining **post-bifurcation behavior** and **imperfection sensitivity** in applications as well as in providing efficient **numerical tools** for finding solutions near the singular points of complex nonlinear systems with a **high degree of initial symmetry**





Method is a straightforward generalization of simple mode case:

• **About an arbitrary point $\lambda = \lambda_c + \Delta\lambda$ of the principal solution $u^0(\lambda)$ project difference $\Delta u = u - u^0$ along m-dimensional null space $\mathcal{N}$ and its orthogonal complement $\mathcal{N}^{\perp}$. Subsequently expand corresponding equilibrium equations about λ_c**

• **From critical point u_c, λ_c at most $2^m - 1$ (asymmetric case) or $(3^m - 1)/2$ (symmetric case) bifurcated equilibrium paths emerge (because initial tangents are solutions of m 2nd or 3rd order polynomial equations with m variables**

ENERGY AND PRINCIPAL SOLUTION

$\mathcal{E}(u,\lambda)$: energy at displacement $u(\mathbf{x}) \in U$ and load $\lambda \geq 0$

$\mathcal{E}(0,\lambda) = 0, \ \forall \lambda$: zero energy at zero displacement

$\mathcal{E},_u(u,\lambda)\delta u = 0, \ \forall \ \delta u \in U$: equilibrium statement

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$\mathcal{E},_u(\overset{0}{u}(\lambda),\lambda)\delta u = 0, \ \forall \lambda;$ principal solution $\overset{0}{u}(\lambda), \ (\overset{0}{u}(0) = 0)$

$\overset{0}{u}(\lambda)$ stable near $\lambda = 0$, i.e. min. eigenvalue of $\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda) \equiv \mathcal{E},^0_{uu}$ is $\overset{0}{\beta} > 0$

$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u \geq \overset{0}{\beta}(\lambda) \parallel \delta u \parallel^2, \ \overset{0}{\beta}(\lambda) > 0; \quad \exists \epsilon > 0, \ \forall \ \lambda \in [0,\epsilon]$

NOTE: Unique & stable principal solution near zero load assumed (realistic structures)

CRITICAL LOAD AND EIGENMODES

$$\frac{d}{d\lambda}\left[\mathcal{E}_{,u}(\overset{0}{u}(\lambda),\lambda)\delta u\right] = 0; \quad \text{i.e. differentiate principal solution with respect to } \lambda$$

$$\left[\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)(d\overset{0}{u}/d\lambda) + \mathcal{E}_{,u\lambda}(\overset{0}{u}(\lambda),\lambda)\right]\delta u = 0$$

$$d\overset{0}{u}/d\lambda \text{ exists if } (\mathcal{E}^{0}_{,uu})^{-1} \text{ exists, which is the case as long as :}$$

$$\mathcal{E}^{0}_{,uu} \equiv \mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda) \text{ is positive definite, invertible, i.e. } \overset{0}{\beta}(\lambda) > 0, \text{ for } \lambda \in [0,\lambda_c)$$

As λ increases away from 0, the lowest λ that $\mathcal{E}^{0}_{,\mathrm{uu}}$ loses positive definiteness is : λ_{c}

NOTE: Principal solution has a singular point at the critical load λ_c

CRITICAL LOAD AND EIGENMODES

$$(\mathcal{E},_{uu}^{c} \overset{i}{u})\delta u = 0, \quad m \text{ distinct eigenmodes } \overset{i}{u}, \quad \text{where : } \mathcal{E},_{uu}^{c} \equiv \mathcal{E},_{uu} (\overset{0}{u}(\lambda_c), \lambda_c)$$

$$(\overset{i}{u}, \overset{j}{u}) = \delta_{ij};\ 1 \leq i, j \leq m, \quad \{\overset{i}{u}\} \text{ orthonormal basis of null space } \mathcal{N} \text{ of } \mathcal{E}_{uu}^{c},$$

$$\mathcal{N} \equiv \{u \in U \mid u = \textstyle\sum_{i=1}^{m} \xi_i \overset{i}{u},\ \xi_i \in \mathbb{R}\}, \quad \mathcal{N}^{\perp} \equiv \{v \in U \mid (v, \overset{i}{u}) = 0,\ 1 \leq i \leq m\}.$$

In all directions not belonging to $\mathcal{N}$, operator $\mathcal{E},_{uu}^{c}$ is still positive definite :

$$(\mathcal{E},_{uu}^{c} \delta v)\delta v \geq \gamma \parallel \delta v \parallel^2, \quad \exists \gamma > 0, \quad \forall \delta v \in \mathcal{N}^{\perp}.$$

$\mathcal{E},_{u\lambda}^{c} \overset{i}{u} = 0,\ 1 \leq i \leq m; \quad m-$tuple bifurcation point at : (u_c, λ_c) is assumed.

NOTE: multiple singular point at λ_c is assumed to be a bifurcation point

LSK ASYMPTOTICS – EQUILIBRIUM SOLUTIONS

$$u = \overset{0}{u}(\lambda) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v; \quad \xi_i \in \mathbb{R}, \quad v \in \mathcal{N}^{\perp}, \quad \xi_i \equiv (u - \overset{0}{u}, \overset{i}{u})$$

$$\mathcal{E}_{,v}\, \delta v = 0 \implies \mathcal{E}_{,u}\, (\overset{0}{u}(\lambda_c + \Delta\lambda) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v(\xi_i, \Delta\lambda), \lambda_c + \Delta\lambda)\delta v = 0; \text{ equilibrium in } \mathcal{N}^{\perp}$$

Expand about (u_c, λ_c) to find $v(\xi_i, \Delta\lambda)$, where :

$$v(\xi_i, \Delta\lambda) = \sum_{i=1}^{m} \xi_i v_i + \Delta\lambda v_\lambda + \frac{1}{2}(\sum_{i=1}^{m}\sum_{j=1}^{m} \xi_i \xi_j v_{ij} + 2\Delta\lambda \sum_{i=1}^{m} \xi_i v_{i\lambda} + (\Delta\lambda)^2 v_{\lambda\lambda}) + \ldots$$

$$\mathcal{E}_{,\xi_i} = 0 \implies \mathcal{E}_{,u}\, (\overset{0}{u}(\lambda_c + \Delta\lambda) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v(\xi_i, \Delta\lambda), \lambda_c + \Delta\lambda)\overset{i}{u} = 0; \text{ equilibrium in } \mathcal{N}$$

Expand about (u_c, λ_c), using $v(\xi_i, \Delta\lambda)$, to find m equations for ξ_i in $\mathcal{N}$

LSK ASYMPTOTICS – EQUILIBRIUM SOLUTIONS

$$O(\xi_i) \;:\; (\mathcal{E},^c_{uu} v_i)\delta v = 0 \implies v_i = 0, \quad (\mathcal{E},^c_{uu} \text{ positive definite in} : \mathcal{N}^{\perp})$$

$$O(\Delta\lambda) \;:\; (\mathcal{E},^c_{uu} v_\lambda + \mathcal{E},^c_{uu} (d\overset{0}{u}/d\lambda)_c + \mathcal{E},^c_{u\lambda})\delta v = (\mathcal{E},^c_{uu} v_\lambda)\delta v = 0 \implies v_\lambda = 0, \quad \text{(same)}$$

$$O(\xi_i \xi_j) \;:\; (\mathcal{E},^c_{uu} v_{ij} + (\mathcal{E},^c_{uuu} \overset{i}{u})\overset{j}{u})\delta v = 0$$

$$O(\xi_i \Delta\lambda) \;:\; (\mathcal{E},^c_{uu} v_{i\lambda} + (\mathcal{E},^c_{uuu} (d\overset{0}{u}/d\lambda)_c + \mathcal{E},^c_{uu\lambda})\overset{i}{u})\delta v = 0$$

$$O((\Delta\lambda)^2) \;:\; (\mathcal{E},^c_{uu} v_{\lambda\lambda} + (\mathcal{E},^c_{uuu} (d\overset{0}{u}/d\lambda)_c)(d\overset{0}{u}/d\lambda)_c + 2\mathcal{E},^c_{uu\lambda} (d\overset{0}{u}/d\lambda)_c +$$

$$\mathcal{E},^c_{u\lambda\lambda} + \mathcal{E},^c_{uu} (d^2\overset{0}{u}/d\lambda^2)_c)\delta v = (\mathcal{E},^c_{uu} v_{\lambda\lambda})\delta v = 0 \implies v_{\lambda\lambda} = 0, \quad \text{(same)}$$

NOTE A : $v(0, \Delta\lambda) = u - \overset{0}{u}(\lambda) = 0 \implies v_\lambda = v_{\lambda\lambda} = v_{\lambda\lambda\lambda} = ... = 0$

NOTE B : Highlighted terms λ derivatives of principal equilibrium : $\mathcal{E},_u (\overset{0}{u}(\lambda), \lambda)\delta u = 0$

LSK ASYMPTOTICS – EQUILIBRIUM SOLUTIONS

use : $v(\xi_i, \Delta\lambda) = \sum_{i=1}^{m} \xi_i v_i + \Delta\lambda v_\lambda + \frac{1}{2}(\sum_{i=1}^{m}\sum_{j=1}^{m} \xi_i \xi_j v_{ij} + 2\Delta\lambda \sum_{i=1}^{m} \xi_i v_{i\lambda} + (\Delta\lambda)^2 v_{\lambda\lambda}) + ...$

into : $\mathcal{E},_{\xi_i} = \mathcal{E},_u (\overset{0}{u}(\lambda_c + \Delta\lambda) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v(\xi_i, \Delta\lambda), \lambda_c + \Delta\lambda)\overset{i}{u} = 0;$ equilibrium in $\mathcal{N}$

expand above equilibrium equations $\mathcal{E},_{\xi_\mathrm{i}} = 0$ about $(\mathrm{u_c}, \lambda_\mathrm{c})$ in powers of ξ_i, and $\Delta\lambda$

$$\frac{1}{2}[\ \sum_{j=1}^{m}\sum_{k=1}^{m} \xi_j \xi_k \mathcal{E}_{ijk} + 2\Delta\lambda \sum_{j=1}^{m} \xi_j \mathcal{E}_{ij\lambda}\ ] + \frac{1}{6}[\ \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} \xi_j \xi_k \xi_l \mathcal{E}_{ijkl} + ...\ ]\ + ... = 0,\ \ m \text{ equs.}$$

$$\mathcal{E}_{ijk} \equiv ((\mathcal{E},^{c}_{uuu} \overset{i}{u})\overset{j}{u})\overset{k}{u},$$

$$\mathcal{E}_{ij\lambda} \equiv ((d\mathcal{E},_{uu} / d\lambda)_c)\overset{i}{u})\overset{j}{u} = ((\mathcal{E},^{c}_{uuu} (d\overset{0}{u}/d\lambda)_c + \mathcal{E},^{c}_{uu\lambda})\overset{i}{u})\overset{j}{u}$$

$$\mathcal{E}_{ijkl} \equiv (((\mathcal{E},^{c}_{uuuu} \overset{j}{u})\overset{k}{u})\overset{l}{u} + (\mathcal{E},^{c}_{uuu} v_{jk})\overset{l}{u} + (\mathcal{E},^{c}_{uuu} v_{kl})\overset{j}{u} + (\mathcal{E},^{c}_{uuu} v_{lj})\overset{k}{u})\overset{i}{u}$$

LSK ASYMPTOTICS – EQUILIBRIUM SOLUTIONS

Introduce bifurcation amplitude ξ :

$$\xi \equiv (u - \overset{0}{u}, \sum_{i=1}^{m} \alpha_i^1 \overset{i}{u}),$$

Parametrize equilibrium solutions :

$$\xi_i(\xi) = \alpha_i^1 \xi + \alpha_i^2 \frac{\xi^2}{2} + ...$$

$$\Delta\lambda(\xi) = \lambda_1 \xi + \lambda_2 \frac{\xi^2}{2} + ...$$

Use $\xi_{\mathrm{i}}(\xi),\ \Delta\lambda(\xi)$ into system of m equilibrium equations

$$\frac{1}{2}[\sum_{j=1}^{m}\sum_{k=1}^{m} \xi_j \xi_k \mathcal{E}_{ijk} + 2\Delta\lambda \sum_{j=1}^{m} \xi_j \mathcal{E}_{ij\lambda}\ ] + \frac{1}{6}[\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} \xi_j \xi_k \xi_l \mathcal{E}_{ijkl} + ...\ ]\ + ... = 0.$$

LSK ASYMPTOTICS – EQUILIBRIUM SOLUTIONS

case (i) : $\mathcal{E}_{ijk} \neq 0,\ \exists(i,j,k)$

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\alpha_j^1\alpha_k^1\mathcal{E}_{ijk} + 2\lambda_1\sum_{j=1}^{m}\alpha_j^1\mathcal{E}_{ij\lambda} = 0, \quad \sum_{i=1}^{m}(\alpha_i^1)^2 = 1$$

ξ expansion to higher order possible if : Det $[B_{ij}] \neq 0, \quad B_{ij} \equiv \sum_{k=1}^{m}\alpha_k^1\mathcal{E}_{ijk} + \lambda_1\mathcal{E}_{ij\lambda}$

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case (ii) : $\mathcal{E}_{ijk} = 0,\ \forall(i,j,k)$

$$\lambda_1 = 0, \quad \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\alpha_j^1\alpha_k^1\alpha_l^1\mathcal{E}_{ijkl} + 3\lambda_2\sum_{j=1}^{m}\alpha_j^1\mathcal{E}_{ij\lambda} = 0, \quad \sum_{i=1}^{m}(\alpha_i^1)^2 = 1$$

ξ expansion to higher order possible if : Det $[B_{ij}] \neq 0, \quad B_{ij} \equiv \sum_{k=1}^{m}\sum_{l=1}^{m}\alpha_k^1\alpha_l^1\mathcal{E}_{ijkl} + \lambda_2\mathcal{E}_{ij\lambda}$

LSK ASYMPTOTICS – STABILITY OF PRINCIPAL PATH

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)\overset{0}{x}_i(\lambda))\delta u = \overset{0}{\beta}_i(\lambda)(\overset{0}{x}_i(\lambda),\delta u)\ (!);\quad 1 \leq i \leq m,$$

$$\overset{0}{\beta}_i(\lambda):\ m \text{ lowest eigenvalues},\ \overset{0}{x}_i(\lambda):\ \text{corresp. eigenvectors of } \mathcal{E},^{0}_{uu};\ (\overset{0}{x}_i(\lambda),\overset{0}{x}_i(\lambda)) = 1$$

$$\text{Evaluate at } \lambda_c:\ (\mathcal{E},^{c}_{uu}\overset{0}{x}_i(\lambda_c))\delta u = 0;\quad \overset{0}{\beta}_i(\lambda_c) = 0,\quad \overset{0}{x}_i(\lambda_c) = \overset{i}{u};\quad 1 \leq i \leq m,$$

$$\text{Differentiate at } \lambda_c:\ ((\mathcal{E},^{c}_{uuu}(d\overset{0}{u}/d\lambda)_c + \mathcal{E},^{c}_{uu\lambda})\overset{i}{u} + \mathcal{E},^{c}_{uu}(d\overset{0}{x}_i/d\lambda)_c)\delta u = (d\overset{0}{\beta}_i/d\lambda)_c(\overset{i}{u},\delta u)\ (!)$$

$$\text{Substitute}:\ \delta u = \overset{j}{u},\ \text{recall}:\ (\overset{i}{u},\overset{j}{u}) = \delta_{ij} \implies ((d\mathcal{E},_{uu}/d\lambda)_c\overset{i}{u})\overset{j}{u} = (d\overset{0}{\beta}_i/d\lambda)_c\delta_{ij}\ (!)$$

$$\text{Assumption}:\ (d\overset{0}{\beta}_i/d\lambda)_c < 0;\ (\text{recall}:\ \overset{0}{\beta}_i(\lambda) > 0,\ \forall\lambda \in [0,\lambda_c),\ \text{holds in most applications})$$

NOTE : The symbol (!) at an equation's end denotes no sum over repeated indexes

LSK ASYMPTOTICS – STABILITY OF BIFURCATED BRANCH

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda_c + \Delta\lambda(\xi)) + \sum_{i=1}^{m} \xi_i(\xi)\overset{i}{u} + v(\xi_i(\xi), \Delta\lambda(\xi)), \lambda_c + \Delta\lambda(\xi))x(\xi))\delta u = \beta(\xi)(x(\xi), \delta u)$$

Use : $\beta(\xi) = \xi\beta_1 + \frac{\xi^2}{2}\beta_2 + ...,\ x(\xi) = x_0 + \xi x_1 + \frac{\xi^2}{2}x_2 + ...,$ expand about : (u_c, λ_c)

$$O(1) \ :\quad (\mathcal{E},^c_{uu} x_0)\delta u = 0,\ (x_0, x_0) = 1;\ \Longrightarrow\ \mathcal{N} \ni x_0 = \sum_{i=1}^{m} \chi_i \overset{i}{u},\ (\sum_{i=1}^{m} \chi_i^2 = 1)$$

Assume : $\exists(i,j,k)\ ((\mathcal{E},^c_{uuu} \overset{i}{u})\overset{j}{u})\overset{k}{u} \neq 0,\quad$ substitute : $\delta u = \overset{i}{u},$

$$O(\xi) \ :\quad ((\mathcal{E},^c_{uuu}(\lambda_1(d\overset{0}{u}/d\lambda)_c + \sum_{k=1}^{m} \alpha^1_k \overset{k}{u}) + \lambda_1 \mathcal{E},^c_{uu\lambda})(\sum_{j=1}^{m} \chi_j \overset{j}{u}) + \mathcal{E},^c_{uu} x_1)\delta u =$$

$$= \beta_1((\sum_{j=1}^{m} \chi_j \overset{j}{u}), \delta u) \ \Longrightarrow\ \boxed{\sum_{j=1}^{m} B_{ij}\chi_j = \beta_1 \chi_i, \quad B_{ij} \equiv \sum_{k=1}^{m} \alpha^1_k \mathcal{E}_{ijk} + \lambda_1 \mathcal{E}_{ij\lambda}}$$

Branch is unstable when not all eigenvalues β_1 of B_{ij} have same sign

LSK ASYMPTOTICS – STABILITY OF BIFURCATED BRANCH

$$\text{Assume}: \ \forall(i,j,k)\ ((\mathcal{E},^{c}_{uuu} \overset{i}{u})\overset{j}{u})\overset{k}{u} = 0 \implies \lambda_1 = 0, \ \text{substitute}: \ \delta u = \delta v,$$

$$O(\xi) \ : \ (\sum_{j=1}^{m}\sum_{k=1}^{m} \chi_j \alpha^1_k (\mathcal{E},^{c}_{uuu} \overset{j}{u})\overset{k}{u} + \mathcal{E},^{c}_{uu} x_1)\delta v = 0, \ x_1 \in \mathcal{N}^{\perp} \implies x_1 = \sum_{i=1}^{m}\sum_{j=1}^{m} \chi_i \alpha^1_j v_{ij}$$

$$O(\xi^2) \ : \ (((\mathcal{E},^{c}_{uuuu} (\sum_{k=1}^{m} \alpha^1_k \overset{k}{u}))(\sum_{l=1}^{m} \alpha^1_l \overset{l}{u}) + \mathcal{E},^{c}_{uuu} (\lambda_2 (d\overset{0}{u}/d\lambda)_c + \sum_{k=1}^{m}\sum_{l=1}^{m} \alpha^1_k \alpha^1_l v_{kl} + \sum_{k=1}^{m} \alpha^2_k \overset{k}{u})$$

$$+\lambda_2 \mathcal{E},^{c}_{uu\lambda})(\sum_{j=1}^{m} \chi_j \overset{j}{u}) + 2(\mathcal{E},^{c}_{uuu} (\sum_{k=1}^{m} \alpha^1_k \overset{k}{u}))x_1 + \mathcal{E},^{c}_{uu} x_2)\delta u = \beta_2((\sum_{j=1}^{m} \chi_j \overset{j}{u}), \delta u)$$

$$\text{Use}: \ \delta u = \overset{i}{u}, \ \text{recall } x_1 \implies \boxed{\sum_{j=1}^{m} B_{ij}\chi_j = \beta_2 \chi_i, \ B_{ij} \equiv \sum_{k=1}^{m}\sum_{l=1}^{m} \alpha^1_k \alpha^1_l \mathcal{E}_{ijkl} + \lambda_2 \mathcal{E}_{ij\lambda}}$$

Branch is stable when all eigenvalues β_2 of B_{ij} are positive $(\beta_2 > 0)$

LSK ASYMPTOTICS – MUTIPLE MODE CASE

$$\mathcal{E}_{,u}(u,\lambda)\delta u = 0,\ \forall\ \delta u \in U$$

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c),\lambda_c)\overset{i}{u})\delta u = 0,\ \ (\overset{i}{u},\overset{j}{u}) = \delta_{ij};\ 1 \le i,j \le m$$

$$\mathcal{E}^{c}_{,u\lambda}\overset{i}{u} = 0,\ 1 \le i \le m;\quad \text{m}-\text{tuple bifurcation point at}:\ (u_c,\lambda_c)$$

$\text{case (i)}:\ \ \mathcal{E}_{ijk} \neq 0,\ \exists(i,j,k)$

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\alpha^1_j\alpha^1_k\mathcal{E}_{ijk} + 2\lambda_1\sum_{j=1}^{m}\alpha^1_j\mathcal{E}_{ij\lambda} = 0,\quad \sum_{i=1}^{m}(\alpha^1_i)^2 = 1$$

$$B_{ij} \equiv \sum_{k=1}^{m}\alpha^1_k\mathcal{E}_{ijk} + \lambda_1\mathcal{E}_{ij\lambda}$$

Branch unstable for different sign eigenvalues

$\text{case (ii)}:\ \ \mathcal{E}_{ijk} = 0,\ \forall(i,j,k)$

$$\lambda_1 = 0,\quad \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\alpha^1_j\alpha^1_k\alpha^1_l\mathcal{E}_{ijkl} + 3\lambda_2\sum_{j=1}^{m}\alpha^1_j\mathcal{E}_{ij\lambda} = 0,\quad \sum_{i=1}^{m}(\alpha^1_i)^2 = 1$$

$$B_{ij} \equiv \sum_{k=1}^{m}\sum_{l=1}^{m}\alpha^1_k\alpha^1_l\mathcal{E}_{ijkl} + \lambda_2\mathcal{E}_{ij\lambda}$$

Branch stable for all positive eigenvalues

$$\mathcal{E}_{ijk} \equiv ((\mathcal{E}^{c}_{,uuu}\overset{i}{u})\overset{j}{u})\overset{k}{u},\qquad (\mathcal{E}^{c}_{,uu}v_{ij} + (\mathcal{E}^{c}_{,uuu}\overset{i}{u})\overset{j}{u})\delta v = 0$$

$$\mathcal{E}_{ij\lambda} \equiv ((d\mathcal{E}_{,uu}/d\lambda)_c)\overset{i}{u})\overset{j}{u} = ((\mathcal{E}^{c}_{,uuu}(d\overset{0}{u}/d\lambda)_c + \mathcal{E}^{c}_{,uu\lambda})\overset{i}{u})\overset{j}{u}$$

$$\mathcal{E}_{ijkl} \equiv (((\mathcal{E}^{c}_{,uuuu}\overset{j}{u})\overset{k}{u})\overset{l}{u} + (\mathcal{E}^{c}_{,uuu}v_{jk})\overset{l}{u} + (\mathcal{E}^{c}_{,uuu}v_{kl})\overset{j}{u} + (\mathcal{E}^{c}_{,uuu}v_{lj})\overset{k}{u})\overset{i}{u}$$

LSK ASYMPTOTICS – MUTIPLE MODE CASE

- Due to the presence of several equilibrium solutions emerging from the critical point, imperfect structure's response can be quite complicated

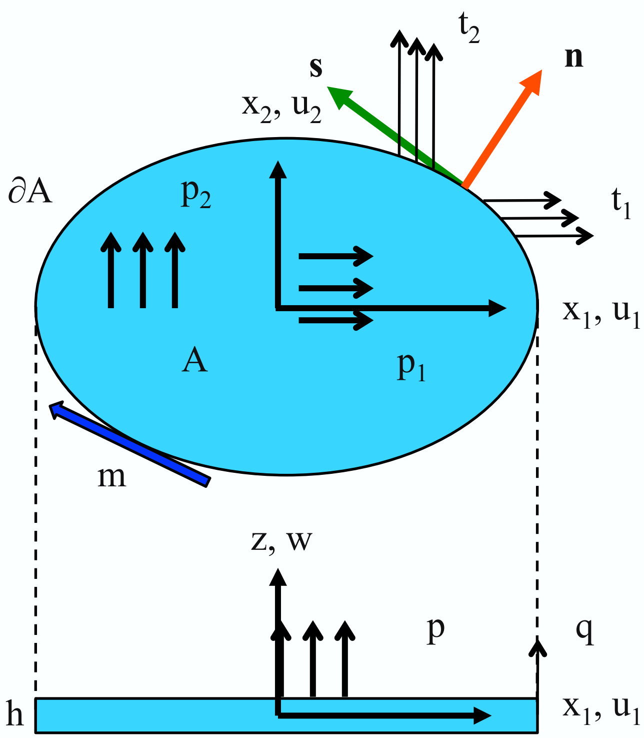
- Unlike the simple bifurcation case, it is very difficult to guess what imperfection shape to chose so as to predict the worst possible system response

- Of interest is worst imperfection shape, i.e. shape for which imperfect structure's equilibrium solution has maximum drop $\Delta\lambda_s$ for a given imperfection amplitude

- Worst imperfection shape is found to be the eigenmode corresponding to lowest $\lambda_2<0$ (for symmetric case) or highest $|\lambda_1|$ (for asymmetric case) of the perfect structure. Hence knowledge of perfect structure's bifurcation solves imperfection sensitivity problem

- NOTE: Depending on the nature of imperfection, there are situations where the imperfect structure has several, closely spaced bifurcation points of a lower order instead of a single bifurcation point of a higher order (nearly simultaneous bifurcation case). There is also the possibility that the imperfect structure has no bifurcation points near the critical load, just a highly complicated pattern of limit loads, hence the impossibility of guessing correctly the shape of the worst imperfection

VON KARMAN PLATE – ASSUMPTIONS



Assumptions:

- **Small strains**
- **Plane stress state**
- **Linearly elastic response**
- **Linear strain distribution though thickness**
- **Normals to mid-plane stay normal after deformation**
- **Moderate in-plane rotations of mid-plane**

VON KARMAN PLATE – KINEMATICS & CONSTITUTIVE LAW

$$\text{Plane stress state}: \; \sigma_{\alpha\beta}(x_1, x_2, z) = L_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}(x_1, x_2, z); \; (\text{Greek indexes}: \; 1, 2)$$

$$\text{Plane stress moduli}: \; L_{\alpha\beta\gamma\delta} = \frac{E}{1-\nu^2}\left[\frac{1-\nu}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) + \nu\delta_{\alpha\beta}\delta_{\gamma\delta}\right]$$

$$\text{Strain distribution}: \; \varepsilon_{\alpha\beta}(x_1, x_2, z) = E_{\alpha\beta}(x_1, x_2) + zK_{\alpha\beta}(x_1, x_2)$$

$$\text{Membrane strains}: \; E_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + \frac{1}{2}w_{,\alpha}\, w_{,\beta}\,; \quad (f_{,\alpha} \equiv \partial f/\partial x_\alpha)$$

$$\text{Curvature strains}: \; K_{\alpha\beta} = -w_{,\alpha\beta}$$

$$\text{Membrane resultants}: \; N_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}\, dz = hL_{\alpha\beta\gamma\delta}E_{\gamma\delta}$$

$$\text{Moment resultants}: \; M_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}\, z\, dz = \frac{h^3}{12}L_{\alpha\beta\gamma\delta}K_{\gamma\delta}$$

EXAMPLE - II – SIMPLE MODE

VON KARMAN PLATE – ENERGY

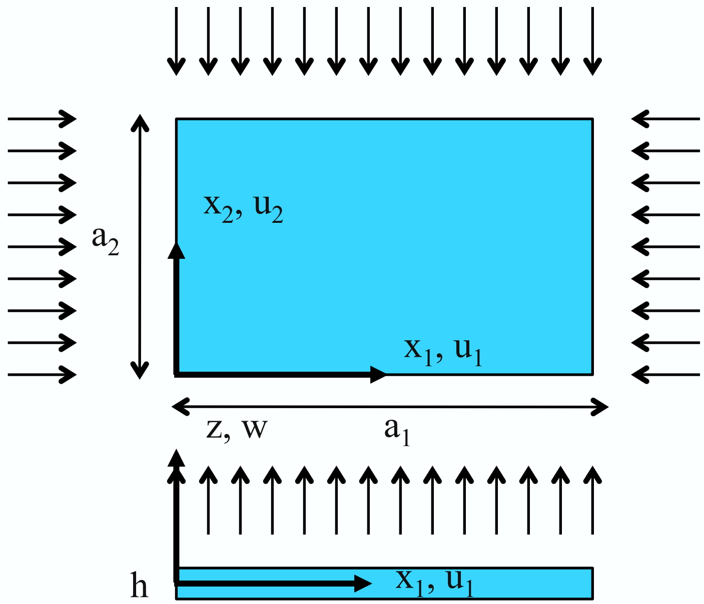
$$\text{Internal energy}: \ \mathcal{E}_{int} = \int_A [\frac{1}{2} N_{\alpha\beta} E_{\alpha\beta} + \frac{1}{2} M_{\alpha\beta} K_{\alpha\beta}) h] dA$$

$$\text{External energy}: \ \mathcal{E}_{ext} = -\int_A [p_\alpha u_\alpha + pw]\, dA - \int_{\partial A} [t_\alpha u_\alpha + qw + m(-w,_n)]\, ds$$

$$\text{Total energy}: \ \mathcal{E} = \mathcal{E}_{int} + \mathcal{E}_{ext}$$

$$\mathcal{E} = \frac{1}{2} \int_A [L_{\alpha\beta\gamma\delta} (E_{\alpha\beta} E_{\gamma\delta} + \frac{h^2}{12} K_{\alpha\beta} K_{\gamma\delta}) h] dA - \\ - \int_A [p_\alpha u_\alpha + pw]\, dA - \int_{\partial A} [t_\alpha u_\alpha + qw + m(-w,_n)]\, ds$$

PLATE BUCKLING – PROBLEM SETTING



$$\mathcal{E}_{ext} = -\int_{\partial A} [t_\alpha u_\alpha]\, ds; \ \ (p_\alpha = p = q = m = 0)$$

$$t_\alpha = n_\beta \lambda \overset{0}{\sigma}_{\alpha\beta} h; \ \ (\text{prestress}: \ \lambda \overset{0}{\sigma}_{\alpha\beta})$$

$$\mathcal{E}_{ext} = -\lambda \int_A [\overset{0}{\sigma}_{\alpha\beta} u_{\alpha,\beta}] h dA,$$

$$\mathcal{E} = \mathcal{E}_{int} + \mathcal{E}_{ext}$$

$$\mathcal{E}(u,\lambda) = \int_A \left[\frac{1}{2} L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}E_{\gamma\delta} + \frac{h^2}{12} K_{\alpha\beta}K_{\gamma\delta}) - \lambda \overset{0}{\sigma}_{\alpha\beta} u_{\alpha,\beta} \right] h dA, \quad u \equiv (u_1, u_2, w)$$

$$w(0,x_2) = w(a_1,x_2) = w(x_1,0) = w(x_1,a_2) = 0; \quad (w = 0, \ \text{simple support on } \partial A)$$

$$u_1(0,x_2) = u_{1,2}(a_1,x_2) = u_2(x_1,0) = u_{2,1}(x_1,a_2) = 0; \quad (u_{\alpha,s} = 0, \ \text{straight edges on } \partial A)$$

EXAMPLE – MULTIPLE MODE

PLATE BUCKLING – EQUILIBRIUM

Equilibrium equations :

$$\mathcal{E},_u \delta u = \int_A \left[L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}\delta E_{\gamma\delta} + \frac{h^2}{12} K_{\alpha\beta}\delta K_{\gamma\delta}) - \lambda \overset{0}{\sigma}_{\alpha\beta}\delta u_{\alpha,\beta} \right] h dA = 0$$

$$\delta E_{\alpha\beta} = \frac{1}{2}(\delta u_{\alpha,\beta} + \delta u_{\beta,\alpha}) + \frac{1}{2}(w,_\alpha \delta w,_\beta + w,_\beta \delta w,_\alpha), \quad \delta K_{\alpha\beta} = -\delta w,_{\alpha\beta}$$

Principal solution is flat plate configuration :

$$\overset{0}{N}_{\alpha\beta} = \lambda h \overset{0}{\sigma}_{\alpha\beta}, \quad (\overset{0}{\sigma}_{11}, \overset{0}{\sigma}_{22} \neq 0, \quad \overset{0}{\sigma}_{12} = \overset{0}{\sigma}_{21} = 0); \quad \overset{0}{M}_{\alpha\beta} = 0$$

$$\overset{0}{u}_1 = (\lambda x_1/E)(\overset{0}{\sigma}_{11} - \nu \overset{0}{\sigma}_{22}), \quad \overset{0}{u}_2 = (\lambda x_2/E)(\overset{0}{\sigma}_{22} - \nu \overset{0}{\sigma}_{11}); \quad \overset{0}{w} = 0$$

PLATE BUCKLING – MULTIPLE MODES

$$(\mathcal{E}^c,_{uu}\overset{i}{u})\delta u = \int_A \left[L_{\alpha\beta\gamma\delta}(\overset{i}{u}_{\alpha,\beta}\delta u_{\gamma,\delta} + \frac{h^2}{12}\overset{i}{w},_{\alpha\beta}\delta w,_{\gamma\delta}) + \lambda_c \overset{0}{\sigma}_{\alpha\beta}\overset{i}{w},_{\alpha}\delta w,_{\beta} \right] h dA = 0$$

Integration by parts gives following Euler – Lagrange system $(\overset{i}{u} \equiv (\overset{i}{u}_1, \overset{i}{u}_2, \overset{i}{w}))$

δ_{u_α} : $\quad (L_{\alpha\beta\gamma\delta}\overset{i}{u}_{\gamma,\delta}),_{\beta} = 0 \text{ in } A,$

$L_{12\gamma\delta}\overset{i}{u}_{\gamma,\delta} = 0 \text{ on } \partial A, \quad \overset{i}{u}_1(0,x_2) = \overset{i}{u}_{1,2}(a_1,x_2) = \overset{i}{u}_2(x_1,0) = \overset{i}{u}_{2,1}(x_1,a_2) = 0,$

Solution : $\overset{i}{u}_\alpha = 0$

δw : $\quad (h^2/12)L_{\alpha\beta\gamma\delta}\overset{i}{w},_{\alpha\beta\gamma\delta} - \lambda\overset{0}{\sigma}_{\alpha\beta}\overset{i}{w},_{\alpha\beta} = 0 \text{ in } A,$

$\overset{i}{w} = 0 \text{ on : } \partial A : \quad \overset{i}{w},_{11}(0,x_2) = \overset{i}{w},_{11}(a_1,x_2) = \overset{i}{w},_{22}(x_1,0) = \overset{i}{w},_{22}(x_1,a_2) = 0$

Solution : $\overset{i}{w} = h\sin(m_i\pi x_1/a_1)\sin(n_i\pi x_2/a_2)$

EXAMPLE – MULTIPLE MODE

PLATE BUCKLING – MULTIPLE MODES

$$\mathcal{E}^c{}_{,u\lambda}\overset{i}{u} = -\int_A [\overset{0}{\sigma}_{\alpha\beta}\overset{i}{u}_{\alpha,\beta}] h dA = 0 \implies \text{Bifurcation}$$

$$\lambda_c = \frac{Eh^2}{12(1-\nu^2)} \min_{m_i, n_i \in \mathbb{N}} \left\{ -\frac{\left[(m_i\pi/a_1)^2 + (n_i\pi/a_2)^2\right]^2}{\overset{0}{\sigma}_{11}(m_i\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n_i\pi/a_2)^2} \right\} : \quad \text{Critical load}$$

$$\overset{0}{\sigma}_{11}(m_i\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n_i\pi/a_2)^2 < 0 \quad \text{at least one stress component compressive}$$

$$\text{Consider}: \; a_1 = \sqrt{2}a, \; a_2 = a; \quad \overset{0}{\sigma}_{11} = -1, \; \overset{0}{\sigma}_{22} = 0 \implies \text{Double mode}$$

$$(m_1, \; n_1) = (1, \; 1), \quad (m_2, \; n_2) = (2, \; 1) \text{ at } \lambda_c = \frac{3\pi^2}{8}\frac{E}{1-\nu^2}\left(\frac{h}{a}\right)^2$$

$$\overset{1}{u} = (0, \; 0, \; h\sin(\pi_1/a_1)\sin(\pi_2/a_2)), \; \overset{2}{u} = (0, \; 0, \; h\sin(2\pi_1/a_1)\sin(\pi_2/a_2)); \; (\text{to constant}).$$

EXAMPLE – MULTIPLE MODE

PLATE BUCKLING – MULTIPLE BIFURCATION PATHS

Bifurcation is symmetric (recall $\overset{i}{u} = 0,\ i = 1, 2$) since :

$$\mathcal{E}_{ijk} \equiv ((\mathcal{E}^c,_{uuu} \overset{i}{u})\overset{j}{u})\overset{k}{u} = \int_A [L_{\alpha\beta\gamma\delta}(\overset{i}{w},_\alpha \overset{j}{w},_\beta \overset{k}{u}_{\gamma,\delta} + \overset{j}{w},_\alpha \overset{k}{w},_\beta \overset{i}{u}_{\gamma,\delta} + \overset{k}{w},_\alpha \overset{i}{w},_\beta \overset{j}{u}_{\gamma,\delta})]hdA = 0$$

Need to calculate higher coefficients :

$$\mathcal{E}_{ijkl} \equiv (((\mathcal{E},^c_{uuuu})) + (\mathcal{E},^c_{uuu} v_{jk}) + (\mathcal{E},^c_{uuu} v_{kl}) + (\mathcal{E},^c_{uuu} v_{lj}))$$

which in turn require finding v_{ij}, given by :

$$[(\mathcal{E}^c,_{uuu} \overset{i}{u})\overset{j}{u} + \mathcal{E},^c_{uu} v_{ij})\delta v = \int_A [L_{\alpha\beta\gamma\delta}(\overset{i}{w},_\gamma \overset{j}{w},_\delta \delta u_{\alpha,\beta} + \overset{i}{u}_{\gamma,\delta}\overset{j}{w},_\alpha \delta w,_\beta + \overset{j}{u}_{\gamma,\delta}\overset{i}{w},_\alpha \delta w,_\beta)]hdA$$

$$+ \int_A [L_{\alpha\beta\gamma\delta}(\overset{ij}{u}_{\gamma,\delta}\delta u_{\alpha,\beta} + \frac{h^2}{12}\overset{ij}{w},_{\gamma\delta} \delta w,_{\alpha\beta}) + \lambda_c \overset{0}{\sigma}_{\alpha\beta}\overset{ij}{w},_\alpha \delta w,_\beta]hdA = 0$$

PLATE BUCKLING – MULTIPLE BIFURCATION PATHS

To solve $[(\mathcal{E}^c,_{uuu} \overset{i}{u})\overset{j}{u} + \mathcal{E},^c_{uu} v_{ij})\delta v = 0,$

Integration by parts gives following Euler – Lagrange system for $v_{ij} \equiv (\overset{ij}{u}_1, \overset{ij}{u}_2, \overset{ij}{w})$:

$$\delta u_\alpha : \quad L_{\alpha\beta\gamma\delta}(\overset{ij}{u}_{\gamma,\delta} + \overset{i}{w},_\gamma \overset{j}{w},_\delta),_\beta = 0 \text{ in } A,$$

$$L_{12\gamma\delta}(\overset{ij}{u}_{\gamma,\delta} + \overset{i}{w},_\gamma \overset{j}{w},_\delta) = 0 \text{ on } \partial A,$$

$$\overset{ij}{u}_1(0, x_2) = \overset{ij}{u}_{1,2}(a_1, x_2) = \overset{ij}{u}_2(x_1, 0) = \overset{ij}{u}_{2,1}(x_1, a_2) = 0$$

$$\delta w : \quad \frac{h^2}{12} L_{\alpha\beta\gamma\delta} \overset{ij}{w},_{\alpha\beta\gamma\delta} - \lambda_c \overset{0}{\sigma}_{\alpha\beta} \overset{ij}{w},_{\alpha\beta} = 0 \text{ in } A,$$

$$\overset{ij}{w} = 0 \text{ on } \partial A,$$

$$\overset{ij}{w},_{11} (0, x_2) = \overset{ij}{w},_{11} (a_1, x_2) = \overset{ij}{w},_{22} (x_1, 0) = \overset{ij}{w},_{22} (x_1, a_2) = 0.$$

PLATE BUCKLING – MULTIPLE BIFURCATION PATHS

Uniqueness of eigenmode $\Longrightarrow$ $\boxed{\overset{ij}{w} = 0}$

Introduce Airy functions $\overset{ij}{f}$: $\overset{ij}{s}_{11} \equiv \overset{ij}{f}_{,22}\,, \quad \overset{ij}{s}_{22} \equiv \overset{ij}{f}_{,11}\,, \quad \overset{ij}{s}_{12} = \overset{ij}{s}_{21} \equiv -\overset{ij}{f}_{,12}\,; \quad (\overset{ij}{s}_{\alpha\beta,\beta} = 0)$

$$\overset{ij}{s}_{\alpha\beta} \equiv L_{\alpha\beta\gamma\delta}(\overset{ij}{u}_{\gamma,\delta} + \overset{i}{w}_{,\gamma}\,\overset{j}{w}_{,\delta}) = L_{\alpha\beta\gamma\delta}\overset{ij}{e}_{\gamma\delta}, \quad \overset{ij}{e}_{\alpha\beta} \equiv \frac{1}{2}(\overset{ij}{u}_{\alpha,\beta} + \overset{ij}{u}_{\beta,\alpha}) + \frac{1}{2}(\overset{i}{w}_{,\alpha}\,\overset{j}{w}_{,\beta} + \overset{j}{w}_{,\alpha}\,\overset{i}{w}_{,\beta})$$

expressing $\overset{ij}{e}_{ij}$ i.t.o. $\overset{ij}{s}_{ij}$: $\overset{ij}{e}_{11} = \frac{1}{E}(\overset{ij}{s}_{11} - \nu\overset{ij}{s}_{22}), \ \overset{ij}{e}_{22} = \frac{1}{E}(\overset{ij}{s}_{22} - \nu\overset{ij}{s}_{11}), \ \overset{ij}{e}_{12} = \overset{ij}{e}_{21} = \frac{1+\nu}{E}\overset{ij}{s}_{12}$

Compatibility : $\overset{ij}{e}_{11,22} + \overset{ij}{e}_{22,11} - 2\overset{ij}{e}_{12,12} = 2\overset{i}{w}_{,12}\,\overset{j}{w}_{,12} - \overset{i}{w}_{,11}\,\overset{j}{w}_{22} - \overset{j}{w}_{,11}\,\overset{i}{w}_{,22} \Longrightarrow$

$$\boxed{\frac{1}{E}\nabla^4(\overset{ij}{f}) = 2\overset{i}{w}_{,12}\,\overset{j}{w}_{,12} - \overset{i}{w}_{,11}\,\overset{j}{w}_{22} - \overset{j}{w}_{,11}\,\overset{i}{w}_{,22}}$$

PLATE BUCKLING – MULTIPLE BIFURCATION PATHS

Solution for the biharmonic operator equations gives : $\overset{ij}{f}$

$$\overset{11}{f} = \frac{Eh^2}{16}\left[\left(\frac{a_1}{a_2}\right)^2 \cos(2\pi x_1/a_1) + \left(\frac{a_2}{a_1}\right)^2 \cos(2\pi x_2/a_2)\right]$$

$$\overset{22}{f} = \frac{Eh^2}{16}\left[\left(\frac{a_1}{2a_2}\right)^2 \cos(4\pi x_1/a_1) + \left(\frac{2a_2}{a_1}\right)^2 \cos(2\pi x_2/a_2)\right]$$

$$\overset{12}{f} = \frac{Eh^2}{4}\left\{\frac{\cos(3\pi x_1/a_1)}{(3a_2/a_1)^2} - \frac{\cos(\pi x_1/a_1)}{(a_2/a_1)^2} + \right.$$
$$\left. + \left[\frac{9\cos(\pi x_1/a_1)}{[(a_2/a_1)+(4a_1/a_2)]^2} - \frac{\cos(3\pi x_1/a_1)}{[(9a_2/a_1)+(4a_1/a_2)]^2}\right]\cos(2\pi x_2/a_2)\right\}$$

EXAMPLE – MULTIPLE MODE

PLATE BUCKLING – MULTIPLE BIFURCATION PATHS

$$\mathcal{E}_{ijkl} = \int_A [L_{\alpha\beta\gamma\delta}[\overset{i}{w}_{,\alpha} \overset{j}{w}_{,\beta} (\overset{kl}{u}_{\gamma,\delta} + \overset{k}{w}_{,\gamma} \overset{l}{w}_{,\delta}) + \\ + \overset{i}{w}_{,\alpha} \overset{l}{w}_{,\beta} (\overset{jk}{u}_{\gamma,\delta} + \overset{j}{w}_{,\gamma} \overset{k}{w}_{,\delta}) + \overset{i}{w}_{,\alpha} \overset{k}{w}_{,\beta} (\overset{jl}{u}_{\gamma,\delta} + \overset{j}{w}_{,\gamma} \overset{l}{w}_{,\delta})] h dA$$

$$\mathcal{E}_{ijkl} = \int_A [\overset{kl}{s}_{\alpha\beta} \overset{i}{w}_{,\alpha} \overset{j}{w}_{,\beta} + \overset{kj}{s}_{\alpha\beta} \overset{i}{w}_{,\alpha} \overset{l}{w}_{,\beta} + \overset{jl}{s}_{\alpha\beta} \overset{i}{w}_{,\alpha} \overset{k}{w}_{,\beta}] h dA$$

Notice : $\mathcal{E}_{ijkl}$ are completely symmetric with respect to all indexes

Where : $\mathcal{E}_{1112} = \mathcal{E}_{1222} = 0, \ \mathcal{E}_{1111} = \alpha, \ \mathcal{E}_{2222} = 4\alpha, \ \mathcal{E}_{1122} = \frac{4}{3}\alpha(1+\delta); \quad \alpha, \delta > 0$

$$\mathcal{E}_{ij\lambda} = \int_A [\overset{0}{\sigma}_{\alpha\beta} \overset{i}{w}_{,\alpha} \overset{j}{w}_{,\beta}] h dA, \ \text{With :} \ \mathcal{E}_{11\lambda} = -\gamma, \ \mathcal{E}_{22\lambda} = -4\gamma, \ \mathcal{E}_{12\lambda} = \mathcal{E}_{21\lambda} = 0; \ \gamma > 0$$

Here : $\alpha = (15\sqrt{2}/128) E h^5 \pi^4 / a^2, \quad \gamma = (\sqrt{2}/8) h^3 \pi^2, \quad \delta = 226/2601$

EXAMPLE – MULTIPLE MODE

PLATE BUCKLING – MULTIPLE BIFURCATION PATHS

Symmetric bifurcation : $\mathcal{E}_{ijk} = 0,\ \forall(i,j,k)$; initial tangents α_i^1 from :

$$\mathcal{E}_{1111}(\alpha_1^1)^3 + 3\mathcal{E}_{1122}(\alpha_1^1)(\alpha_2^1)^2 + 3\lambda_2\mathcal{E}_{11\lambda}(\alpha_1^1) = 0$$

$$3\mathcal{E}_{2211}(\alpha_1^1)^2(\alpha_2^1) + \mathcal{E}_{2222}(\alpha_2^1)^3 + 3\lambda_2\mathcal{E}_{22\lambda}(\alpha_2^1) = 0$$

Four different equilibrium paths found :

$$P1: \ \alpha_1^1 = 1, \quad \alpha_2^1 = 0; \quad \lambda_2 = \alpha/3\gamma$$

$$P2: \ \alpha_1^1 = 0, \quad \alpha_2^1 = 1; \quad \lambda_2 = \alpha/3\gamma$$

$$P3: \ \alpha_2^1/\alpha_1^1 = [\delta/(3+4\delta)]^{1/2}; \qquad \lambda_2 = (\alpha/3\gamma)[1 + \delta(3+4\delta)/(3+5\delta)]$$

$$P4: \ \alpha_2^1/\alpha_1^1 = -[\delta/(3+4\delta)]^{1/2}; \quad \lambda_2 = (\alpha/3\gamma)[1 + \delta(3+4\delta)/(3+5\delta)]$$

PLATE BUCKLING – STABILITY OF BIFURCATION PATHS

Stability depends on positive definiteness of : $B_{ij} = \sum_{k=1}^{m}\sum_{l=1}^{m} \alpha_k^1 \alpha_l^1 \mathcal{E}_{ijkl} + \lambda_2 \mathcal{E}_{ij\lambda}$

$$P1: \quad B_{ij} = \begin{bmatrix} \mathcal{E}_{1111} + \lambda_2 \mathcal{E}_{11\lambda} & 0 \\ 0 & \mathcal{E}_{1122} + \lambda_2 \mathcal{E}_{22\lambda} \end{bmatrix} = \frac{2\alpha}{3} \begin{bmatrix} 1 & 0 \\ 0 & 2\delta \end{bmatrix} \Longrightarrow \text{Stable}$$

$$P2: \quad B_{ij} = \begin{bmatrix} \mathcal{E}_{1122} + \lambda_2 \mathcal{E}_{11\lambda} & 0 \\ 0 & \mathcal{E}_{2222} + \lambda_2 \mathcal{E}_{22\lambda} \end{bmatrix} = \frac{\alpha}{3} \begin{bmatrix} 3+4\delta & 0 \\ 0 & 8 \end{bmatrix} \Longrightarrow \text{Stable}$$

$$P3\&4: B_{ij} = \begin{bmatrix} \mathcal{E}_{1111}(\alpha_1^1)^2 + \mathcal{E}_{1122}(\alpha_2^1)^2 + \lambda_2 \mathcal{E}_{11\lambda} & 2\mathcal{E}_{1122}\alpha_1^1\alpha_2^1 \\ 2\mathcal{E}_{1122}\alpha_1^1\alpha_2^1 & \mathcal{E}_{2211}(\alpha_1^1)^2 + \mathcal{E}_{2222}(\alpha_2^1)^2 + \lambda_2 \mathcal{E}_{22\lambda} \end{bmatrix} =$$

$$= \frac{2\alpha}{3(3+5\delta)} \begin{bmatrix} 3+4\delta & \pm 4(1+\delta)\sqrt{\delta(3+4\delta)} \\ \pm 4(1+\delta)\sqrt{\delta(3+4\delta)} & 4\delta \end{bmatrix} \Longrightarrow \text{Unstable}$$

PLATE BUCKLING – NEAR SIMULTANEOUS BIFURCATION

$$\lambda_{ci} = \frac{Eh^2}{12(1-\nu^2)} \min_{m_i, n_i \in \mathbb{N}} \left\{ - \frac{\left[(m_i\pi/a_1)^2 + (n_i\pi/a_2)^2\right]^2}{\overset{0}{\sigma}_{11}(m_i\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n_i\pi/a_2)^2} \right\} : \quad \text{Critical loads}$$

$$\overset{0}{\sigma}_{11}(m_i\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n_i\pi/a_2)^2 < 0 \quad \text{at least one stress component compressive}$$

$$\text{Take}: \ a_1 = \sqrt{2}a, \ a_2 = a(1-\varepsilon); \quad \overset{0}{\sigma}_{11} = -1, \ \overset{0}{\sigma}_{22} = 0 \implies \text{Nearly simultaneous modes}$$

$$m_1 = 1, \ n_1 = 1, \ \lambda_{c1} = \lambda_c(1 + \frac{8}{3}\varepsilon + O(\varepsilon^2)); \ m_2 = 2, \ n_2 = 1, \ \lambda_{c2} = \lambda_c(1 + \frac{4}{3}\varepsilon + O(\varepsilon^2)),$$

$$\lambda_c \equiv \frac{3\pi^2}{8}\frac{E}{1-\nu^2}\left(\frac{h}{a}\right)^2, \ \text{critical load for simultaneous modes}: \ (a_1 = \sqrt{2}a, \ a_2 = a)$$

$$\overset{1}{u} = (0, \ 0, \ h\sin(\pi_1/a_1)\sin(\pi_2/a_2)), \ \overset{2}{u} = (0, \ 0, \ h\sin(2\pi_1/a_1)\sin(\pi_2/a_2)); \ (\text{to constant})$$

NOTE : eigenmodes $\overset{1}{u}$, $\overset{2}{u}$ are simple bifurcation modes.