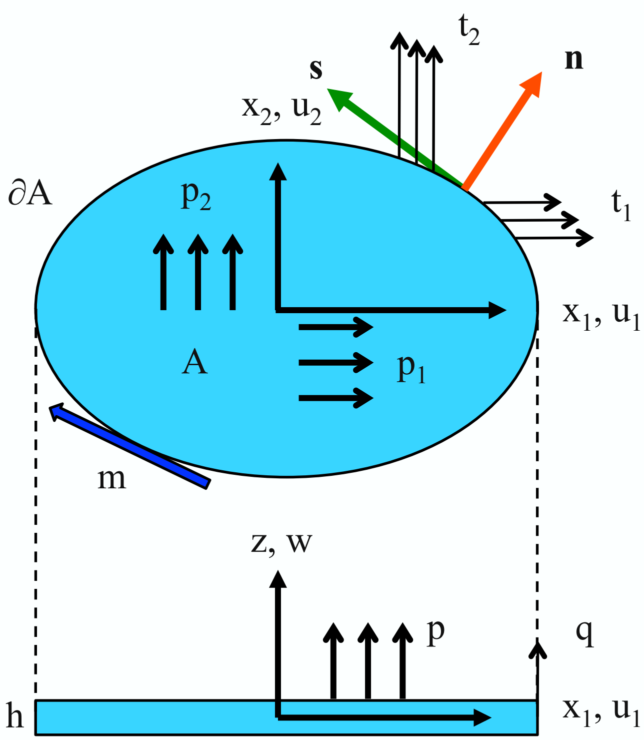
VON KARMAN PLATE – ASSUMPTIONS



Assumptions:

- Small strains
- Plane stress state
- Linearly elastic response
- Linear strain distribution though thickness
- Normals to mid-plane stay normal after deformation
- Moderate in-plane rotations of mid-plane

VON KARMAN PLATE – KINEMATICS & CONSTITUTIVE LAW

$$\text{Plane stress state}: \ \sigma_{\alpha\beta}(x_1, x_2, z) = L_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}(x_1, x_2, z); \ (\text{Greek indexes}: \ 1, 2)$$

$$\text{Plane stress moduli}: \ L_{\alpha\beta\gamma\delta} = \frac{E}{1-\nu^2}\left[\frac{1-\nu}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) + \nu\delta_{\alpha\beta}\delta_{\gamma\delta}\right]$$

$$\text{Strain distribution}: \ \varepsilon_{\alpha\beta}(x_1, x_2, z) = E_{\alpha\beta}(x_1, x_2) + zK_{\alpha\beta}(x_1, x_2)$$

$$\text{Membrane strains}: \ E_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + \frac{1}{2}w_{,\alpha}\, w_{,\beta}\,; \quad (f_{,\alpha} \equiv \partial f/\partial x_\alpha)$$

$$\text{Curvature strains}: \ K_{\alpha\beta} = -w_{,\alpha\beta}$$

$$\text{Membrane resultants}: \ N_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}\, dz = hL_{\alpha\beta\gamma\delta}E_{\gamma\delta}$$

$$\text{Moment resultants}: \ M_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}\, z\, dz = \frac{h^3}{12}L_{\alpha\beta\gamma\delta}K_{\gamma\delta}$$

VON KARMAN PLATE – ENERGY

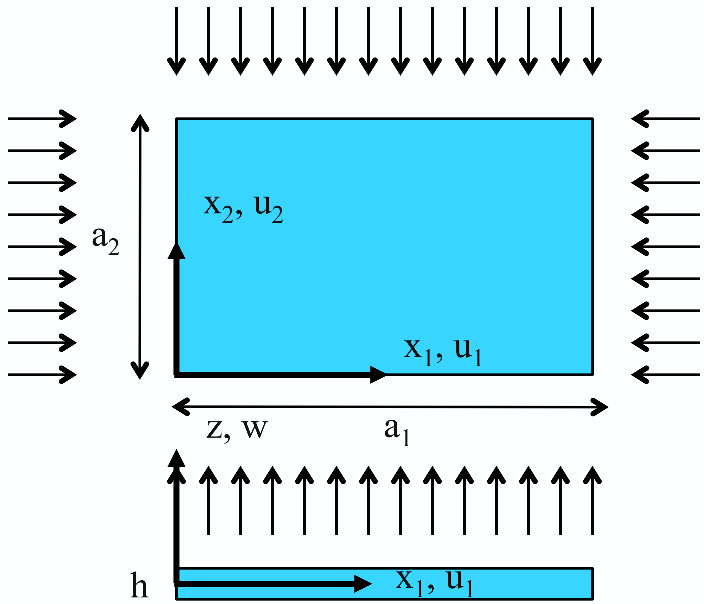
$$\text{Internal energy}: \ \mathcal{E}_{int} = \int_A [\frac{1}{2} N_{\alpha\beta} E_{\alpha\beta} + \frac{1}{2} M_{\alpha\beta} K_{\alpha\beta}) h] dA$$

$$\text{External energy}: \ \mathcal{E}_{ext} = -\int_A [p_\alpha u_\alpha + pw]\, dA - \int_{\partial A} [t_\alpha u_\alpha + qw + m(-w,_n)]\, ds$$

$$\text{Total energy}: \ \mathcal{E} = \mathcal{E}_{int} + \mathcal{E}_{ext}$$

$$\mathcal{E} = \frac{1}{2} \int_A [L_{\alpha\beta\gamma\delta}(E_{\alpha\beta} E_{\gamma\delta} + \frac{h^2}{12} K_{\alpha\beta} K_{\gamma\delta}) h] dA - $$
$$- \int_A [p_\alpha u_\alpha + pw]\, dA - \int_{\partial A} [t_\alpha u_\alpha + qw + m(-w,_n)]\, ds$$

PLATE BUCKLING EXAMPLE – PROBLEM SETTING



$$\mathcal{E}_{ext} = -\int_{\partial A} [t_\alpha u_\alpha]\, ds; \;\; (p_\alpha = p = q = m = 0)$$

$$t_\alpha = n_\beta \lambda \overset{0}{\sigma}_{\alpha\beta} h; \;\; (\text{prestress}: \;\; \lambda \overset{0}{\sigma}_{\alpha\beta})$$

$$\mathcal{E}_{ext} = -\lambda \int_A [\overset{0}{\sigma}_{\alpha\beta} u_{\alpha,\beta}] h dA,$$

$$\mathcal{E} = \mathcal{E}_{int} + \mathcal{E}_{ext}$$

$$\mathcal{E}(u,\lambda) = \int_A \left[\frac{1}{2} L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}E_{\gamma\delta} + \frac{h^2}{12} K_{\alpha\beta}K_{\gamma\delta}) - \lambda \overset{0}{\sigma}_{\alpha\beta} u_{\alpha,\beta} \right] h dA, \quad u \equiv (u_1, u_2, w)$$

$$w(0,x_2) = w(a_1,x_2) = w(x_1,0) = w(x_1,a_2) = 0; \quad (w = 0, \text{ simple support on } \partial A)$$

$$u_1(0,x_2) = u_{1,2}(a_1,x_2) = u_2(x_1,0) = u_{2,1}(x_1,a_2) = 0; \quad (u_{\alpha,s} = 0, \text{ straight edges on } \partial A)$$

PLATE BUCKLING EXAMPLE – EQUILIBRIUM

Equilibrium equations :

$$\mathcal{E},_u \delta u = \int_A \left[L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}\delta E_{\gamma\delta} + \frac{h^2}{12} K_{\alpha\beta}\delta K_{\gamma\delta}) - \lambda \overset{0}{\sigma}_{\alpha\beta}\delta u_{\alpha,\beta} \right] h dA = 0$$

$$\delta E_{\alpha\beta} = \frac{1}{2}(\delta u_{\alpha,\beta} + \delta u_{\beta,\alpha}) + \frac{1}{2}(w,_\alpha \delta w,_\beta + w,_\beta \delta w,_\alpha), \quad \delta K_{\alpha\beta} = -\delta w,_{\alpha\beta}$$

Principal solution is flat plate configuration :

$$\overset{0}{N}_{\alpha\beta} = \lambda h \overset{0}{\sigma}_{\alpha\beta}, \quad (\overset{0}{\sigma}_{11}, \overset{0}{\sigma}_{22} \neq 0, \quad \overset{0}{\sigma}_{12} = \overset{0}{\sigma}_{21} = 0); \quad \overset{0}{M}_{\alpha\beta} = 0$$

$$\overset{0}{u}_1 = (\lambda x_1/E)(\overset{0}{\sigma}_{11} - \nu\overset{0}{\sigma}_{22}), \quad \overset{0}{u}_2 = (\lambda x_2/E)(\overset{0}{\sigma}_{22} - \nu\overset{0}{\sigma}_{11}); \quad \overset{0}{w} = 0$$

PLATE BUCKLING EXAMPLE – BIFURCATION

$$(\mathcal{E}^c,_{uu}\overset{1}{u})\delta u = \int_A \left[L_{\alpha\beta\gamma\delta}(\overset{1}{u}_{\alpha,\beta}\delta u_{\gamma,\delta} + \frac{h^2}{12}\overset{1}{w},_{\alpha\beta}\delta w,_{\gamma\delta}) + \lambda\overset{0}{\sigma}_{\alpha\beta}\overset{1}{w},_{\alpha}\delta w,_{\beta} \right] hdA = 0$$

Integration by parts gives following Euler – Lagrange system $(\overset{1}{u} \equiv (\overset{1}{u}_1, \overset{1}{u}_2, \overset{1}{w}))$

δ_{u_α} : $(L_{\alpha\beta\gamma\delta}\overset{1}{u}_{\gamma,\delta}),_{\beta} = 0 \text{ in } A,$

$$L_{12\gamma\delta}\overset{1}{u}_{\gamma,\delta} = 0 \text{ on } \partial A, \quad \overset{1}{u}_1(0,x_2) = \overset{1}{u}_{1,2}(a_1,x_2) = \overset{1}{u}_2(x_1,0) = \overset{1}{u}_{2,1}(x_1,a_2) = 0,$$

Solution : $\overset{1}{u}_\alpha = 0$

δw : $(h^2/12)L_{\alpha\beta\gamma\delta}\overset{1}{w},_{\alpha\beta\gamma\delta} - \lambda\overset{0}{\sigma}_{\alpha\beta}\overset{1}{w},_{\alpha\beta} = 0 \text{ in } A,$

$$\overset{1}{w} = 0 \text{ on : } \partial A : \quad \overset{1}{w},_{11}(0,x_2) = \overset{1}{w},_{11}(a_1,x_2) = \overset{1}{w},_{22}(x_1,0) = \overset{1}{w},_{22}(x_1,a_2) = 0$$

Solution : $\overset{1}{w} = h\sin(m\pi x_1/a_1)\sin(n\pi x_2/a_2)$

EXAMPLE - II – SIMPLE MODE

PLATE BUCKLING EXAMPLE – BIFURCATION

$$\mathcal{E}^c{,_{u\lambda}} \overset{1}{u} = -\int_A [\overset{0}{\sigma}_{\alpha\beta} \overset{1}{u}_{\alpha,\beta}] h dA = 0 \implies \text{Bifurcation}$$

$$\lambda_c = \frac{Eh^2}{12(1-\nu^2)} \min_{m,n\in\mathbb{N}} \left\{ -\frac{\left[(m\pi/a_1)^2 + (n\pi/a_2)^2\right]^2}{\overset{0}{\sigma}_{11}(m\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n\pi/a_2)^2} \right\} : \quad \text{Critical load}$$

$$\overset{0}{\sigma}_{11}(m\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n\pi/a_2)^2 < 0 \quad \text{at least one stress component compressive}$$

$(m_c,\ n_c)$ at λ_c : depend on geometry – not unique pair necessarily

$(m_c,\ n_c)$ at λ_c : unique pair $\implies$ simple bifurcation

PLATE BUCKLING EXAMPLE – PRINCIPAL PATH STABILITY

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u = \int_A \left[L_{\alpha\beta\gamma\delta}(\delta u_{\alpha,\beta}\delta u_{\gamma,\delta} + \frac{h^2}{12}\delta w_{,\alpha\beta}\delta w_{,\gamma\delta}) + \lambda \overset{0}{\sigma}_{\alpha\beta}\delta w_{,\alpha}\,\delta w_{,\beta} \right] h dA$$

$$\delta w = h \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \delta w_{mn} \sin(m\pi x_1/a_1)\sin(n\pi x_2/a_2) : \text{ Fourier representation of arbitrary } \delta w$$

$$(\mathcal{E}_{,uu}^{0})\delta u)\delta u = \int_A \left[\frac{h^2}{12} L_{\alpha\beta\gamma\delta}\delta w_{,\alpha\beta}\delta w_{,\gamma\delta} + \lambda \overset{0}{\sigma}_{\alpha\beta}\delta w_{,\alpha}\,\delta w_{,\beta} \right] h dA = \frac{a_1 a_2 h^3}{4}\frac{Eh^2}{12(1-\nu^2)} \times$$

$$\times \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (\delta w_{mn})^2 \left[\left\{ \frac{\left[(m\pi/a_1)^2 + (n\pi/a_2)^2\right]^2}{-\overset{0}{\sigma}_{11}(m\pi/a_1)^2 - \overset{0}{\sigma}_{22}(n\pi/a_2)^2} \right\} - \lambda \right] \left[-\overset{0}{\sigma}_{11}\left(\frac{m\pi}{a_1}\right)^2 - \overset{0}{\sigma}_{22}\left(\frac{n\pi}{a_2}\right)^2 \right]$$

$\lambda \in [0, \lambda_c) \implies (\mathcal{E}_{,uu}^{0})\delta u)\delta u > 0 \implies$ principal path is stable for loads below λ_c

PLATE BUCKLING EXAMPLE – POSTBUCKLING

$$((\mathcal{E}^c{}_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = \int_A 3[L_{\alpha\beta\gamma\delta}(\overset{1}{w}_{,\alpha}\overset{1}{w}_{,\beta}\overset{1}{u}_{\gamma,\delta})]hdA = 0 \implies \text{symmetric bifurcation}$$

$$((\mathcal{E}^c{}_{,uuu}\overset{1}{u})\overset{1}{u} + \mathcal{E}^c{}_{,uu}v_{\xi\xi})\ \delta v = \int_A [L_{\alpha\beta\gamma\delta}(\overset{1}{w}_{,\gamma}\overset{1}{w}_{,\delta}\delta u_{\alpha,\beta} + 2\overset{1}{u}_{\gamma,\delta}\overset{1}{w}_{,\alpha}\delta w_{,\beta})]hdA$$

$$+ \int_A L_{\alpha\beta\gamma\delta}(\overset{2}{u}_{\gamma,\delta}\delta u_{\alpha,\beta} + \frac{h^2}{12}\overset{2}{w}_{,\gamma\delta}\delta w_{,\alpha\beta}) + \lambda_c\overset{0}{\sigma}_{\alpha\beta}\overset{2}{w}_{,\alpha}\delta w_{,\beta}]hdA = 0$$

Integration by parts gives following Euler – Lagrange system $(v_{\xi\xi} \equiv (\overset{2}{u}_1, \overset{2}{u}_2, \overset{2}{w}))$

$$\delta u_\alpha : \quad (L_{\alpha\beta\gamma\delta}(\overset{2}{u}_{\gamma,\delta} + \overset{1}{w}_{,\gamma}\overset{1}{w}_{,\delta}))_{,\beta} = 0 \text{ in } A;$$

$$L_{12\gamma\delta}(\overset{2}{u}_{\gamma,\delta} + \overset{1}{w}_{,\gamma}\overset{1}{w}_{,\delta}) = 0 \text{ on } \partial A,\ \overset{2}{u}_1(0,x_2) = \overset{2}{u}_{1,2}(a_1,x_2) = \overset{2}{u}_2(x_1,0) = \overset{2}{u}_{2,1}(x_1,a_2) = 0$$

$$\delta w : \quad (h^2/12)L_{\alpha\beta\gamma\delta}\overset{2}{w}_{,\alpha\beta\gamma\delta} - \lambda_c\overset{0}{\sigma}_{\alpha\beta}\overset{2}{w}_{,\alpha\beta} = 0 \text{ in } A;$$

$$\overset{2}{w} = 0 \text{ on } \partial A, \quad \overset{2}{w}_{,11}(0,x_2) = \overset{2}{w}_{,11}(a_1,x_2) = \overset{2}{w}_{,22}(x_1,0) = \overset{2}{w}_{,22}(x_1,a_2) = 0$$

PLATE BUCKLING EXAMPLE – POSTBUCKLING

Uniqueness of eigenmode $\Longrightarrow \overset{2}{w} = 0$

Introduce Airy function f : $s_{11} \equiv f_{,22}$, $s_{22} \equiv f_{,11}$, $s_{12} = s_{21} \equiv -f_{,12}$; $\quad (s_{\alpha\beta,\beta} = 0)$

$$s_{\alpha\beta} \equiv L_{\alpha\beta\gamma\delta}(\overset{2}{u}_{\gamma,\delta} + \overset{1}{w}_{,\gamma}\overset{1}{w}_{,\delta}) = L_{\alpha\beta\gamma\delta}e_{\gamma\delta}, \quad e_{\alpha\beta} \equiv \frac{1}{2}(\overset{2}{u}_{\alpha,\beta} + \overset{2}{u}_{\beta,\alpha}) + \overset{1}{w}_{,\alpha}\overset{1}{w}_{,\beta}$$

Notice : $e_{11} = \dfrac{1}{E}(s_{11} - \nu s_{22}), \quad e_{22} = \dfrac{1}{E}(s_{22} - \nu s_{11}), \quad e_{12} = e_{21} = \dfrac{1+\nu}{E} s_{12}$

Compatibility : $e_{11,22} + e_{22,11} - 2e_{12,12} = 2[(\overset{1}{w}_{,12})^2 - \overset{1}{w}_{,11}\overset{1}{w}_{,22}] \Longrightarrow$

$$\frac{1}{E}\nabla^4 f = h^2(m_c\pi/a_1)^2(n_c\pi/a_2)^2[\cos(2m_c\pi x_1/a_1) + \cos(2n_c\pi x_2/a_2)]$$

Solution : $f = \dfrac{Eh^2}{16}\left[\dfrac{(n_c/a_2)^2}{(m_c/a_1)^2}\cos(2m_c\pi x_1/a_1) + \dfrac{(m_c/a_1)^2}{(n_c/a_2)^2}\cos(2n_c\pi x_2/a_2)\right]$

EXAMPLE - II – SIMPLE MODE

PLATE BUCKLING EXAMPLE – POSTBUCKLING

$$\text{from general LSK theory recall}: \lambda_2 = -\frac{1}{3}\frac{(((\mathcal{E}^c{}_{,uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E}^c{}_{,uuu}\, v_{\xi\xi})\overset{1}{u})\overset{1}{u}}{((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}}$$

$$(((\mathcal{E}^c{}_{,uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E}^c{}_{,uuu}\, v_{\xi\xi})\overset{1}{u})\overset{1}{u} = 3\int_A [L_{\alpha\beta\gamma\delta}(\overset{2}{u}_{\gamma,\delta} + \overset{1}{w}_{,\gamma}\overset{1}{w}_{,\delta})\overset{1}{w}_{,\alpha}\overset{1}{w}_{,\beta}]h dA =$$

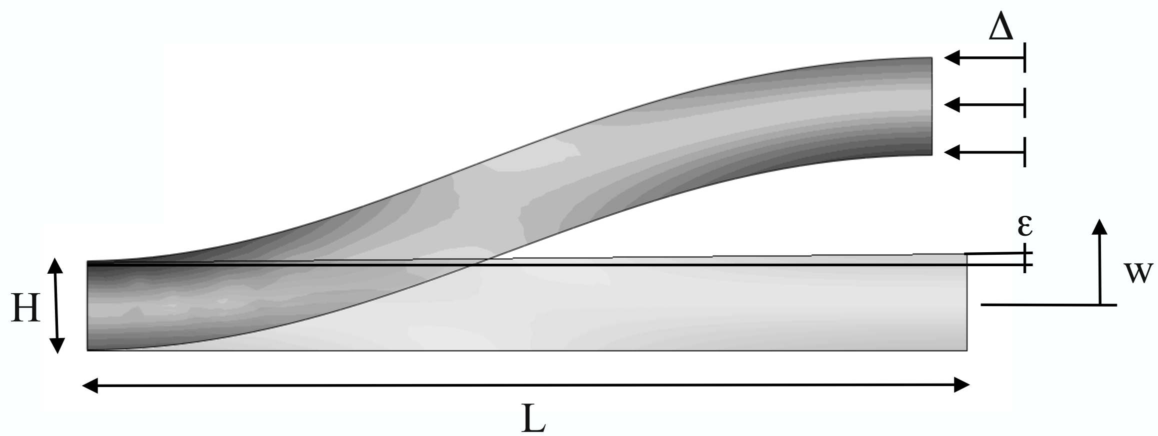
$$= 3\int_A [s_{\alpha\beta}\overset{1}{w}_{,\alpha}\overset{1}{w}_{,\beta}]h dA = \frac{3}{32}Eh^5 a_1 a_2[(m_c\pi/a_1)^4 + (n_c\pi/a_2)^4]$$

$$((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = \int_A [\overset{0}{\sigma}_{\alpha\beta}\overset{1}{w}_{,\alpha}\overset{1}{w}_{,\beta}]h dA$$

$$= \frac{h^3}{4}a_1 a_2[\overset{0}{\sigma}_{11}(m_c\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n_c\pi/a_2)^2] < 0$$

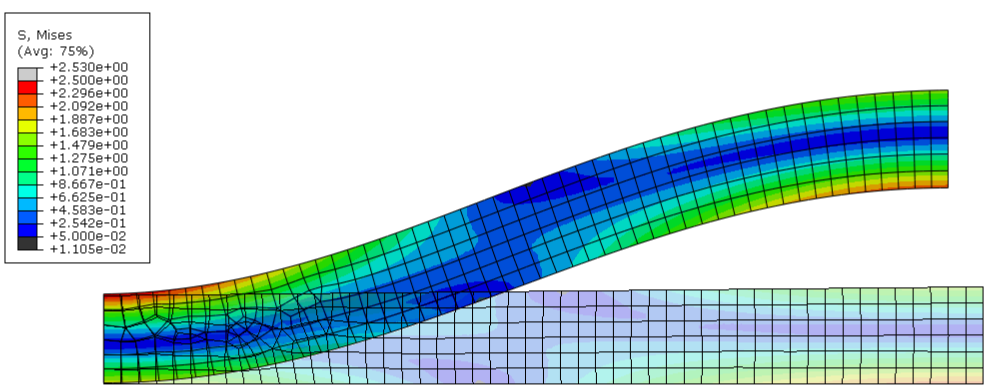
$$\lambda_2 = -\frac{Eh^2}{8}\;\frac{(m_c\pi/a_1)^4 + (n_c\pi/a_2)^4}{\overset{0}{\sigma}_{11}(m_c\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n_c\pi/a_2)^2} = \frac{3}{2}\lambda_c(1-\nu^2)\frac{(m_c\pi/a_1)^4 + (n_c\pi/a_2)^4}{[(m_c\pi/a_1)^2 + (n_c\pi/a_2)^2]^2} > 0$$

CONCEPT OF IMPERFECTION – ILLUSTRATION



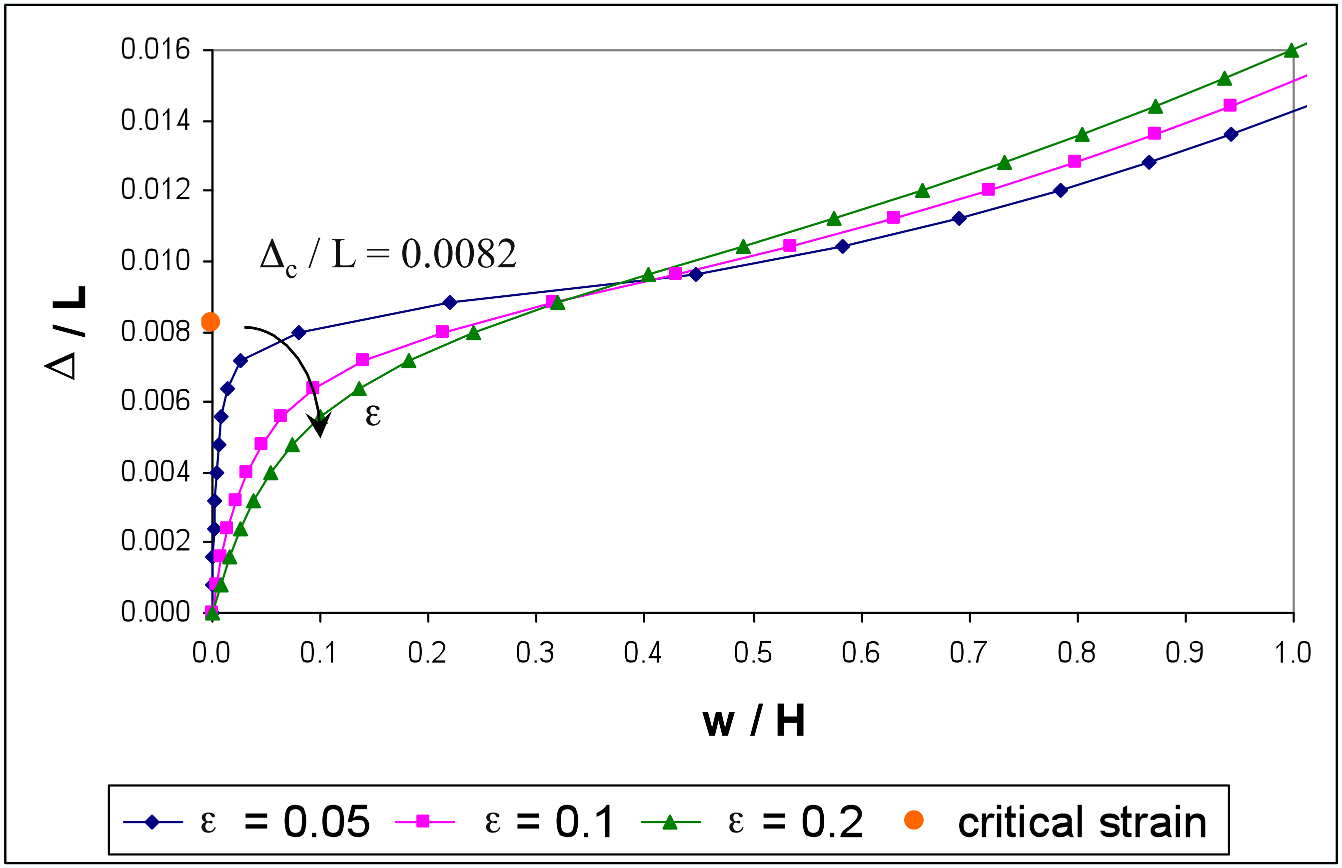
Beam slenderness: $L/H = 10$

Imperfection $\varepsilon = \Delta H/H$



Deformation of an axially compressed beam that has a slight geometric imperfection (beam is trapezoidal with an imperfection angle $\varepsilon = \Delta H/H$)

CONCEPT OF IMPERFECTION – ILLUSTRATION



As imperfection amplitude ε decreases, equilibrium approaches perfect case

LSK ASYMPTOTICS – IMPERFECT CASE – RESULTS

• **Behavior of imperfect structure near critical load can be analyzed by asymptotic expansion of equilibrium solution, just like in the perfect case**

• **IDEA: Study the projection of equilibrium solution along the (one-dimensional) null space of the perfect system's stability operator about its critical point**

• **Method finds (for a given imperfection shape) the relation at equilibrium between imperfection amplitude ε, distance from critical load $\Delta\lambda$ and projection ξ along eigenmode**

• **Method determines the limit points (where stability changes) of imperfect structures and the load drop $\Delta\lambda_s$ from the critical load of the perfect structure λ_c to the maximum load of the imperfect one λ_s**

• **NOTE: Imperfection can occur in any property (material or geometric) that destroys the symmetry of the system**

• **NOTE: By introducing concept of imperfection amplitude ε (scalar quantity) only one extra parameter is introduced in the asymptotic expansions**

LSK ASYMPTOTICS – IMPERFECT CASE – EQUILIBRIUM

$\overline{\mathcal{E}}(u, \lambda, w)$: imperfect system's energy; $u(\mathbf{x}) \in U, w(\mathbf{x}) \in W$ ($w(\mathbf{x})$: imperfection),

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$\overline{\mathcal{E}}(u, \lambda, 0) = \mathcal{E}(u, \lambda)$: reduces to perfect system's energy when imperfection $w(\mathbf{x}) = 0$,

$\overline{\mathcal{E}}(0, \lambda, w) = 0$: zero energy when displacement $u(\mathbf{x}) = 0$.

$\epsilon \equiv \| w \|$: imperfection amplitude; $\overline{w}(\mathbf{x}) \equiv w(\mathbf{x})/\epsilon$: imperfection shape ($\| \overline{w} \| = 1$).

$\overline{\mathcal{E}}_{,u}(u, \lambda, w)\delta u = 0$: equilibrium of imperfect system

$\overline{\mathcal{E}}_{,u}(u(\lambda, w), \lambda, w)\delta u = 0$, $u(\lambda, w)$: solution through $\lambda = 0$; $u(0, w) = 0$, $u(\lambda, 0) = \overset{0}{u}(\lambda)$

NOTE: Imperfection amplitude can be controlled (manufacturing tolerances), shape cannot!

LSK ASYMPTOTICS – IMPERFECT CASE – EQUILIBRIUM

$$u = \overset{0}{u}(\lambda) + \xi \overset{1}{u} + v, \quad v \in \mathcal{N}^{\perp}, \quad \xi \in \mathbb{R}; \quad \xi \equiv (u - \overset{0}{u}, \overset{1}{u}), \quad \Delta\lambda \equiv \lambda - \lambda_c$$

$$\overline{\mathcal{E}}_{,v}\, \delta v = 0 \implies \overline{\mathcal{E}}_{,u}\, (\overset{0}{u}(\lambda_c + \Delta\lambda) + \xi \overset{1}{u} + v, \lambda_c + \Delta\lambda, \epsilon \overline{w}) \delta v = 0; \quad \text{equilibrium in } \mathcal{N}^{\perp},$$

Expand about $(u, \lambda, w) = (u_c, \lambda_c, 0)$ to find $v(\xi, \Delta\lambda, \epsilon)$, where :

$$v(\xi, \Delta\lambda, \epsilon) = \quad \xi v_{\xi} + \Delta\lambda v_{\lambda} + \epsilon v_{\epsilon} +$$

$$\frac{1}{2}[\xi^2 v_{\xi\xi} + 2\xi\Delta\lambda v_{\xi\lambda} + 2\xi\epsilon v_{\xi\epsilon} + (\Delta\lambda)^2 v_{\lambda\lambda} + 2\Delta\lambda\epsilon v_{\lambda\epsilon} + \epsilon^2 v_{\epsilon\epsilon}] + \ldots$$

Expand about $(u, \lambda, w) = (u_c, \lambda_c, 0)$, using $v(\xi, \Delta\lambda, \epsilon)$, to find $\epsilon(\xi, \Delta\lambda)$ from :

$$\overline{\mathcal{E}}_{,\xi} = 0 \implies \overline{\mathcal{E}}_{,u}\, (\overset{0}{u}(\lambda_c + \Delta\lambda) + \xi \overset{1}{u} + v, \lambda_c + \Delta\lambda, \epsilon \overline{w}) \overset{1}{u} = 0; \quad \text{equilibrium in } \mathcal{N}.$$

LSK ASYMPTOTICS – IMPERFECT CASE – EQUILIBRIUM

$$O(\xi) \;:\; (\mathcal{E},^c_{uu} v_\xi)\delta v = 0 \implies v_\xi = 0, \quad (\mathcal{E},^c_{uu} \text{ unique eigenvalue} : \overset{1}{u})$$

$$O(\Delta\lambda) \;:\; (\mathcal{E},^c_{uu} v_\lambda + \boxed{\mathcal{E},^c_{uu} (d\overset{0}{u}/d\lambda)_c + \mathcal{E},^c_{u\lambda}})\delta v = (\mathcal{E},^c_{uu} v_\lambda)\delta v = 0 \implies v_\lambda = 0, \quad \text{(same)}$$

$$O(\xi^2) \;:\; (\mathcal{E},^c_{uu} v_{\xi\xi} + (\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\delta v = 0$$

$$O(\xi\Delta\lambda) \;:\; (\mathcal{E},^c_{uu} v_{\xi\lambda} + (\mathcal{E},^c_{uuu} (d\overset{0}{u}/d\lambda)_c + \mathcal{E},^c_{uu\lambda})\overset{1}{u})\delta v = 0$$

$$O((\Delta\lambda)^2) \;:\; (\mathcal{E},^c_{uu} v_{\lambda\lambda} + \boxed{(\mathcal{E},^c_{uuu} (d\overset{0}{u}/d\lambda)_c)(d\overset{0}{u}/d\lambda)_c + 2\mathcal{E},^c_{uu\lambda} (d\overset{0}{u}/d\lambda)_c +}$$

$$\boxed{\mathcal{E},^c_{u\lambda\lambda} + \mathcal{E},^c_{uu} (d^2\overset{0}{u}/d\lambda^2)_c})\delta v = (\mathcal{E},^c_{uu} v_{\lambda\lambda})\delta v = 0 \implies v_{\lambda\lambda} = 0, \quad \text{(same)}$$

NOTE A : $O(\xi^n \Delta\lambda^m \epsilon^0)$ terms same as perfect case since at $\epsilon = 0 :\; \overline{\mathcal{E}}(u, \lambda, w) = \mathcal{E}(u, \lambda)$

NOTE B : Highlighted terms λ derivatives of principal equilibrium : $\boxed{\mathcal{E},_u (\overset{0}{u}(\lambda), \lambda)\delta u = 0}$

LSK ASYMPTOTICS – IMPERFECT CASE – EQUILIBRIUM

$$O(\epsilon) \ : \quad (\mathcal{E},^{c}_{uu} v_\epsilon + \overline{\mathcal{E}},^{c}_{uw} \overline{w})\delta v = 0$$

$$O(\xi\epsilon) \ : \quad (\mathcal{E},^{c}_{uu} v_{\xi\epsilon} + (\mathcal{E},^{c}_{uuu} v_\epsilon + \overline{\mathcal{E}},^{c}_{uuw} \overline{w})\overset{1}{u})\delta v = 0$$

$$O(\epsilon\Delta\lambda) \ : \quad (\mathcal{E},^{c}_{uu} v_{\lambda\epsilon} + (d\mathcal{E},_{uu} / d\lambda)_c v_\epsilon + (d\overline{\mathcal{E}},_{uw} / d\lambda)_c \overline{w})\delta v = 0$$

$$O(\epsilon^2) \ : \quad (\mathcal{E},^{c}_{uu} v_{\epsilon\epsilon} + (\mathcal{E},^{c}_{uuu} v_\epsilon)v_\epsilon + 2(\overline{\mathcal{E}},^{c}_{uuw} \overline{w})v_\epsilon + (\overline{\mathcal{E}},^{c}_{uww} \overline{w})\overline{w})\delta v = 0$$

$$\begin{aligned} 0 = \quad & \epsilon(\overline{\mathcal{E}},^{c}_{uw} \overline{w})\overset{1}{u} + \frac{1}{2}[\xi^2((\mathcal{E},^{c}_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} + 2\xi\Delta\lambda((d\mathcal{E},_{uu} / d\lambda)_c \overset{1}{u})\overset{1}{u} + \\ & 2\xi\epsilon((\mathcal{E}^c_{uuu} v_\epsilon + \overline{\mathcal{E}},_{uuw})\overline{w}\overset{1}{u})\overset{1}{u} + 2\epsilon\Delta\lambda((d\mathcal{E},_{uu} / d\lambda)_c v_\epsilon + (d\overline{\mathcal{E}},_{uw} / d\lambda)_c \overline{w})\overset{1}{u} + \\ & \epsilon^2((\mathcal{E}^c_{uuu} v_\epsilon)v_\epsilon + 2(\overline{\mathcal{E}},^{c}_{uuw} \overline{w})v_\epsilon + (\overline{\mathcal{E}},^{c}_{uww} \overline{w})\overline{w})\overset{1}{u}] + \\ & \frac{1}{6}[\xi^3((((\mathcal{E},^{c}_{uuuu} \overset{1}{u})\overset{1}{u} + 3\mathcal{E},^{c}_{uu} v_{\xi\xi})\overset{1}{u})\overset{1}{u} + \ldots] + \ldots \end{aligned}$$

LSK ASYMPTOTICS – IMPERFECT CASE – EQUILIBRIUM

Assuming $(\overline{\mathcal{E}},^c_{uw}\,\overline{w})\overset{1}{u} \neq 0,$ use imperfection amplitude $\epsilon(\xi, \Delta\lambda)$ in equilibrium : $\overline{\mathcal{E}}_\xi = 0$

$$\epsilon(\xi, \Delta\lambda) = [((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}/(\overline{\mathcal{E}},^c_{uw}\,\overline{w})\overset{1}{u}] \begin{cases} (\lambda_1\xi^2 - \xi\Delta\lambda) + \dots & \text{for} \quad ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0 \\ (\dfrac{\lambda_2}{2}\xi^3 - \xi\Delta\lambda) + \dots & \text{for} \quad ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0 \end{cases}$$

- **To avoid fractional expansions, one must find the level sets of the imperfection amplitude, i.e. express ε in terms of ξ and $\Delta\lambda$.**

- **There is no sensitivity to an imperfection w unless it has a component along the critical mode of the corresponding perfect system**

- **NOTE: For typical system with random imperfection shape, it always has a component along the critical mode of the corresponding perfect system**

LSK ASYMPTOTICS – IMPERFECT CASE – STABILITY

$$(\overline{\mathcal{E}},_{uu}(\overset{0}{u}(\lambda_c + \Delta\lambda) + \xi\overset{1}{u} + v(\xi, \Delta\lambda, \epsilon(\xi, \Delta\lambda)), \lambda_c + \Delta\lambda, \epsilon(\xi, \Delta\lambda)\overline{w})\overline{x}(\xi, \Delta\lambda))\delta u =$$

$$= \overline{\beta}(\xi, \Delta\lambda)(\overline{x}(\xi, \delta\lambda), \delta u), \text{ with } \overline{\beta} \text{ eigenvalue, } \overline{x} \ (\| \overline{x} \| = 1) \text{ eigenvector of : } \overline{\mathcal{E}},_{uu}$$

$$\text{Where : } \overline{\beta}(\xi, \Delta\lambda) = \xi\overline{\beta}_\xi + \Delta\lambda\overline{\beta}_\lambda + \frac{1}{2}(\xi^2\overline{\beta}_{\xi\xi} + 2\xi\Delta\lambda\overline{\beta}_{\xi\lambda} + (\Delta\lambda)^2\overline{\beta}_{\lambda\lambda}) + ...$$

$$\text{Where : } \overline{x}(\xi, \Delta\lambda) = \overline{x}_0 + \xi\overline{x}_\xi + \Delta\lambda\overline{x}_\lambda + \frac{1}{2}(\xi^2\overline{x}_{\xi\xi} + 2\xi\Delta\lambda\overline{x}_{\xi\lambda} + (\Delta\lambda)^2\overline{x}_{\lambda\lambda}) + ...$$

$$O(1) \ : \ (\mathcal{E},^c_{uu}\overline{x}_0)\delta u = 0, \quad (\overline{x}_0, \overline{x}_0) = 1 \implies \overline{x}_0 = \overset{1}{u}$$

$$O(\xi) \ : \ (\mathcal{E},^c_{uu}\overline{x}_\xi + (\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\delta u = \overline{\beta}_\xi(\overset{1}{u}, \delta u)$$

$$\text{using : } \delta u = \overset{1}{u} \implies \overline{\beta}_\xi = ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}$$

$$\text{using : } \delta u = \delta v \implies (\mathcal{E},^c_{uu}\overline{x}_\xi + (\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\delta v = 0 \implies \overline{x}_\xi = v_{\xi\xi}$$

LSK ASYMPTOTICS – IMPERFECT CASE – STABILITY

$$O(\Delta\lambda) \quad : \quad (\mathcal{E},^c_{uu} \overline{x}_\lambda + (d\mathcal{E},_{uu}/d\lambda)_c \overset{1}{u})\delta u = \overline{\beta}_\lambda(\overset{1}{u}, \delta u)$$

$$\text{using : } \delta u = \overset{1}{u} \implies \overline{\beta}_\lambda = ((d\mathcal{E},_{uu}/d\lambda)_c \overset{1}{u})\overset{1}{u}$$

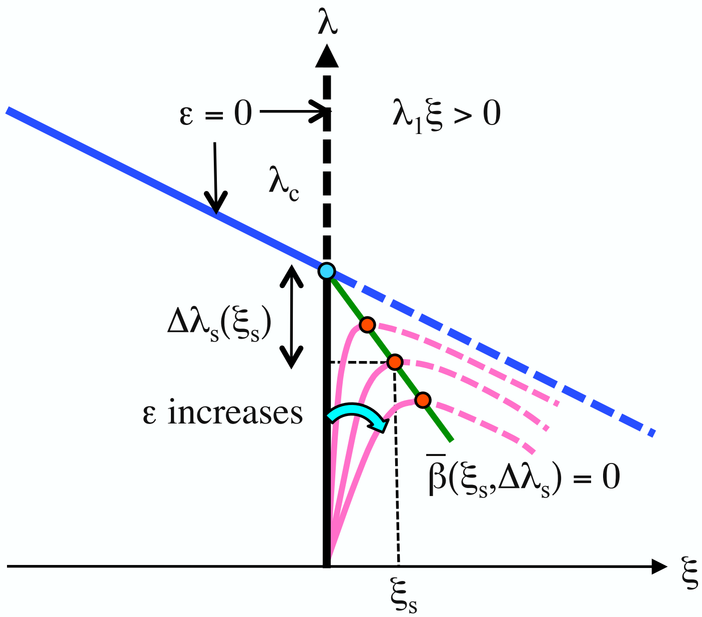
$$\text{using : } \delta u = \delta v \implies (\mathcal{E},^c_{uu} \overline{x}_\lambda + (d\mathcal{E},_{uu}/d\lambda)_c \overset{1}{u})\delta u = 0 \implies \overline{x}_\lambda = v_{\xi\lambda}$$

$$O(\xi^2) \quad : \quad (\frac{1}{2}(((\mathcal{E},^c_{uuuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} + \mathcal{E},^c_{uuu} v_{\xi\xi})\overset{1}{u} + (\mathcal{E},^c_{uuu} v_{\xi\xi})\overset{1}{u} + \frac{1}{2}\mathcal{E},^c_{uu} \overline{x}_{\xi\xi})\delta u = \frac{1}{2}\overline{\beta}_{\xi\xi}(\overset{1}{u}, \delta u)$$

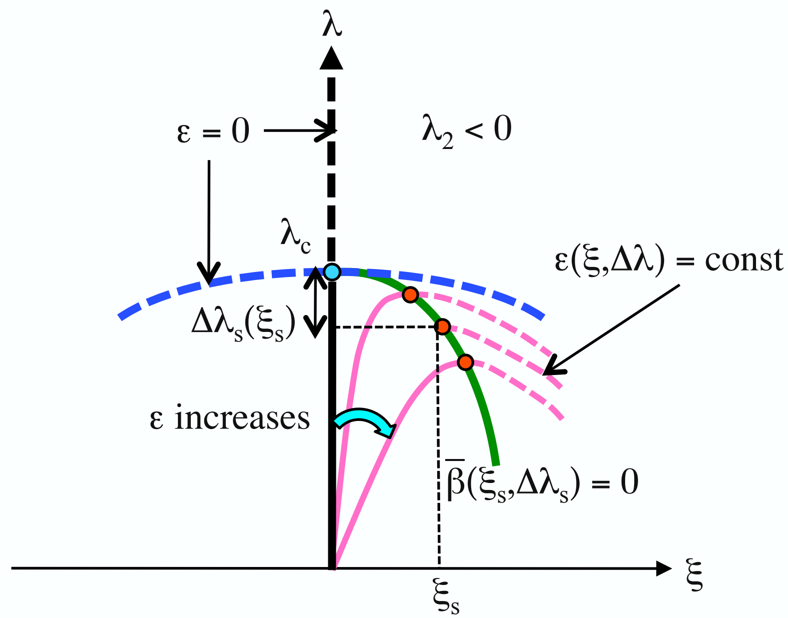
$$\text{using : } \delta u = \overset{1}{u} \implies \overline{\beta}_{\xi\xi} = (((\mathcal{E}^c_{uuuu}\overset{1}{u})\overset{1}{u} + 3\mathcal{E}^c_{uuu} v_{\xi\xi})\overset{1}{u})\overset{1}{u}$$

$$\overline{\beta}(\xi, \Delta\lambda) = [((d\mathcal{E},_{uu}/d\lambda)_c \overset{1}{u})\overset{1}{u}] \begin{cases} (-2\lambda_1\xi + \Delta\lambda) + ... & \text{for} \quad ((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0 \\ (-\frac{3}{2}\lambda_2\xi^2 + \Delta\lambda) + ... & \text{for} \quad ((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} = 0 \end{cases}$$

LSK ASYMPTOTICS – IMPERFECT CASE – RESULTS



Asymmetric bifurcation – Imperfect Case



Symmetric bifurcation – Imperfect Case

LSK ASYMPTOTICS – IMPERFECT CASE – LOAD DROP

$$\overline{\beta}(\xi, \Delta\lambda_s(\xi)) = 0, \quad \Delta\lambda_s(\xi) = s_1\xi + \frac{s_2}{2!}\xi^2 + \frac{s_3}{3!}\xi^3 + O(\xi^4), \quad \text{locus of limit points}$$

$$\Delta\lambda_s(\xi) = \begin{cases} 2\lambda_1\xi + O(\xi^2) & \text{for} \quad ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u} \neq 0 \\ \frac{3}{2}\lambda_2\xi^2 + O(\xi^3) & \text{for} \quad ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u} = 0 \end{cases}$$

Using the expansion for the imperfection amplitude $\epsilon(\xi, \Delta\lambda_s)$; Assume : $\epsilon > 0$

$$\Delta\lambda_s(\epsilon) = \begin{cases} -2(|\lambda_1|)^{1/2}\left[\dfrac{\epsilon|(\overline{\mathcal{E}},^c_{uw}\overline{w})\overset{1}{u}|}{((-d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}}\right]^{1/2} + O(\epsilon) \text{ for } ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0, \\ \dfrac{3}{2}(\lambda_2)^{1/3}\left[\dfrac{\epsilon(\overline{\mathcal{E}},^c_{uw}\overline{w})\overset{1}{u}}{((-d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}}\right]^{2/3} + O(\epsilon) \text{ for } ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0. \end{cases}$$

IMPORTANT : $\Delta\lambda_s(\epsilon) < 0$ if $\lambda_1(\overline{\mathcal{E}},^c_{uw}\overline{w})\overset{1}{u} < 0$ (assym. bif.) or $\lambda_2 < 0$ (sym. bif).

LSK ASYMPTOTICS – IMPERFECT CASE – REVIEW

- **Behavior of imperfect structure near critical load can be analyzed by asymptotic expansion of equilibrium solution.**

- **Bifurcation point in perfect case is replaced by limit points in imperfect case.**

- **Method determines the limit points (where stability changes) of imperfect structure and the load drop $\Delta\lambda_s$ from the critical load of the perfect structure λ_c to the maximum load of the imperfect one λ_s.**

- **Recall that in real structures the amplitude ε can be controlled but not the shape w.**

- **Load drop is maximized when imperfection has the shape of the eigenmode.**

- **Structures are imperfection sensitive (i.e. $\Delta\lambda_s < 0$) for asymmetric or symmetric subcritical bifurcations (load max exists for the equilibrium path through zero load); $\Delta\lambda_s = O(\varepsilon^{1/2})$ for asymmetric case and $\Delta\lambda_s = O(\varepsilon^{2/3})$ for subcritical symmetric case.**

- **NOTE: Imperfection can occur in any property (material or geometric) that destroys the symmetry of the system; all related asymptotic analyses are equivalent.**