LSK (LYAPUNOV - SCHMIDT - KOITER) ASYMPTOTICS

• General asymptotic method to study motion of systems (discrete or continuous) near singular points. Here the method is applied to the equilibrium of conservative elastic systems.

• IDEA: Study the projection of equilibrium equations along the finite dimensional null space of the system's stability operator at critical point. This way the study of a large problem is reduced to the study of a nonlinear system of m equations, where m is the multiplicity of the stability operator's eigenvalue at the critical point.

• Method follows asymptotically equilibrium paths emerging from bifurcation points (simple or multiple) of perfect systems and determines their stability.

• Method also investigates the equilibrium and stability of imperfect systems, near critical points of their perfect counterparts, for small imperfection amplitudes.

• NOTE: Method is useful in determining post-bifurcation behavior and imperfection sensitivity in applications as well as in providing efficient numerical tools for finding solutions near the singular points of complex nonlinear systems.

PRELIMINARIES – FUNCTIONALS & THEIR DERIVATIVES

$\mathcal{E}(u)$: functional (i.e. function of a function) $\mathcal{E} \in \mathbb{R}$ depends on $u \in U$

U : infinite dimensional (Hilbert) space (special case : finite (Euclidean) space $U = \mathbb{R}^n$)

(u_1, u_2) : inner product of $u_i \in U$, $\quad \| u \| \equiv (u, u)^{1/2}$: norm of u

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$\mathcal{E},_u$: linear operator $U \longrightarrow \mathbb{R}$ is *Frechet* derivative $\mathcal{E},_u [\alpha u_1 + \beta u_2] = \alpha \mathcal{E},_u [u_1] + \beta \mathcal{E},_u [u_2]$

$\lim_{\|\delta u\| \to 0} \mid \mathcal{E}(u + \delta u) - \mathcal{E}(u) - \mathcal{E},_u [\delta u] \mid / \| \delta u \| = 0$: Definition Frechet

Special case : $\delta u = \epsilon v$, $\| v \| = 1$, *Gateau* derivative (*Frechet* $\Longrightarrow$ *Gateau*)

$\lim_{\epsilon \to 0} \mid \mathcal{E}(u + \epsilon v) - \mathcal{E}(u) - \epsilon \mathcal{E},_u [v] \mid / \epsilon = 0$: Definition Gateau

$\mathcal{E},_u [v] \equiv \mathcal{E},_u v$: used henceforth for notational simplicity

CALCULATING DERIVATIVES OF ANY ORDER

$\mathcal{E}_{,u}\, v = [\partial \mathcal{E}(u + \epsilon v)/\partial \epsilon]_{\epsilon=0}$: practical way to calculate first derivative

In analogy, the second derivative $\mathcal{E}_{,uu}$ is a bilinear operator $(\mathcal{E}_{,u}\, v)w$

$$(\mathcal{E}_{,uu}\, v)w = (\mathcal{E}_{,uu}\, w)v = [\partial^2 \mathcal{E}(u + \epsilon v + \zeta w)/\partial \epsilon \partial \zeta]_{\epsilon=\zeta=0}$$

The third derivative $\mathcal{E}_{,uuu}$ is a symmetric trilinear operator $((\mathcal{E}_{,uuu}\, v)w)z$

$$((\mathcal{E}_{,uuu}\, v)w)z = [\partial^3 \mathcal{E}(u + \epsilon v + \zeta w + \eta z)/\partial \epsilon \partial \zeta \partial \eta]_{\epsilon=\zeta=\eta=0}$$

Using funcional derivatives, Taylor series expansion of $\mathcal{E}$ is found :

$$\mathcal{E}(u + \delta u) \approx \mathcal{E}(u) + \frac{1}{1!}\mathcal{E}_{,u}\, \delta u + \frac{1}{2!}(\mathcal{E}_{,uu}\, \delta u)\delta u + \frac{1}{3!}((\mathcal{E}_{,uuu}\, \delta u)\delta u)\delta u + ...$$

CALCULATING FUNCTIONAL DERIVATIVES - EXAMPLES

$$\mathcal{E}(u) \equiv \int_0^1 [\exp(w) + (2+\cos(x))w^2 + 3(v,_x^2 + v^2)]dx, \quad u = (v(x), w(x));$$

$$\text{where}: \quad w \in H^0[0,1], \ (\text{i.e.} \int_0^1 w^2 < \infty); \ v \in H^1[0,1], \ (\text{i.e.} \int_0^1 [v^2 + v,_x^2]dx < \infty)$$

$$\mathcal{E},_u u_1 = \frac{\partial}{\partial \epsilon_1} \left\{ \int_0^1 [\exp(w + \epsilon_1 w_1) + (2+\cos x)(w+\epsilon_1 w_1)^2 + \right.$$

$$\left. +3((v,_x + \epsilon_1 v_{1,x})^2 + (v + \epsilon_1 v_1)^2)]dx \right\}_{\epsilon_1 = 0}$$

$$\mathcal{E},_u u_1 = \int_0^1 [\exp(w)w_1 + 2(2+\cos x)ww_1 + 6(v,_x v_{1,x} + vv_1)]dx$$

FUNCTIONALS & THEIR DERIVATIVES

$$(\mathcal{E}_{,uu}\, u_1)u_2 = \frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}\left\{\int_0^1 [\exp(w+\epsilon_1 w_1+\epsilon_2 w_2) + (2+\cos x)(w+\epsilon_1 w_1+\epsilon_2 w_2)^2\right.$$

$$\left. +3((v_{,x}+\epsilon_1 v_{1,x}+\epsilon_2 v_{2,x})^2 + (v+\epsilon_1 v_1+\epsilon_2 v_2)^2)]dx\right\}_{\epsilon_1=\epsilon_2=0}$$

$$(\mathcal{E}_{,uu}\, u_1)u_2 = \int_0^1 [\exp(w)w_1 w_2 + 2(2+\cos x)w_1 w_2 + 6(v_{1,x}v_{2,x} + v_1 v_2)]dx$$

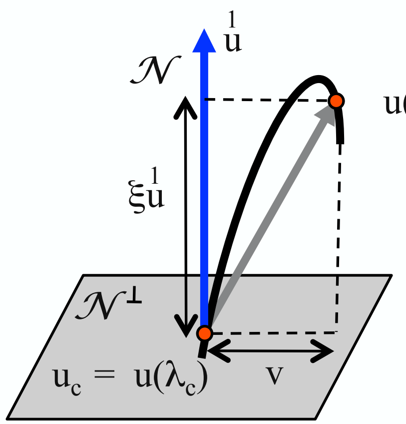
Similarly one obtains for higher derivatives...

$$((\mathcal{E}_{,uuu}\, u_1)u_2)u_3 = \int_0^1 [\exp(w)w_1 w_2 w_3]dx$$

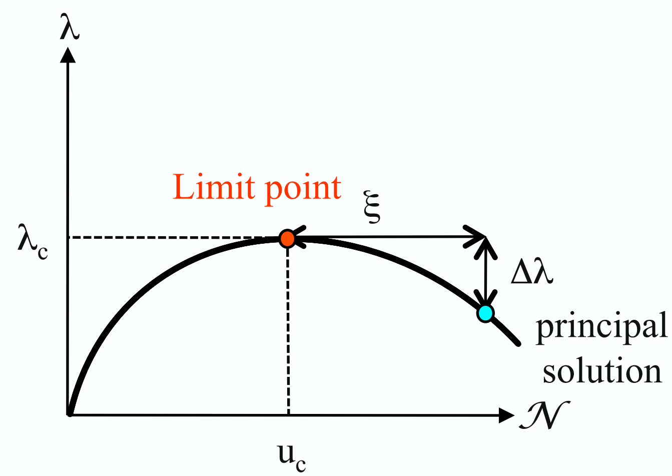
$$(((\mathcal{E}_{,uuuu}\, u_1)u_2)u_3)u_4 = \int_0^1 [\exp(w)w_1 w_2 w_3 w_4]dx$$

PERFECT STRUCTURE – SIMPLE MODE

BIFURCATION VS LIMIT POINT



$$u(\lambda_c + \Delta\lambda) = u_c + \xi\overset{1}{u} + v(\xi, \Delta\lambda)$$

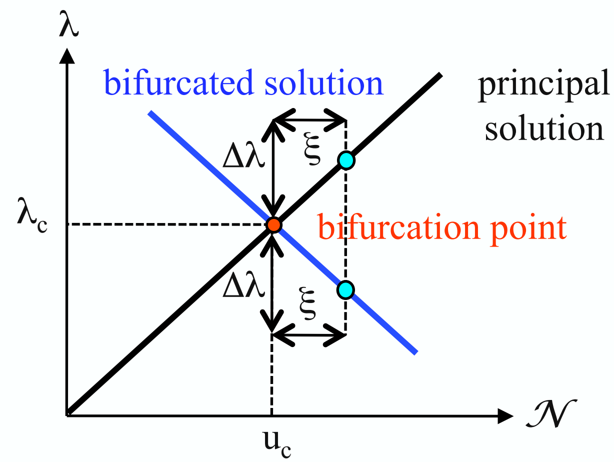


- **About critical point u_c project solution increment Δu along null space $\mathcal{N}$ and its complement $\mathcal{N}^{\perp}$.**

- **Solve equilibrium in $\mathcal{N}^{\perp}$ and use $v(\xi, \Delta\lambda)$ to find equilibrium in $\mathcal{N}$ from which you determine $\Delta\lambda$ as a function of ξ**

- **If $\Delta\lambda(\xi)$ is unique: limit point**

- **If $\Delta\lambda(\xi)$ is not unique: bifurcation**



BIFURCATION VS LIMIT POINT

$\mathcal{E}(u,\lambda)$: energy at displacement $u(\mathbf{x}) \in U$ and load $\lambda \geq 0$

$\mathcal{E}(0,\lambda) = 0, \ \forall\lambda$: zero energy at zero displacement

$\mathcal{E}_{,u}(u,\lambda)\delta u = 0, \ \forall\ \delta u \in U$: equilibrium statement

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$\mathcal{E}_{,u}(\overset{0}{u}(\lambda),\lambda)\delta u = 0, \ \forall\lambda;$ principal solution $\overset{0}{u}(\lambda), \ (\overset{0}{u}(0) = 0)$

$\overset{0}{u}(\lambda)$ stable near $\lambda = 0$, i.e. min. eigenvalue of $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda) \equiv \mathcal{E}_{,uu}^{0}$ is $\overset{0}{\beta} > 0$

$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u \geq \overset{0}{\beta}(\lambda) \parallel \delta u \parallel^2, \ \overset{0}{\beta}(\lambda) > 0; \quad \exists\epsilon > 0, \ \forall\ \lambda \in [0,\epsilon]$

NOTE: Unique & stable principal solution near zero load assumed (realistic structures)

BIFURCATION VS LIMIT POINT

$$\left[\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)(d\overset{0}{u}/d\lambda)+\mathcal{E},_{u\lambda}(\overset{0}{u}(\lambda),\lambda)\right]\delta u=0 \implies d\overset{0}{u}/d\lambda \text{ exists if } (\mathcal{E},^{0}_{uu})^{-1} \text{ exists}$$

$$\mathcal{E},^{0}_{uu}\equiv\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda) \text{ is positive definite, invertible, i.e. } \overset{0}{\beta}(\lambda)>0, \text{ for } \lambda\in[0,\lambda_c)$$

As load increases away from 0, the lowest load that $\overset{0}{\mathrm{u}}$ cannot be continued is : λ_{c}

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda_c),\lambda_c)\overset{1}{u})\delta u=0;\ \lambda_c: \text{ critical load, } \overset{1}{u}: \text{mode } (\|\overset{1}{u}\|=1, \text{ assumed unique})$$

In all directions orthogonal to null space $\mathcal{N}$ of $\mathcal{E}^c,_{uu}$ operator still positive definite :

$$(\mathcal{E},^{c}_{uu}\delta v)\delta v\geq\gamma\|\delta v\|^2,\quad \exists\gamma>0,\quad \forall\delta v\in\mathcal{N}^{\perp};\quad \text{where}: \mathcal{E},^{c}_{uu}\equiv\mathcal{E},_{uu}(\overset{0}{u}(\lambda_c),\lambda_c)$$

$$\mathcal{N}\equiv\{u\in U \mid u=\mu\overset{1}{u},\ \forall\mu\in\mathbb{R}\},\ \mathcal{N}^{\perp}\equiv\{v\in U \mid (v,\overset{1}{u})=0\};\quad (\mathcal{N}\oplus\mathcal{N}^{\perp}=U)$$

BIFURCATION VS LIMIT POINT

$$u = u_c + \xi \overset{1}{u} + v, \quad v \in \mathcal{N}^{\perp}, \quad \xi \in \mathbb{R}$$

$$\mathcal{E}_{,v}\, \delta v = 0 \implies \mathcal{E}_{,u}\, (u_c + \xi \overset{1}{u} + v, \lambda_c + \Delta\lambda)\delta v = 0; \quad \text{equilibrium in } \mathcal{N}^{\perp}$$

Expand about (u_c, λ_c) to find $v(\xi, \Delta\lambda)$, where :

$$v(\xi, \Delta\lambda) = \xi v_{\xi} + \Delta\lambda v_{\lambda} + \frac{1}{2!}\left[\xi^2 v_{\xi\xi} + 2\xi\Delta\lambda v_{\xi\lambda} + (\Delta\lambda)^2 v_{\lambda\lambda}\right] + ...$$

$$\mathcal{E}_{,\xi} = 0 \implies \mathcal{E}_{,u}\, (u_c + \xi \overset{1}{u} + v, \lambda_c + \Delta\lambda)\overset{1}{u} = 0; \quad \text{equilibrium in } \mathcal{N}$$

Expand about (u_c, λ_c), using $v(\xi, \Delta\lambda)$, to find $\Delta\lambda(\xi)$

$\Delta\lambda(\xi)$ is unique $\implies$ limit load, $\Delta\lambda(\xi)$ is not unique $\implies$ bifurcation

BIFURCATION VS LIMIT POINT

$$O(\xi) \ : \ (\mathcal{E},^{c}_{uu} v_\xi)\delta v = 0 \implies v_\xi = 0, \quad (\mathcal{E},^{c}_{uu} \text{ has unique eigenvalue } \overset{1}{u})$$

$$O(\Delta\lambda) \ : \ (\mathcal{E},^{c}_{uu} v_\lambda + \mathcal{E},^{c}_{u\lambda})\delta v = 0 \implies 0 \neq v_\lambda \in \mathcal{N}^\perp, \quad (\mathcal{E},^{c}_{uu} \text{ invertible in } \mathcal{N}^\perp)$$

$$O(\xi^2) \ : \ (\mathcal{E},^{c}_{uu} v_{\xi\xi} + (\mathcal{E},^{c}_{uuu} \overset{1}{u})\overset{1}{u})\delta v = 0 \implies 0 \neq v_{\xi\xi} \in \mathcal{N}^\perp$$

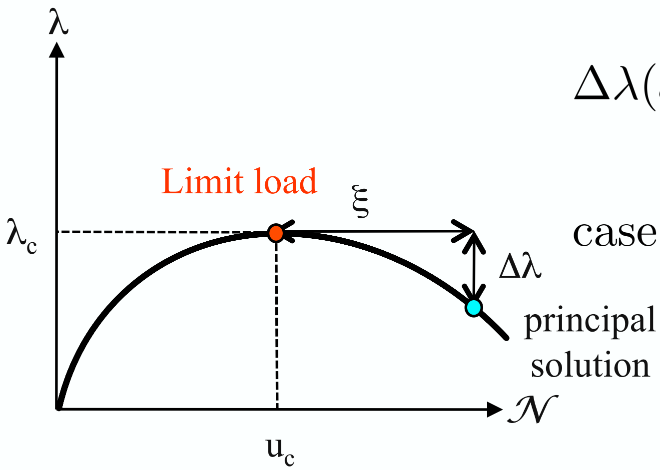
$$O(\xi\Delta\lambda) \ : \ (\mathcal{E},^{c}_{uu} v_{\xi\lambda} + (\mathcal{E},^{c}_{uuu} v_\lambda + \mathcal{E},^{c}_{uu\lambda})\overset{1}{u})\delta v = 0 \implies 0 \neq v_{\xi\lambda} \in \mathcal{N}^\perp$$

$$O((\Delta\lambda)^2) \ : \ (\mathcal{E},^{c}_{uu} v_{\lambda\lambda} + (\mathcal{E},^{c}_{uuu} v_\lambda)v_\lambda + 2\mathcal{E},^{c}_{uu\lambda} v_\lambda + \mathcal{E},^{c}_{u\lambda\lambda})\delta v = 0 \implies 0 \neq v_{\lambda\lambda} \in \mathcal{N}^\perp$$

Using above results in equilibrium along $\mathcal{N}$

$$0 = \Delta\lambda(\mathcal{E},^{c}_{u\lambda} \overset{1}{u}) + \frac{1}{2}[\xi^2((\mathcal{E},^{c}_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} + 2\xi\Delta\lambda((\mathcal{E},^{c}_{uuu} v_\lambda + \mathcal{E},^{c}_{uu\lambda})\overset{1}{u})\overset{1}{u} +$$
$$+(\Delta\lambda)^2((\mathcal{E},^{c}_{uuu} v_\lambda)v_\lambda + 2\mathcal{E},^{c}_{uu\lambda} v_\lambda + \mathcal{E},^{c}_{u\lambda\lambda})\overset{1}{u}] + \ldots$$

BIFURCATION VS LIMIT POINT



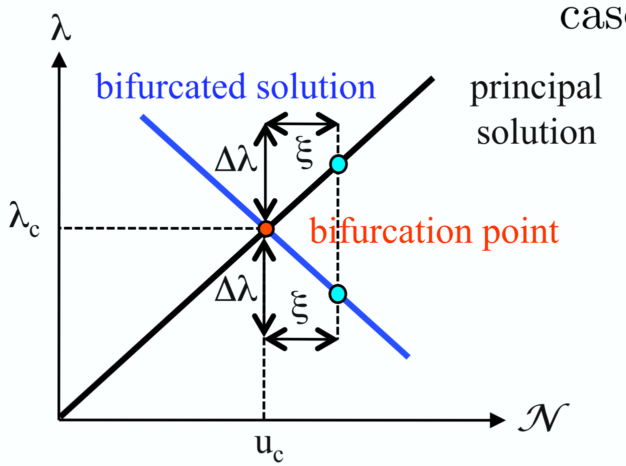
$$\Delta\lambda(\xi) = \quad \lambda_1\xi + \lambda_2\frac{\xi^2}{2!} + \lambda_3\frac{\xi^3}{3!} + \ldots$$

$$\text{case (i)}: \quad \boxed{\mathcal{E},^c_{u\lambda}\overset{1}{u} \neq 0 \implies \text{limit load}:}$$

$$\text{Unique } \Delta\lambda(\xi);$$

$$\lambda_1 = 0, \quad \lambda_2 = -((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}/\mathcal{E},^c_{u\lambda}\overset{1}{u}$$

$$\text{case (ii)}: \quad \boxed{\mathcal{E},^c_{u\lambda}\overset{1}{u} = 0 \implies \text{bifurcation point}:}$$



$$\text{Non-unique } \Delta\lambda(\xi); \quad \lambda_1 \text{ solution of quadratic}:$$

$$(\lambda_1)^2((\mathcal{E},^c_{uuu}v_\lambda)v_\lambda + 2\mathcal{E},^c_{uu\lambda}v_\lambda + \mathcal{E},^c_{u\lambda\lambda})\overset{1}{u} +$$

$$+2\lambda_1((\mathcal{E},^c_{uuu}v_\lambda + \mathcal{E},^c_{uu\lambda})\overset{1}{u})\overset{1}{u} + ((\mathcal{E},^c_{uuu})\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0$$

ASYMPTOTIC EXPANSIONS – SIMPLE BIFURCATION CASE

$$u = \overset{0}{u}(\lambda) + \xi \overset{1}{u} + v, \quad v \in \mathcal{N}^{\perp}, \quad \xi \in \mathbb{R}, \quad \xi \equiv (u - \overset{0}{u}, \overset{1}{u}) : \text{ bifurcation amplitude}$$

$$\mathcal{E}_{,v}\, \delta v = 0 \implies \mathcal{E}_{,u}\, (\overset{0}{u}(\lambda_c + \Delta\lambda) + \xi \overset{1}{u} + v(\xi, \Delta\lambda), \lambda_c + \Delta\lambda)\delta v = 0; \quad \text{equilibrium in } \mathcal{N}^{\perp}$$

Expand about (u_c, λ_c) to find $v(\xi, \Delta\lambda)$, where :

$$v(\xi, \Delta\lambda) = \xi v_{\xi} + \Delta\lambda v_{\lambda} + \frac{1}{2!}\left[\xi^2 v_{\xi\xi} + 2\xi\Delta\lambda v_{\xi\lambda} + (\Delta\lambda)^2 v_{\lambda\lambda}\right] + ...$$

$$\mathcal{E}_{,\xi} = 0 \implies \mathcal{E}_{,u}\, (\overset{0}{u}(\lambda_c + \Delta\lambda) + \xi \overset{1}{u} + v(\xi, \Delta\lambda), \lambda_c + \Delta\lambda)\overset{1}{u} = 0; \quad \text{equilibrium in } \mathcal{N}$$

Expand about (u_c, λ_c), using $v(\xi, \Delta\lambda)$, to find $\Delta\lambda(\xi)$ where :

$$\Delta\lambda(\xi) = \lambda_1 \xi + \lambda_2 \frac{\xi^2}{2!} + \lambda_3 \frac{\xi^3}{3!} + ...$$

ASYMPTOTIC EXPANSIONS – SIMPLE BIFURCATION CASE

$$O(\xi) \;:\; (\mathcal{E},^c_{uu} v_\xi)\delta v = 0 \implies v_\xi = 0, \quad (\mathcal{E},^c_{uu} \text{ unique eigenvalue : } \overset{1}{u})$$

$$O(\Delta\lambda) \;:\; (\mathcal{E},^c_{uu} v_\lambda + \boxed{\mathcal{E},^c_{uu} (d\overset{0}{u}/d\lambda)_c + \mathcal{E},^c_{u\lambda}})\delta v = (\mathcal{E},^c_{uu} v_\lambda)\delta v = 0 \implies v_\lambda = 0, \quad \text{(same)}$$

$$O(\xi^2) \;:\; (\mathcal{E},^c_{uu} v_{\xi\xi} + (\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\delta v = 0$$

$$O(\xi\Delta\lambda) \;:\; (\mathcal{E},^c_{uu} v_{\xi\lambda} + (\mathcal{E},^c_{uuu} (d\overset{0}{u}/d\lambda)_c + \mathcal{E},^c_{uu\lambda})\overset{1}{u})\delta v = 0$$

$$O((\Delta\lambda)^2) \;:\; (\mathcal{E},^c_{uu} v_{\lambda\lambda} + \boxed{(\mathcal{E},^c_{uuu} (d\overset{0}{u}/d\lambda)_c)(d\overset{0}{u}/d\lambda)_c + 2\mathcal{E},^c_{uu\lambda} (d\overset{0}{u}/d\lambda)_c +}$$

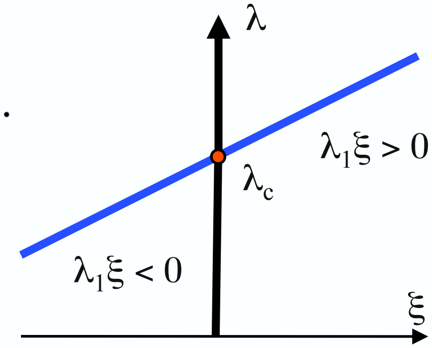
$$\boxed{\mathcal{E},^c_{u\lambda\lambda} + \mathcal{E},^c_{uu} (d^2\overset{0}{u}/d\lambda^2)_c})\delta v = (\mathcal{E},^c_{uu} v_{\lambda\lambda})\delta v = 0 \implies v_{\lambda\lambda} = 0, \quad \text{(same)}$$

NOTE A : $v(0, \Delta\lambda) = u - \overset{0}{u}(\lambda) = 0 \implies v_\lambda = v_{\lambda\lambda} = v_{\lambda\lambda\lambda} = ... = 0$

NOTE B : Highlighted terms λ derivatives of principal equilibrium : $\boxed{\mathcal{E},_u (\overset{0}{u}(\lambda), \lambda)\delta u = 0}$

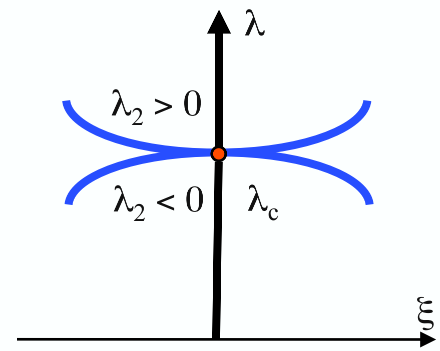
ASYMPTOTIC EXPANSIONS – SIMPLE BIFURCATION CASE

$$0 = \frac{1}{2}[\xi^2((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} + 2\xi\Delta\lambda((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}]+$$
$$\frac{1}{6}[\xi^3((((\mathcal{E},^c_{uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E},^c_{uuu} v_{\xi\xi})\overset{1}{u})\overset{1}{u}) + ...] + ...$$



case (i) : $((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0 \implies$ transcritical bifurcation :

$$\lambda_1 = -\frac{1}{2}((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}/((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$$



case (ii) : $((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0 \implies$ symmetric bifurcation :

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{1}{3}[((((\mathcal{E},^c_{uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E},^c_{uuu} v_{\xi\xi})\overset{1}{u})\overset{1}{u}]/((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$$

STABILITY – SIMPLE BIFURCATION CASE

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)\overset{0}{x}(\lambda))\delta u = \overset{0}{\beta}(\lambda)(\overset{0}{x}(\lambda),\delta u), \quad \text{stability of principal path}$$

$$\overset{0}{\beta}(\lambda):\ \text{eigenvalue},\ \overset{0}{x}(\lambda):\ \text{eigenvector of } \mathcal{E},^{0}_{uu};\quad (\overset{0}{x}(\lambda),\overset{0}{x}(\lambda)) = 1$$

$$\text{Evaluate at } \lambda_c:\ (\mathcal{E},^{c}_{uu}\overset{0}{x}(\lambda_c))\delta u = 0;\quad \overset{0}{\beta}(\lambda_c) = 0,\ (\overset{0}{x}(\lambda_c),\overset{0}{x}(\lambda_c)) = 1 \implies \overset{0}{x}(\lambda_c) = \overset{1}{u}$$

$$\text{Differentiate at } \lambda_c:\ ((\mathcal{E},^{c}_{uuu}(d\overset{0}{u}/d\lambda)_c + \mathcal{E},^{c}_{uu\lambda})\overset{1}{u} + \mathcal{E},^{c}_{uu}(d\overset{0}{x}/d\lambda)_c)\delta u = (d\overset{0}{\beta}/d\lambda)_c(\overset{1}{u},\delta u)$$

$$\text{Substitute}:\ \delta u = \overset{1}{u},\ \text{recall}:\ (\overset{1}{u},\overset{1}{u}) = 1 \implies ((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = (d\overset{0}{\beta}/d\lambda)_c$$

$$\text{Assumption}:\ (d\overset{0}{\beta}/d\lambda)_c < 0;\ (\text{recall}:\ \overset{0}{\beta}(\lambda) > 0,\ \forall\lambda\in[0,\lambda_c),\ \text{holds in most applications})$$

Eigenvalues and eigenvectors of stability operator along principal path

STABILITY – SIMPLE BIFURCATION CASE

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c+\Delta\lambda(\xi))+\xi\overset{1}{u}+v(\xi,\Delta\lambda(\xi)),\lambda_c+\Delta\lambda(\xi))x(\xi))\delta u=\beta(\xi)(x(\xi),\delta u),\ \parallel x(\xi)\parallel=1$$

$$\text{Use}:\ \beta(\xi)=\xi\beta_1+\frac{\xi^2}{2}\beta_2+...,\ x(\xi)=x_0+\xi x_1+\frac{\xi^2}{2}x_2+...,\ \text{expand about}:\ (u_c,\lambda_c)$$

$$O(1)\ :\ (\mathcal{E}^c_{,uu}x_0)\delta u=0,\ (x_0,x_0)=1;\implies x_0=\overset{1}{u}$$

$$\text{Assume}:\ ((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}\neq 0,\quad \text{substitute}:\ \delta u=\overset{1}{u},\quad \boxed{\lambda_1=-\frac{1}{2}\frac{((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}}{((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}}}$$

$$O(\xi)\ :\ ((\mathcal{E}^c_{,uuu}(\lambda_1(d\overset{0}{u}/d\lambda)_c+\overset{1}{u}))\overset{1}{u}+\lambda_1\mathcal{E}^c_{,uu\lambda}\overset{1}{u}+\mathcal{E}^c_{,uu}x_1)\delta u=\beta_1(\overset{1}{u},\delta u),\implies$$

$$\boxed{\beta_1=}\lambda_1((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}+((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}=\boxed{\lambda_1[-((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}]}$$

STABILITY – SIMPLE BIFURCATION CASE

$$\text{Assume}: \ ((\mathcal{E},^{c}_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0, \quad \text{substitute}: \ \delta u = \delta v, \ \lambda_1 = \beta_1 = 0$$

$$O(\xi) \ : \ ((\mathcal{E},^{c}_{uuu}\overset{1}{u})\overset{1}{u} + \mathcal{E},^{c}_{uu} x_1)\delta v = 0, \quad (x_1, \overset{1}{u}) = 0, \implies x_1 = v_{\xi\xi}$$

$$O(\xi^2) \ : \ ((\frac{1}{2}((\mathcal{E},^{c}_{uuuu}\overset{1}{u})\overset{1}{u} + \mathcal{E},^{c}_{uuu}(\lambda_2(d\overset{0}{u}/d\lambda)_c + v_{\xi\xi}) + \lambda_2\mathcal{E},^{c}_{uu\lambda})\overset{1}{u})x_1 +$$

$$+(\mathcal{E},^{c}_{uuu}\overset{1}{u})x_1 + \frac{1}{2}\mathcal{E},^{c}_{uu} x_2)\delta u = \frac{1}{2}\beta_2(\overset{1}{u}, \delta u)$$

$$\text{Use}: \ \delta u = \overset{1}{u}, \ \text{recall}: \ x_1 = v_{\xi\xi}, \quad \boxed{\lambda_2 = -\frac{1}{3}\frac{[(((\mathcal{E},^{c}_{uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E},^{c}_{uuu} v_{\xi\xi})\overset{1}{u})\overset{1}{u}]}{((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}}}$$

$$\boxed{\beta_2 =} (((\mathcal{E},^{c}_{uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + \lambda_2((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} + ((\mathcal{E},^{c}_{uuu} v_{\xi\xi})\overset{1}{u})\overset{1}{u} + 2((\mathcal{E},^{c}_{uuu} x_1)\overset{1}{u})\overset{1}{u}$$

$$= \boxed{2\lambda_2(-((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u})}$$

ENERGY – SIMPLE BIFURCATION CASE

$$\Delta\mathcal{E} \;\equiv\; \mathcal{E}(\overset{0}{u}(\lambda_c+\Delta\lambda)+\xi\overset{1}{u}+v(\xi,\Delta\lambda),\lambda_c+\Delta\lambda)-\mathcal{E}(\overset{0}{u}(\lambda_c+\Delta\lambda),\lambda_c+\Delta\lambda)$$

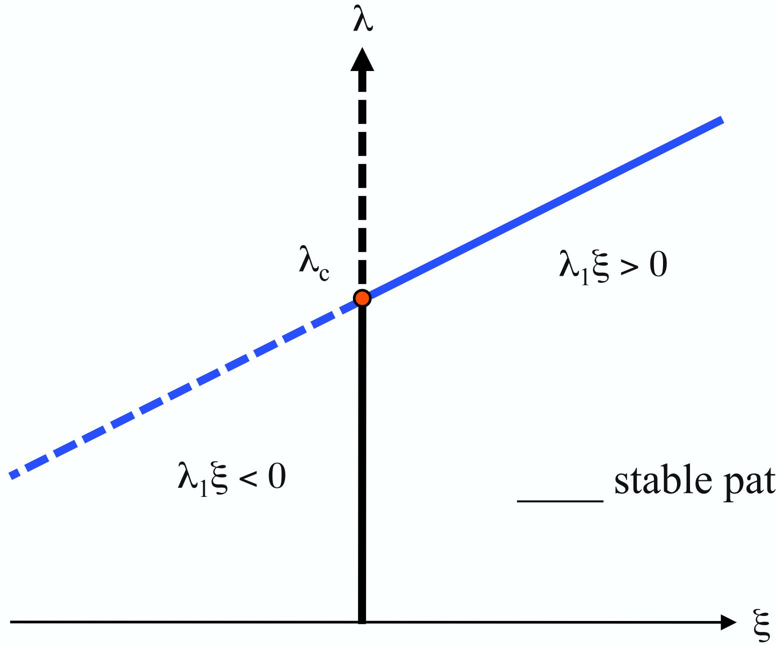
Using previous results from asymptotic expansion about : (u_c, λ_c)

$$\Delta\mathcal{E} = \begin{cases} \dfrac{\xi^3}{6}\lambda_1[-(d\overset{0}{\beta}/d\lambda)_c]+O(\xi^4) & \text{for asymmetric bifurcation : } ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}\neq 0 \\ \dfrac{\xi^4}{8}\lambda_2[-(d\overset{0}{\beta}/d\lambda)_c]+O(\xi^5) & \text{for symmetric bifurcation : } ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}=0 \end{cases}$$

Comparing energy of principal & bifurcated paths for same load

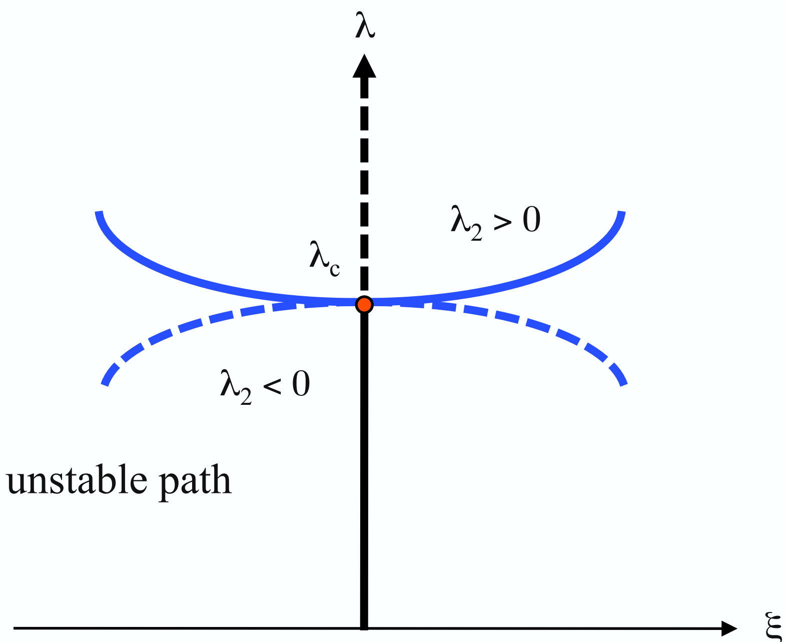
ASYMPTOTIC EXPANSIONS – SIMPLE BIFURCATION CASE

$$\beta(\xi) = \begin{cases} \lambda_1 \xi [-(d\overset{0}{\beta}/d\lambda)_c] + O(\xi^2) & \text{for asymmetric bifurcation} \\ \lambda_2 \xi^2 [-(d\overset{0}{\beta}/d\lambda)_c] + O(\xi^3) & \text{for symmetric bifurcation} \end{cases}$$



ASYMMETRIC BIFURCATION

____ stable path, _ _ _ unstable path



SYMMETRIC BIFURCATION

ASYMPTOTIC EXPANSIONS – SIMPLE BIFURCATION CASE

$$\mathcal{E}_{,u}(u,\lambda)\delta u = 0,\ \forall\ \delta u \in U$$

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c),\lambda_c)\overset{1}{u})\delta u = 0$$

λ_c : critical load, $\overset{1}{u}$: critical mode, unique ($\| \overset{1}{u} \| = 1$)

$\mathcal{E}^{c}_{,u\lambda}\overset{1}{u} \neq 0 \implies$ limit load

$\mathcal{E}^{c}_{,u\lambda}\overset{1}{u} = 0 \implies$ bifurcation point

$((\mathcal{E}^{c}_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0 \implies$ transcritical bifurcation

$$\lambda_1 = -\frac{1}{2}((\mathcal{E}^{c}_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}/((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$$

$((\mathcal{E}^{c}_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0 \implies$ symmetric bifurcation $(\mathcal{E}^{c}_{,uu}v_{\xi\xi} + (\mathcal{E}^{c}_{,uuu}\overset{1}{u})\overset{1}{u})\delta v = 0$

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{1}{3}[(((\mathcal{E}^{c}_{,uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E}^{c}_{,uuu}v_{\xi\xi})\overset{1}{u})\overset{1}{u}]/((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$$

ASYMPTOTIC EXPANSIONS – SIMPLE BIFURCATION CASE

- **LSK asymptotic expansion for simple bifurcation point, reduces infinite dimensional problem to the study of one dimensional problem (projection of equilibrium on the critical operator's null space).**

- **Principal branch changes stability at critical load (from stable at lower loads to unstable).**

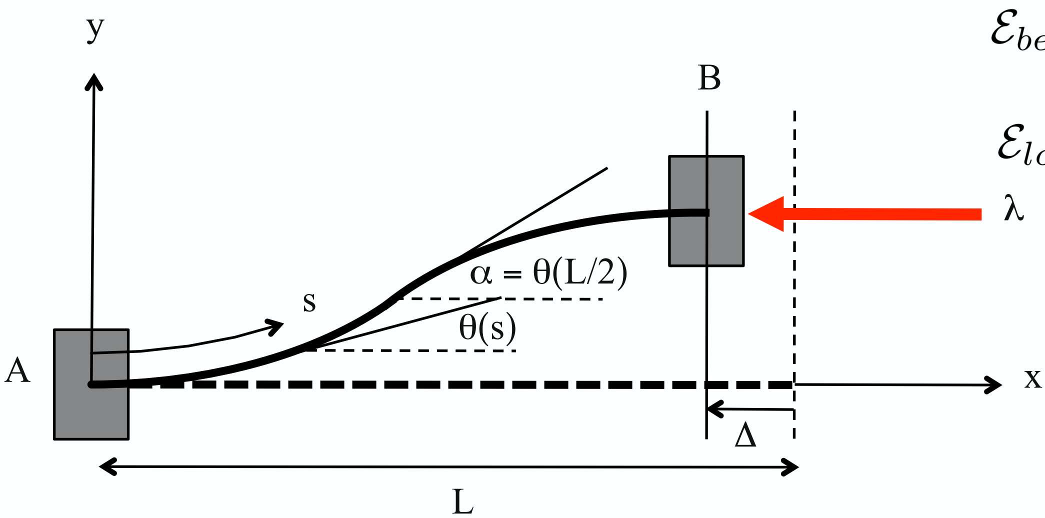
- **For transcritical (asymmetric) bifurcation, supercritical portion of path is stable, subcritical portion of path is unstable.**

- **For symmetric bifurcations, supercritical paths are stable, subcritical paths are unstable.**

- **Stable paths have, for a given load, less energy than their neighboring unstable paths**

NOTE: General case asymptotics obtained here are similar to results of the rigid T model!

ELASTICA EXAMPLE – MODEL SETTING & ENERGY



$$\mathcal{E}_{bend} = \int_0^L (1/2) EI (d\theta/ds)^2$$

$$\mathcal{E}_{load} = -\lambda\Delta$$

$$\Delta = \int_0^L (ds - dx) = \int_0^L (1 - \cos\theta)] ds$$

$$\mathcal{E} = \mathcal{E}_{bend} + \mathcal{E}_{load}$$

$$\mathcal{E}(\theta, \lambda) = \int_0^L \left[\frac{1}{2} EI \left(\frac{d\theta}{ds} \right)^2 - \lambda(1 - \cos\theta) \right] ds, \quad \text{with :} \ \ \theta(0) = \theta(L) = 0$$

$$\theta(s) \in H_0^1[0, L], \quad (\text{i.e. } \int_0^L [(d\theta/ds)^2 + \theta^2]^{1/2} ds < \infty, \ \theta(0) = \theta(L) = 0)$$

$$\text{inner product :} \ \ (\theta_1, \theta_2) \equiv (1/L) \int_0^L [\theta_1(s)\theta_2(s)] ds$$

ELASTICA EXAMPLE – GOVERNING EQUATIONS

$$\mathcal{E},_u \delta u = \int_0^L \left[EI \left(\frac{d\theta}{ds}\right)\left(\frac{d\delta\theta}{ds}\right) - \lambda \sin(\theta)\delta\theta \right] ds = 0, \quad \Longrightarrow \quad \text{(integrate by parts)}$$

$$\mathcal{E},_u \delta u = \int_0^L \left[-EI \left(\frac{d^2\theta}{ds^2}\right) - \lambda \sin(\theta) \right] \delta\theta ds + \left[EI \frac{d\theta}{ds}\delta\theta \right]_{s=L} - \left[EI \frac{d\theta}{ds}\delta\theta \right]_{s=0} = 0$$

Recall : $\delta\theta(s)$ arbitraty function $\in H_0^1[0, L]$, $\delta\theta(0) = \delta\theta(L) = 0, \quad \Longrightarrow$

$$\frac{d^2\theta}{ds^2} + \frac{\lambda}{EI} \sin\theta = 0, \text{ Elastica governing equation (Euler 1744)}; \quad \theta(0) = \theta(L) = 0,$$

$\overset{0}{u}(\lambda) = \overset{0}{\theta}(s, \lambda) = 0 \quad \forall\lambda$, principal solution

ELASTICA EXAMPLE – STABILITY OF PRINCIPAL PATH

$$\overset{0}{\beta}(\lambda) = \min_{(\|\delta u\|=1)} \left[\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u\right] = \int_0^L \left[EI\left(\frac{d(\delta\theta)}{ds}\right)^2 - \lambda(\delta\theta)^2\right] ds, \ \forall \delta\theta \in H_0^1[0,L]$$

$$\delta\theta = \sum_{n=1}^{\infty}\left[\delta\theta_n \sin\left(\frac{n\pi s}{L}\right)\right], \ \left(\frac{1}{2}\sum_{n=1}^{\infty}(\delta\theta_n)^2 = \|\ \delta\theta\ \| = 1\right), \ \text{Fourier representation of } \delta\theta$$

$$\overset{0}{\beta}(\lambda) = \min_{(\delta\theta_n)} \frac{L}{2}\sum_{n=1}^{\infty}\left\{(\delta\theta_n)^2\left[EI\left(\frac{n\pi}{L}\right)^2 - \lambda\right]\right\} = L\left[EI\left(\frac{\pi}{L}\right)^2 - \lambda\right], \ \frac{d\overset{0}{\beta}(\lambda)}{d\lambda} = -L < 0$$

$$\text{Stability of principal branch changes at : } \boxed{\lambda_c = EI\left(\frac{\pi}{L}\right)^2,} \quad \overset{0}{\beta}(\lambda) > 0 \ \forall\lambda \in [0,\lambda_c)$$

ELASTICA EXAMPLE – CRITICAL LOAD AND MODE

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c),\lambda_c)\overset{1}{u})\delta u = \int_0^L \left[EI \left(\frac{d\overset{1}{\theta}}{ds} \right) \left(\frac{d\delta\theta}{ds} \right) - \lambda_c \cos(\overset{0}{\theta})\overset{1}{\theta}\delta\theta \right] ds = 0, \implies \text{(integrate by parts)}$$

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c),\lambda_c)\overset{1}{u})\delta u = \int_0^L \left[-EI \left(\frac{d^2\overset{1}{\theta}}{ds^2} \right) - \lambda_c \overset{1}{\theta} \right] \delta\theta ds + \left[EI \frac{d\overset{1}{\theta}}{ds} \delta\theta \right]_{s=L} - \left[EI \frac{d\overset{1}{\theta}}{ds} \delta\theta \right]_{s=0} = 0$$

Recall : $\delta\theta(s)$ arbitraty function $\in H_0^1[0,L]$; $\delta\theta(0) = \delta\theta(L) = 0, \implies$

$$\frac{d^2\overset{1}{\theta}}{ds^2} + \frac{\lambda_c}{EI}\overset{1}{\theta} = 0, \quad \overset{1}{\theta}(0) = \overset{1}{\theta}(L) = 0; \quad \text{critical load : } \lambda_c, \quad \text{mode : } \overset{1}{\theta} \left(\frac{1}{L}\int_0^L (\overset{1}{\theta})^2 ds = 1 \right)$$

$$\boxed{\lambda_c = EI(\pi/L)^2, \quad \overset{1}{\theta}(s) = \sqrt{2}\sin(\pi s/L);} \; (\aleph_0 \text{ eigenvalues : } EI(n\pi/L)^2, \text{ modes : } \sqrt{2}\sin(n\pi s/L))$$

ELASTICA EXAMPLE – LSK ASYMPTOTICS

$$\mathcal{E},^c_{u\lambda}\overset{1}{u} = -\int_0^L \sin(\overset{0}{\theta}(s))\overset{1}{\theta}(s)ds = 0, \quad ((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = \lambda_c \int_0^L \sin(\overset{0}{\theta}(s))[\overset{1}{\theta}(s)]^3 ds = 0$$

$$\text{Recall}: \ \lambda_2 = -\frac{1}{3}[(((\mathcal{E},^c_{uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E},^c_{uuu} v_{\xi\xi})\overset{1}{u})\overset{1}{u}]/((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$$

$$(((\mathcal{E}^c_{,uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} = \lambda_c \int_0^L \cos(\overset{0}{\theta}(s))[\overset{1}{\theta}(s)]^4 ds = \frac{3}{2}L\lambda_c = \frac{3}{2}EI\frac{\pi^2}{L}$$

$$((\mathcal{E}^c_{,uuu}v_{\xi\xi})\overset{1}{u})\overset{1}{u} = \lambda_c \int_0^L \sin(\overset{0}{\theta}(s))\theta_{\xi\xi}(s)[\overset{1}{\theta}(s)]^2 ds = 0$$

$$((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = -\int_0^L \cos(\overset{0}{\theta}(s))[\overset{1}{\theta}(s)]^2 ds = -L = \frac{d\overset{0}{\beta}(\lambda)}{d\lambda} \quad \text{(checks independently!)}$$

$$\lambda_2 = -\frac{1}{3}(\frac{3}{2}L\lambda_c)/(-L) = \frac{\lambda_c}{2} > 0, \quad \text{stable, supercritical, symmetric, bifurcation}$$

ELASTICA EXAMPLE – REVIEW

- Euler's elastica has a trivial principal solution (straight configuration) that changes stability at the critical load.

- Critical load and mode depend on boundary conditions.

- Bifurcation at critical load is a simple (unique eigenmode), symmetric bifurcation.

- Principal path changes stability at critical load.

- Bifurcated (symmetric) path is supercritical near the critical load, which implies that it is stable.

- Asymptotic results are confirmed by exact solution (in terms of elliptic integrals).

Exact:

$$L(\frac{\lambda}{EI})^{1/2} = 2K(\sin(\frac{\alpha}{2}))$$

$$K(k) = \int_0^1 \frac{dy}{[(1-y^2)(1-k^2y^2)]^{1/2}}$$

Asymptotic: $\lambda = EI(\frac{\pi}{L})^2[1 + \frac{1}{2}\sin^2(\frac{\alpha}{2}) + \ldots]$