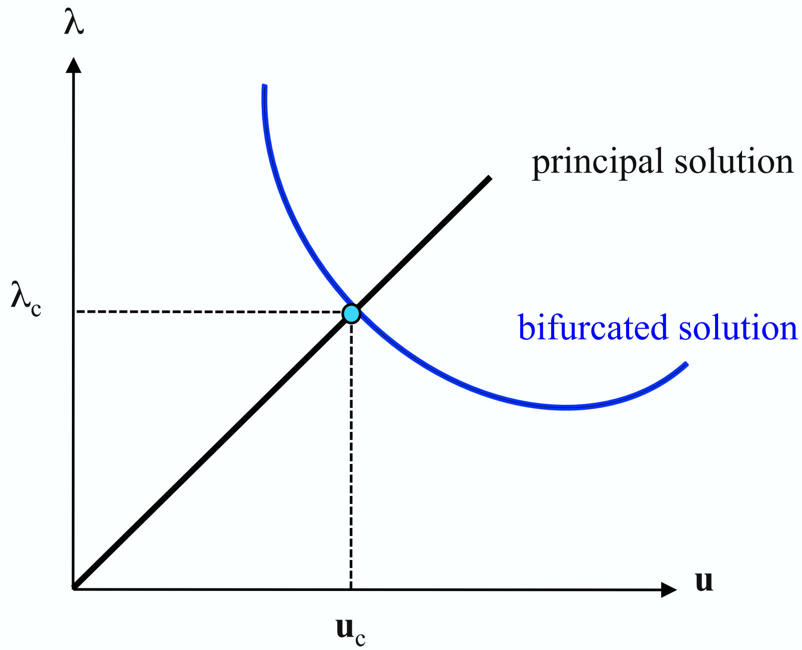
CONCEPT OF A BIFURCATION

FUNDAMENTAL ASSUMPTIONS USED:

- We study finite degree of freedom (u) systems
- Systems are time-independent
- Systems are conservative, i.e. they have an energy which remains constant
- Energy depends on a scalar parameter λ (termed load parameter)
- Systems are highly nonlinear, i.e. energy is non-quadratic function of d.o.f. and as a result for a given λ, multiple equilibrium solutions can be found
- Stability of these equilibrium solutions are examined by investigating if their energy has a local minimum at these solutions

CONCEPT OF A BIFURCATION



BIFURCATION: Loss of uniqueness – as a function of a control parameter – in the solution of a nonlinear system of equations. Bifurcated branch typically emerges as a "fork" from the principal branch.

- System: $\mathbf{f}(\mathbf{u},\lambda) = \mathbf{0}$
- Principal solution starts at $\lambda = 0$, $\mathbf{u} = \mathbf{0}$
- Bifurcated solution emerges from principal one at the critical point λ_c

$\mathcal{E}(\mathbf{u},\lambda)$: energy of system at displacement $\mathbf{u} \in \mathbb{R}^n$ and load $\lambda \geq 0$

$\mathbf{f}(\mathbf{u},\lambda) \equiv \mathcal{E}_{,\mathbf{u}} = \mathbf{0}$, equilibrium is energy extremum : $\mathcal{E}_{,\mathbf{u}} \equiv \partial\mathcal{E}/\partial\mathbf{u}$

$\overset{0}{\mathbf{u}}(\lambda)$: principal solution i.e. $\mathbf{f}(\overset{0}{\mathbf{u}}(\lambda),\lambda) = \mathbf{0}$, $\forall\lambda \geq 0$; $\overset{0}{\mathbf{u}}(0) = 0$

$$\mathcal{E}_{,\mathbf{uu}}\,\Delta\mathbf{u} + \mathcal{E}_{,\mathbf{u}\lambda}\,\Delta\lambda \approx \mathbf{0} \implies \Delta\mathbf{u} \approx -\Delta\lambda[\mathcal{E}_{,\mathbf{uu}}]^{-1}[\mathcal{E}_{,\mathbf{u}\lambda}]; \text{ construct } \overset{0}{\mathbf{u}}(\lambda) \text{ by continuation}$$

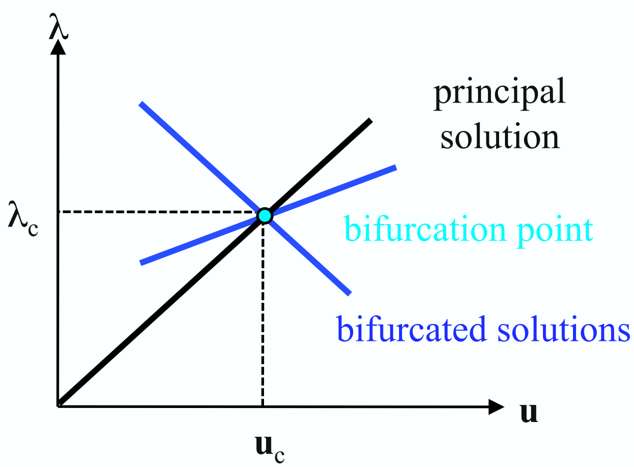
BIFURCATION POINT vs LIMIT LOAD

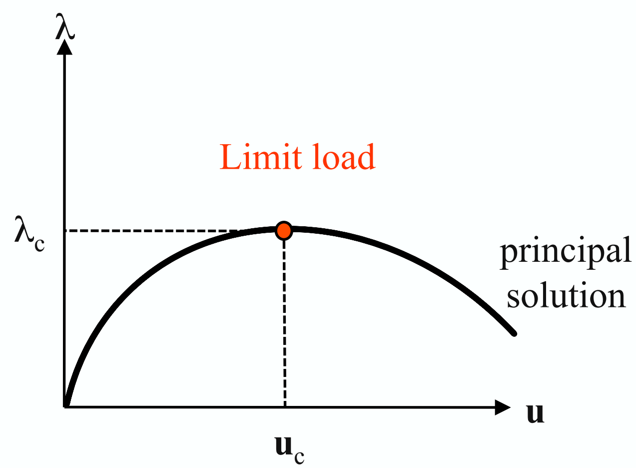
$$\mathcal{E}^c,_{\mathbf{uu}} \equiv \left[\frac{\partial^2 \mathcal{E}(\mathbf{u},\lambda)}{\partial \mathbf{u} \partial \mathbf{u}}\right]_{(\overset{0}{\mathbf{u}}(\lambda_c),\lambda_c)} \quad \text{non} - \text{invertible at } \overset{0}{\mathbf{u}}(\lambda_c) \implies \text{principal solution singularity}$$

$$[\mathcal{E}^c,_{\mathbf{uu}}]\,[\overset{(i)}{\mathbf{u}}] = \mathbf{0},\ i = 1, ..., m; \quad \lambda_c : \text{ critical load}, \quad \overset{(i)}{\mathbf{u}} : \text{ critical mode}, \quad m : \text{multiplicity}$$

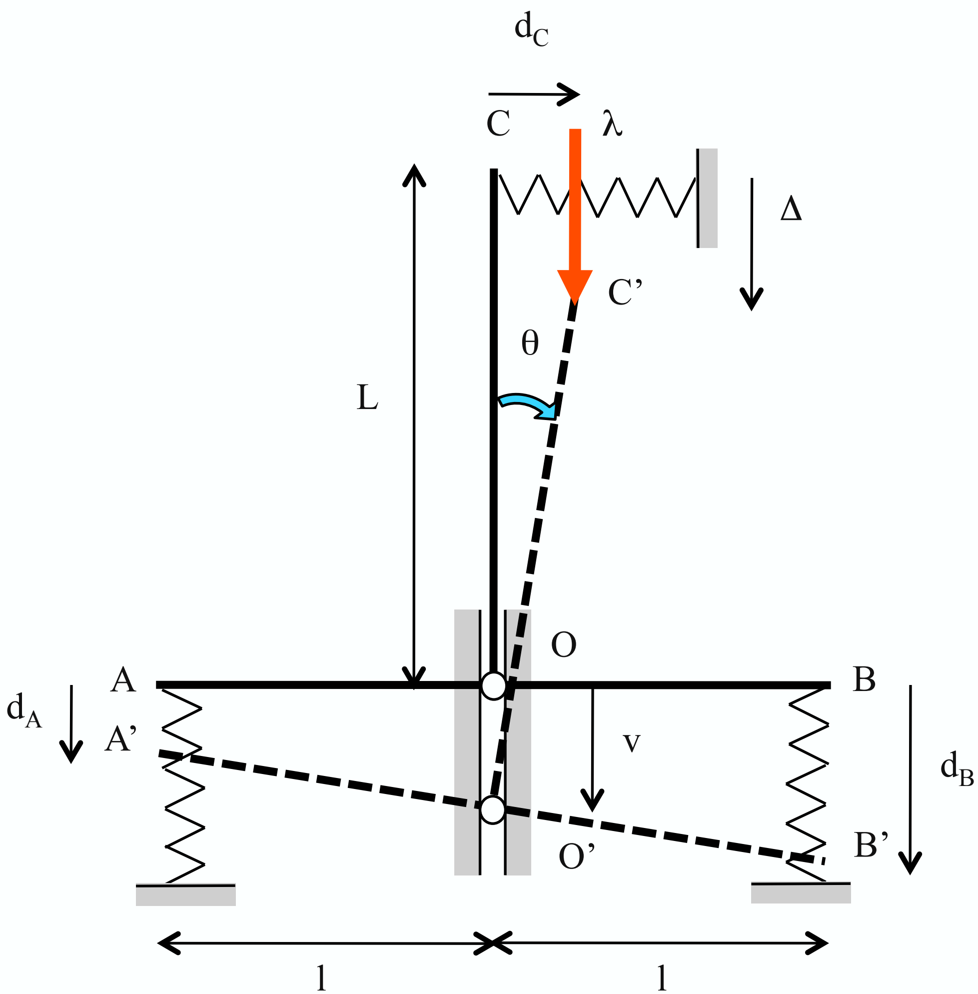
$$\text{if}: \ [\mathcal{E}^c,_{\mathbf{u}\lambda}]\,[\overset{(i)}{\mathbf{u}}] = 0 \quad \implies \quad \text{bifurcation at } \lambda_c$$

$$\text{if}: \ [\mathcal{E}^c,_{\mathbf{u}\lambda}]\,[\overset{(i)}{\mathbf{u}}] \neq 0 \quad \implies \quad \text{limit load at } \lambda_c$$





SIMPLE BIFURCATION EXAMPLE (m=1)



PERFECT RIGID T MODEL

$\mathbf{u} = (v, \theta)$

v: vertical displacement, θ: rotation

- Vertical linear springs at A, B
- Horizontal nonlinear spring at C
- Small rotations approximation
- Frictionless movement

SIMPLE BIFURCATION EXAMPLE (m=1)

KINEMATICS :

$d_A = v - l\theta, \quad d_B = v + l\theta;$ vertical displacements at A, B

$d_C = L\theta;$ horizontal displacement at C

$\Delta = v + L(1 - \cos\theta) \approx v + L\dfrac{\theta^2}{2};$ vertical displacement at C

ENERGY :

$\mathcal{E}_A = \dfrac{E}{2}(d_A)^2, \quad \mathcal{E}_B = \dfrac{E}{2}(d_B)^2;$ energy of springs A, B

$\mathcal{E}_C = \displaystyle\int_0^{d_C} [kx + mx^2 + nx^3]dx = \frac{k}{2}(d_C)^2 + \frac{m}{3}(d_C)^3 + \frac{n}{4}(d_C)^4;$ energy of spring C

$\mathcal{E} = \mathcal{E}_A + \mathcal{E}_B + \mathcal{E}_C - \lambda\Delta;$ total energy of structure

$$\mathcal{E}(\mathbf{u}, \lambda) = E[v^2 + (l\theta)^2] + \frac{k}{2}\,(L\theta)^2 + \frac{m}{3}\,(L\theta)^3 + \frac{n}{4}\,(L\theta)^4 - \lambda\,\left[v + L\frac{\theta^2}{2}\right]$$

PRINCIPAL & BIBURCATED SOLUTIONS OF PERFECT RIGID T MODEL

$$\mathbf{0} = \mathcal{E},_{\mathbf{u}} \equiv (\mathcal{E},_v , \mathcal{E},_\theta) = (0,0) : \quad \text{equilibrium}$$

$$\mathcal{E},_v = 2Ev - \lambda = 0,$$

$$\mathcal{E},_\theta = (2El^2 + kL^2)\theta + mL^3\theta^2 + nL^4\theta^3 - \lambda L\theta = 0$$

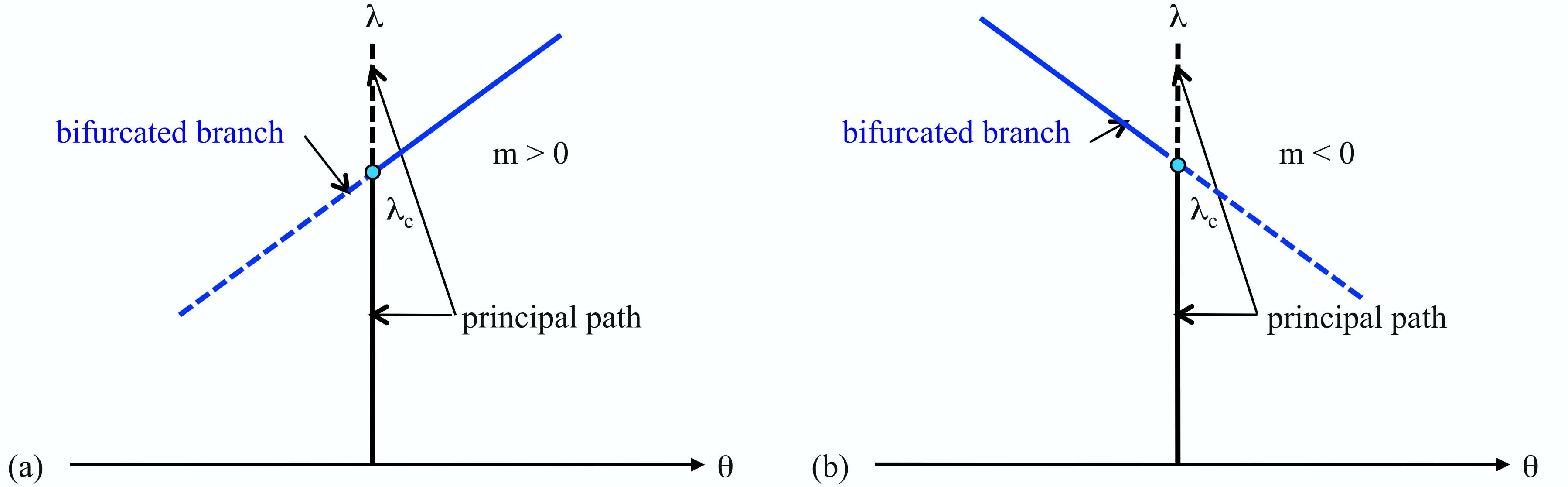
$$\overset{0}{\mathbf{u}} \equiv (\overset{0}{v}, \overset{0}{\theta}) = (\lambda/2E, 0) : \text{ principal solution (straight configuration)}$$

$$\mathbf{u}(\lambda) \equiv (v(\lambda), \theta(\lambda)) = (\lambda/2E, (\lambda - \lambda_c)/mL^2) : \text{ asymmetric bifurcation } (m \neq 0, n = 0)$$

$$\mathbf{u}(\lambda) \equiv (v(\lambda), \theta(\lambda)) = (\lambda/2E, \pm[(|\lambda - \lambda_c|)/nL^3]^{1/2}) : \text{ symmetric bifurcation } (m = 0, n \neq 0)$$

$$\lambda_c \equiv (2El^2 + kL^2)/L : \text{ critical load (where bifurcated solutions emmerge from)}$$

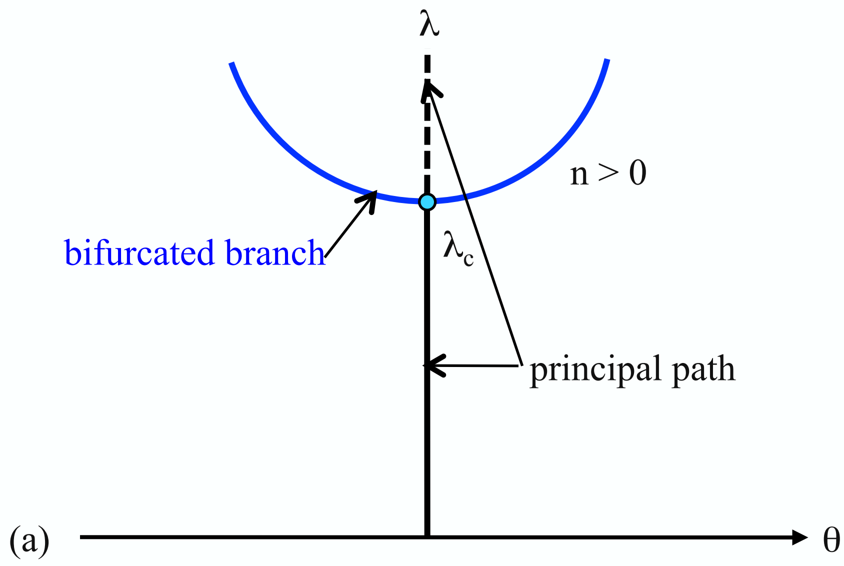
SIMPLE BIFURCATION EXAMPLE (m=1)



____ stable path, _ _ _ unstable path

ASYMMETRIC (TRANSCRITICAL) BIFURCATION

SIMPLE BIFURCATION EXAMPLE (m=1)



_____ stable path, _ _ _ unstable path

SYMMETRIC (SUPERCRITICAL $n>0$ **OR SUBCRITICAL** $n<0$**) BIFURCATION**

SIMPLE BIFURCATION EXAMPLE (m=1)

STABILITY OF SOLUTIONS FOR PERFECT RIGID T MODEL

$$\mathcal{E}_{,\mathbf{uu}} \equiv \begin{bmatrix} \mathcal{E}_{,vv} & \mathcal{E}_{,v\theta} \\ \mathcal{E}_{,\theta v} & \mathcal{E}_{,\theta\theta} \end{bmatrix} = \begin{bmatrix} 2E & 0 \\ 0 & (\lambda_c - \lambda)L + 2mL^3\theta + 3nL^4\theta^2 \end{bmatrix} \quad \mathcal{E}_{,\theta\theta} > 0 \implies \text{stable}$$

$$\begin{bmatrix} 2E & 0 \\ 0 & (\lambda_c - \lambda)L \end{bmatrix}$$ stability of principal path

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$$\begin{bmatrix} 2E & 0 \\ 0 & mL^3\theta \end{bmatrix}$$ stability of asymmetric bifurcated path $(m \neq 0, n = 0)$

$$\begin{bmatrix} 2E & 0 \\ 0 & 2nL^4\theta^2 \end{bmatrix}$$ stability of symmetric bifurcated path $(m = 0, n \neq 0)$

REVIEW OF PERFECT RIGID T MODEL

- Model has a simple bifurcation at the critical load
- Principal branch changes stability at critical load
- Bifurcated branches emerging from critical load are:

a) Stable if load increases (supercritical paths)

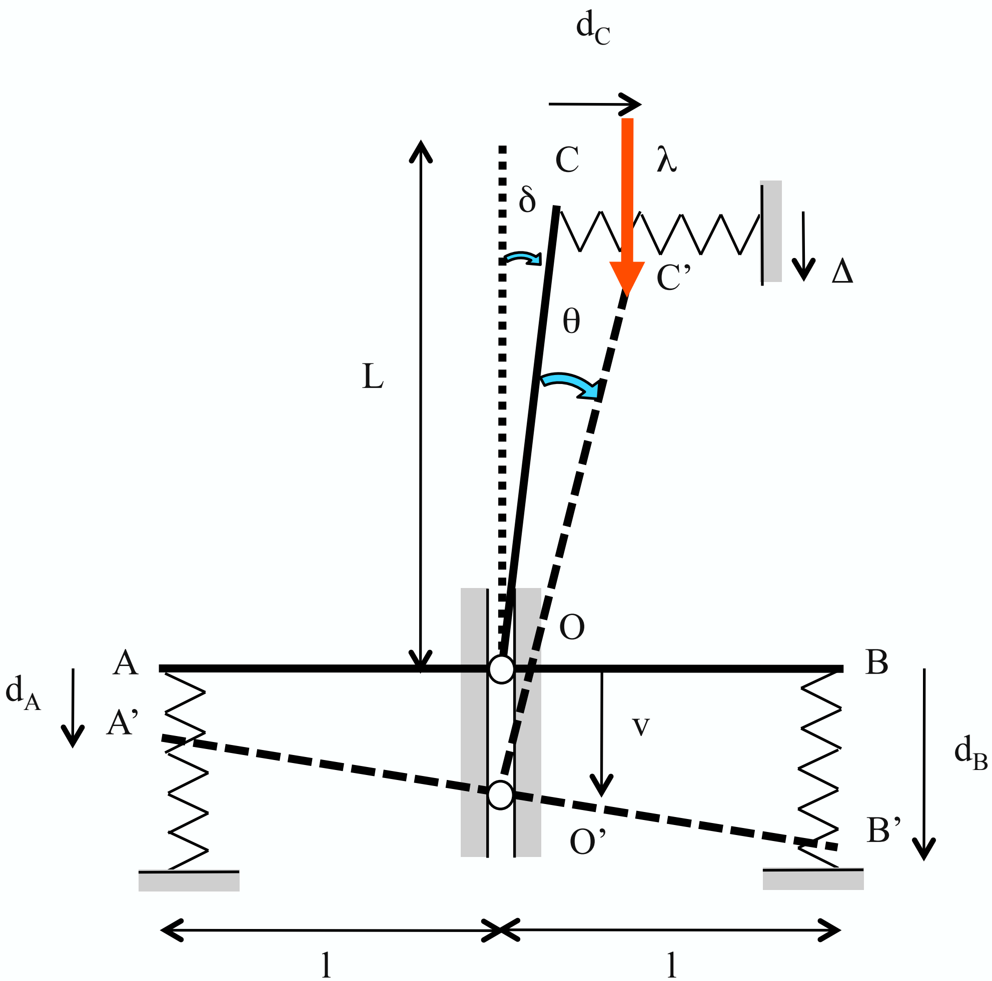
b) Unstable if load decreases (subcritical paths)

c) Stable solutions have, for a given load, less energy than their unstable counterparts at the same load

d) Unstable solutions have, for a given load, more energy than their stable counterparts at the same load

NOTE: Symmetric and asymmetric bifurcations are motivated by applications

SIMPLE BIFURCATION EXAMPLE (m=1)



IMPERFECT RIGID T MODEL

$\mathbf{u} = (v, \theta)$

v: vertical displacement

θ: rotation

δ: initial imperfection

Other assumptions same as in perfect T

NOTE: MANY DIFERENT WAYS TO MAKE AN IMPERFECT RIGID T MODEL, ALL ARE EQUIVALENT

SIMPLE BIFURCATION EXAMPLE (m=1)

KINEMATICS AND ENERGY OF THE IMPERFECT RIGID T MODEL

KINEMATICS :

$$\Delta = v + L[\cos\delta - \cos(\theta + \delta)] \quad \text{New exact vertical displacement at C}$$

$$\approx v + L\left(\frac{\theta^2}{2} + \theta\delta\right); \quad \text{New approximate vertical displacement at C}$$

ENERGY :

$$\overline{\mathcal{E}}(\mathbf{u}, \lambda; \delta) = E[v^2 + (l\theta)^2] + \frac{k}{2}(L\theta)^2 + \frac{m}{3}(L\theta)^3 + \frac{n}{4}(L\theta)^4 - \lambda\left[v + L\left(\frac{\theta^2}{2} + \boxed{\theta\delta}\right)\right]$$

extra term added for the imperfect model

EQUILIBRIUM SOLUTIONS OF IMPERFECT RIGID T MODEL

$$\mathbf{0} = \overline{\mathcal{E}},_{\mathbf{u}} \equiv (\overline{\mathcal{E}},_v , \overline{\mathcal{E}},_\theta) = (0,0) : \quad \text{equilibrium}$$

$$\overline{\mathcal{E}},_v = 2Ev - \lambda = 0,$$

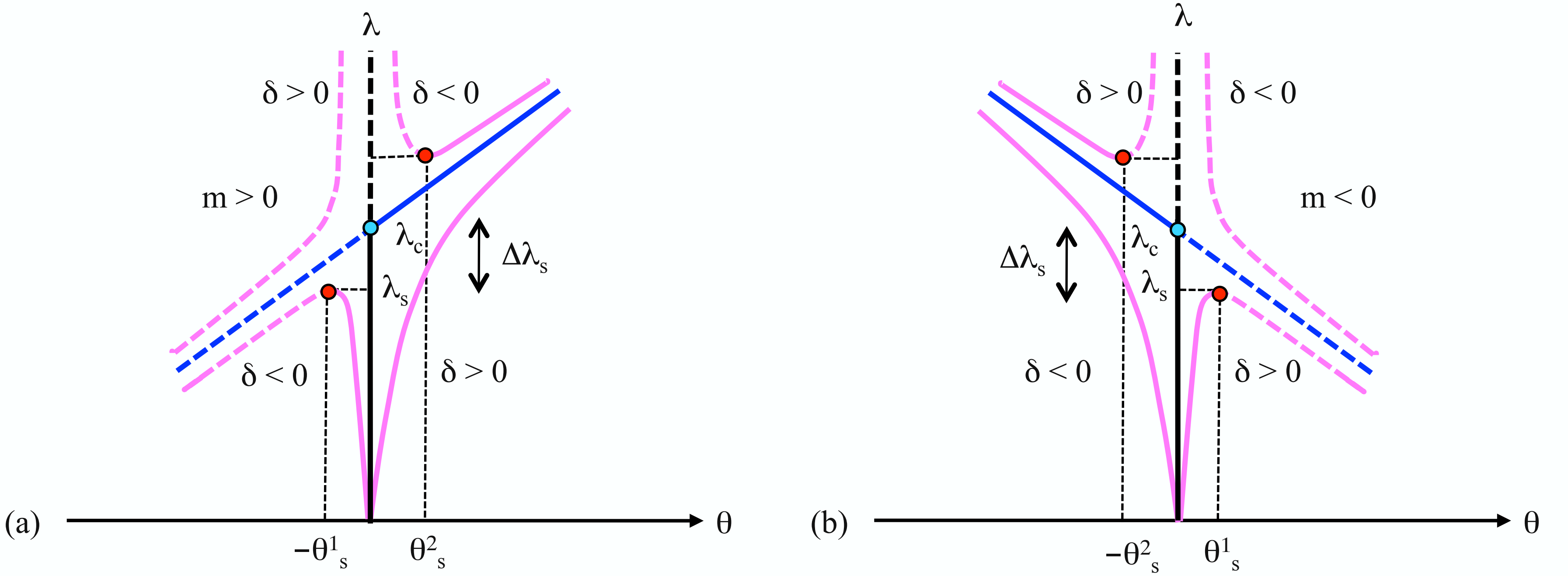
$$\overline{\mathcal{E}},_\theta = (2El^2 + kL^2)\theta + mL^3\theta^2 + nL^4\theta^3 - \lambda L(\theta + \delta) = 0$$

$$v = \lambda/2E, \ \lambda = \frac{\lambda_c\theta + m(L\theta)^2}{\theta + \delta}, \quad \text{for asymmetric case } (m \neq 0, \ n = 0)$$

$$v = \lambda/2E, \ \lambda = \frac{\lambda_c\theta + n(L\theta)^3}{\theta + \delta}, \quad \text{for symmetric case } (m = 0, \ n \neq 0)$$

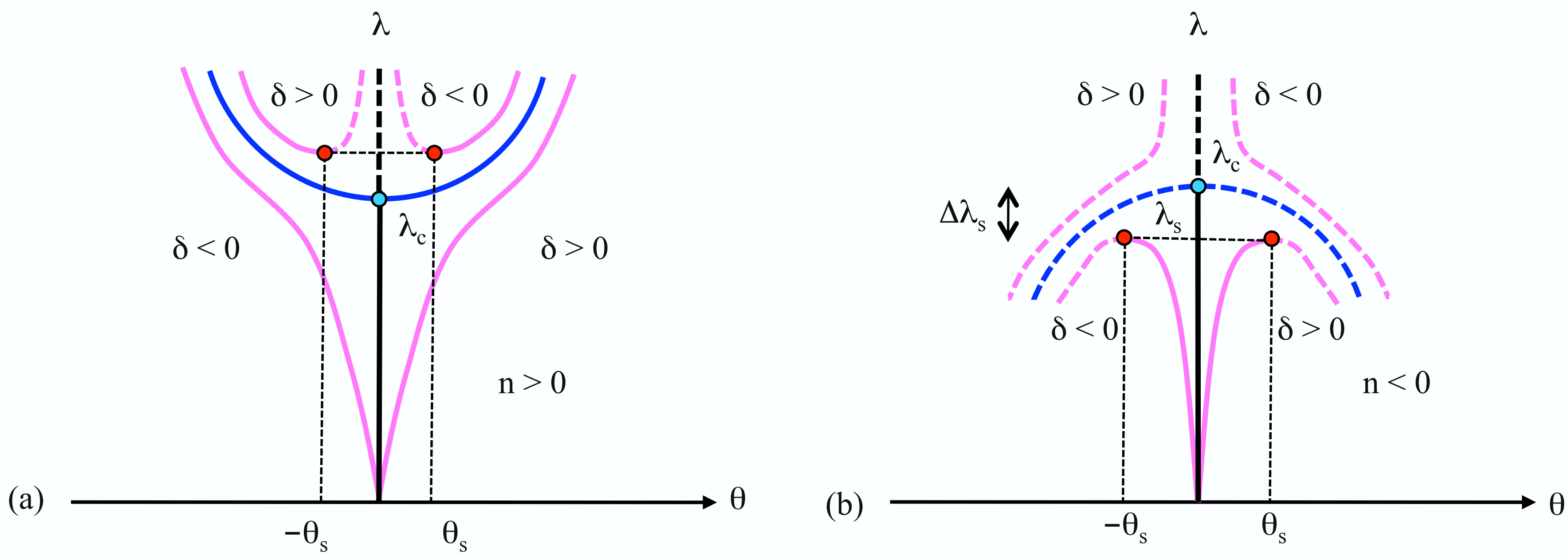
NOTE: IMPERFECT STRUCTURE DOES NOT HAVE BIFURCATION POINTS

ASYMMETRIC
IMPERFECT CASE



_____ stable path, _ _ _ unstable path ● limit load

SYMMETRIC
IMPERFECT CASE



_____ stable path, _ _ _ unstable path ● limit load

STABILITY OF SOLUTIONS FOR IMPERFECT RIGID T MODEL

$$\overline{\mathcal{E}}_{,\mathbf{uu}} \equiv \begin{bmatrix} \overline{\mathcal{E}}_{,vv} & \overline{\mathcal{E}}_{,v\theta} \\ \overline{\mathcal{E}}_{,\theta v} & \overline{\mathcal{E}}_{,\theta\theta} \end{bmatrix} = \begin{bmatrix} 2E & 0 \\ 0 & (\lambda_c - \lambda)L + 2mL^3\theta + 3nL^4\theta^2 \end{bmatrix} \quad \overline{\mathcal{E}}_{,\theta\theta} > 0 \implies \text{stability}$$

$$\begin{bmatrix} 2E & 0 \\ 0 & \dfrac{mL^3\theta(\theta+2\delta)+\lambda_c L\delta}{\theta+\delta} \end{bmatrix} \quad \text{limit load} : \frac{d\lambda}{d\theta} = \frac{mL^2\theta^2 + 2mL^2\theta\delta + \lambda_c\delta}{(\theta+\delta)^2} = 0 \ (n=0)$$

$$\begin{bmatrix} 2E & 0 \\ 0 & \dfrac{nL^4\theta^2(2\theta+3\delta)+\lambda_c L\delta}{\theta+\delta} \end{bmatrix} \quad \text{limit load} : \frac{d\lambda}{d\theta} = \frac{2nL^3\theta^3 + 3nL^3\theta^2\delta + \lambda_c\delta}{(\theta+\delta)^2} \ (m=0)$$

NOTE: STABILITY OF AN EQUILIBRIUM PATH CHANGES AT LIMIT LOAD $(d\lambda/d\theta(\theta_s) = 0)$

REVIEW OF IMPERFECT RIGID T MODEL

• Bifurcation point disappears and limit points appear in some of the equilibrium paths

• The physically relevant solution of the imperfect structure starts from zero load, follows the principal solution until near the critical load and then the closest bifurcation path of its perfect counterpart

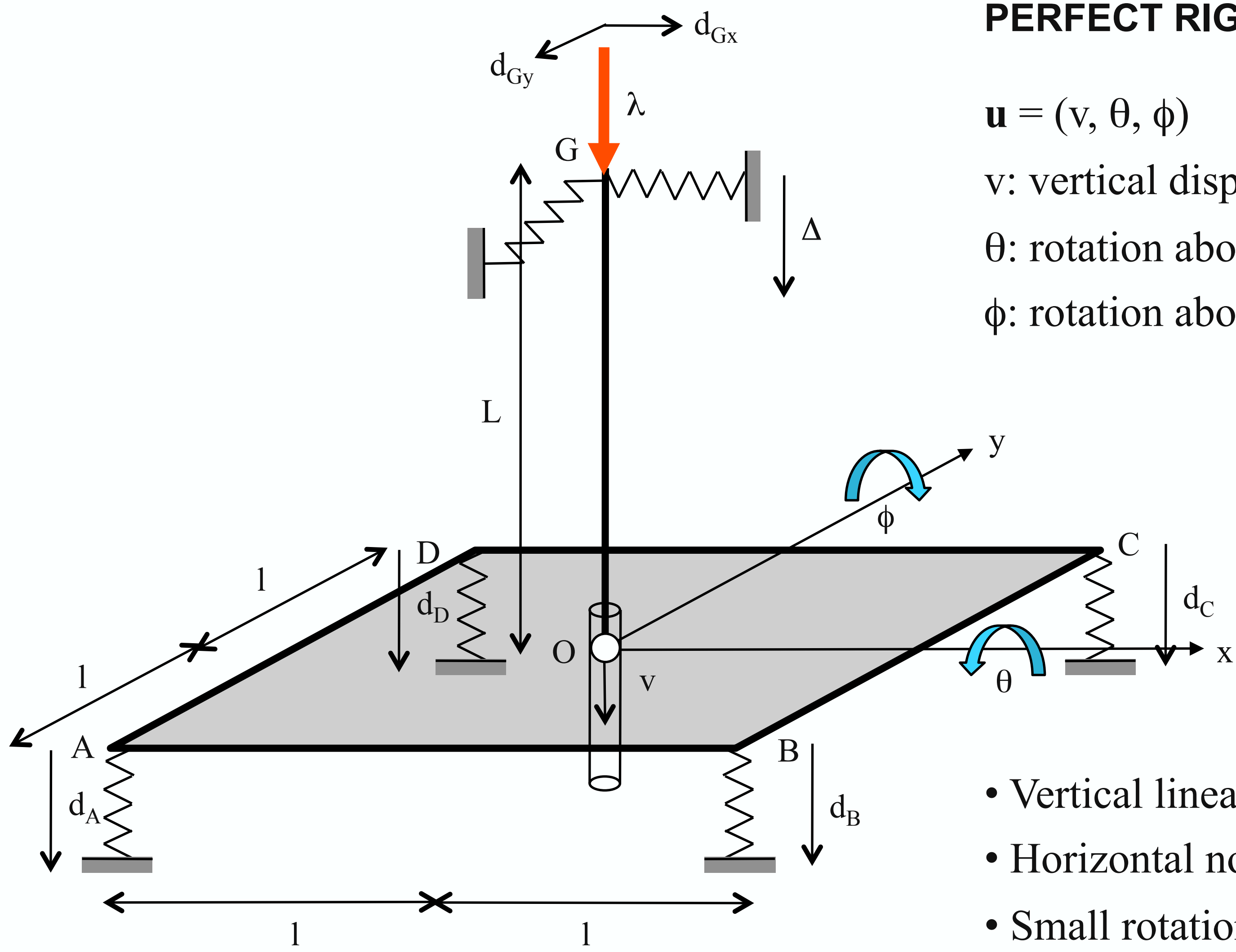
• In applications we can control the amplitude of the imperfection but not its shape.

a) Consequently for a perfect system with asymmetric or symmetric subcritical ($n<0$) bifurcations, its imperfect counterpart will always experience a path through zero load that exhibits limit loads near the critical load

b) For a perfect system with symmetric supercritical ($n>0$) bifurcation, its imperfect counterpart will not have in its path through zero load limit loads near the critical load

NOTE: Branches close to supercritical part of perfect solution cannot be reached in a continuous fashion from zero load due to energy barrier

MULTIPLE BIFURCATION EXAMPLE (m=2)



PERFECT RIGID PLATE MODEL

$\mathbf{u} = (v, \theta, \phi)$

v: vertical displacement

θ: rotation about x axis

ϕ: rotation about y axis

- Vertical linear springs at A, B, C, D
- Horizontal nonlinear springs at G
- Small rotations approximation
- Frictionless movement

MULTIPLE BIFURCATION EXAMPLE (m=2)

KINEMATICS :

$$d_A = v + l\theta - l\phi,\ d_B = v + l\theta + l\phi,\ d_C = v - l\theta + l\phi,\ d_D = v - l\theta - l\phi; \quad \text{at A, B, C, D}$$

$$d_{Gx} = L\phi \equiv d_x, \quad d_{Gy} = L\theta \equiv d_y; \quad \text{horizontal displacements at G}$$

$$\Delta = v + L[1 - (1 - \sin^2\theta - \sin^2\phi)^{1/2}] \approx v + L\left(\theta^2 + \phi^2\right)/2; \quad \text{vertical displacement at G}$$

ENERGY :

$$\mathcal{E}_A = \frac{E}{2}(d_A)^2,\ \mathcal{E}_B = \frac{E}{2}(d_B)^2,\ \mathcal{E}_C = \frac{E}{2}(d_C)^2,\ \mathcal{E}_D = \frac{E}{2}(d_D)^2;\ \text{energy of springs A, B, C, D}$$

$$F_x(d_x, d_y) = -[kd_x + m(d_x^2 + d_y^2) + n(2d_x^3 + 6d_x^2 d_y - 3d_x d_y^2 + 2d_y^3)]; \quad \text{x} - \text{force at G}$$

$$F_y(d_x, d_y) = -[kd_y + 2md_x d_y + n(2d_y^3 + 6d_y^2 d_x - 3d_y d_x^2 + 2d_x^3)]; \quad \text{y} - \text{force at G}$$

$$\mathcal{E}_G = \int_0^{(d_x, d_y)} [F_x(x,y)dx + F_y(x,y)dy];\ \text{energy of spring C};\ (\partial F_x/\partial d_y = \partial F_y/\partial d_x)$$

$$\mathcal{E} = \mathcal{E}_A + \mathcal{E}_B + \mathcal{E}_C + \mathcal{E}_D + \mathcal{E}_G - \lambda\Delta; \quad \text{total energy of structure}$$

ENERGY, EQUILIBRIUM EQUS & PRINCIPAL SOLUTION OF PERFECT RIGID PLATE MODEL

$$\mathcal{E}(\mathbf{u},\lambda) = 2E[v^2 + l^2(\theta^2+\phi^2)] + \frac{kL^2}{2}(\theta^2+\phi^2) + \frac{mL^3}{3}(3\theta^2\phi+\phi^3)$$

$$+\frac{nL^4}{4}(2\theta^4 + 8\theta^3\phi - 6\theta^2\phi^2 + 8\theta\phi^3 + 2\phi^4) - \lambda[v + \frac{L}{2}(\theta^2+\phi^2)]$$

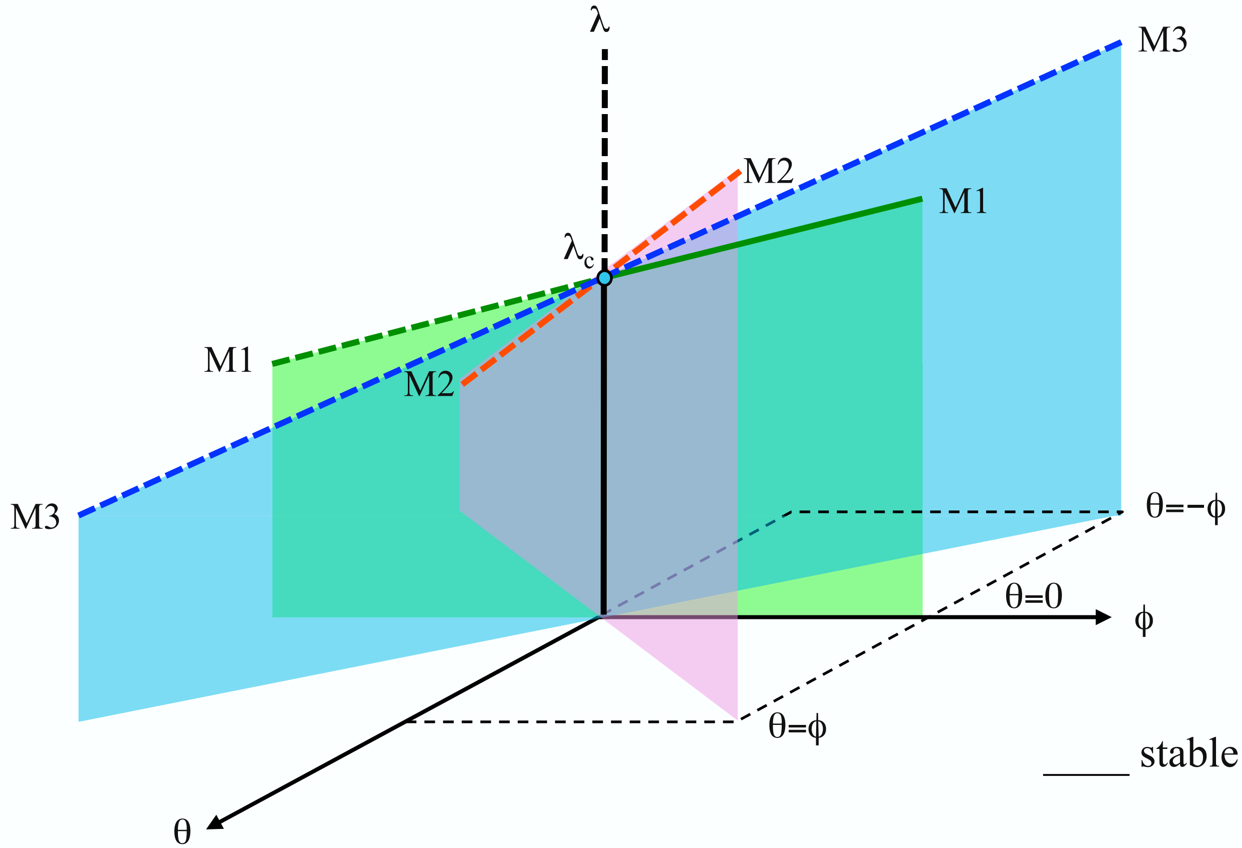
$$\mathbf{0} = \mathcal{E},_{\mathbf{u}} \equiv (\mathcal{E}_v, \mathcal{E}_\theta, \mathcal{E}_\phi) = (0,0,0) : \text{ equilibrium}$$

$$\mathcal{E},_v = 4Ev - \lambda = 0$$

$$\mathcal{E},_\theta = (4El^2 + kL^2 - \lambda L)\theta + 2mL^3\phi\theta + nL^4(2\theta^3 + 6\theta^2\phi - 3\theta\phi^2 + 2\phi^3) = 0$$

$$\mathcal{E},_\phi = (4El^2 + kL^2 - \lambda L)\phi + mL^3(\phi^2+\theta^2) + nL^4(2\phi^3 + 6\phi^2\theta - 3\phi\theta^2 + 2\theta^3) = 0$$

$$\overset{0}{\mathbf{u}} \equiv (\overset{0}{v}, \overset{0}{\theta}, \overset{0}{\phi}) = (\lambda/4E, 0, 0) : \text{ principal solution (straight configuration)}$$



ASYMMETRIC PERFECT PLATE'S EQUILIBRIUM PATHS

____ stable path, _ _ _ unstable path

$$M1: \quad \mathbf{u}(\lambda) \equiv (v(\lambda), \theta(\lambda), \phi(\lambda)) = (\lambda/4E,\ 0,\ (\lambda - \lambda_c)/mL^2)$$

$$M2: \quad \mathbf{u}(\lambda) \equiv (v(\lambda), \theta(\lambda), \phi(\lambda)) = (\lambda/4E,\ (\lambda - \lambda_c)/mL^2,\ (\lambda - \lambda_c)/mL^2)$$

$$M3: \quad \mathbf{u}(\lambda) \equiv (v(\lambda), \theta(\lambda), \phi(\lambda)) = (\lambda/4E,\ (\lambda_c - \lambda)/mL^2,\ (\lambda - \lambda_c)/mL^2)$$

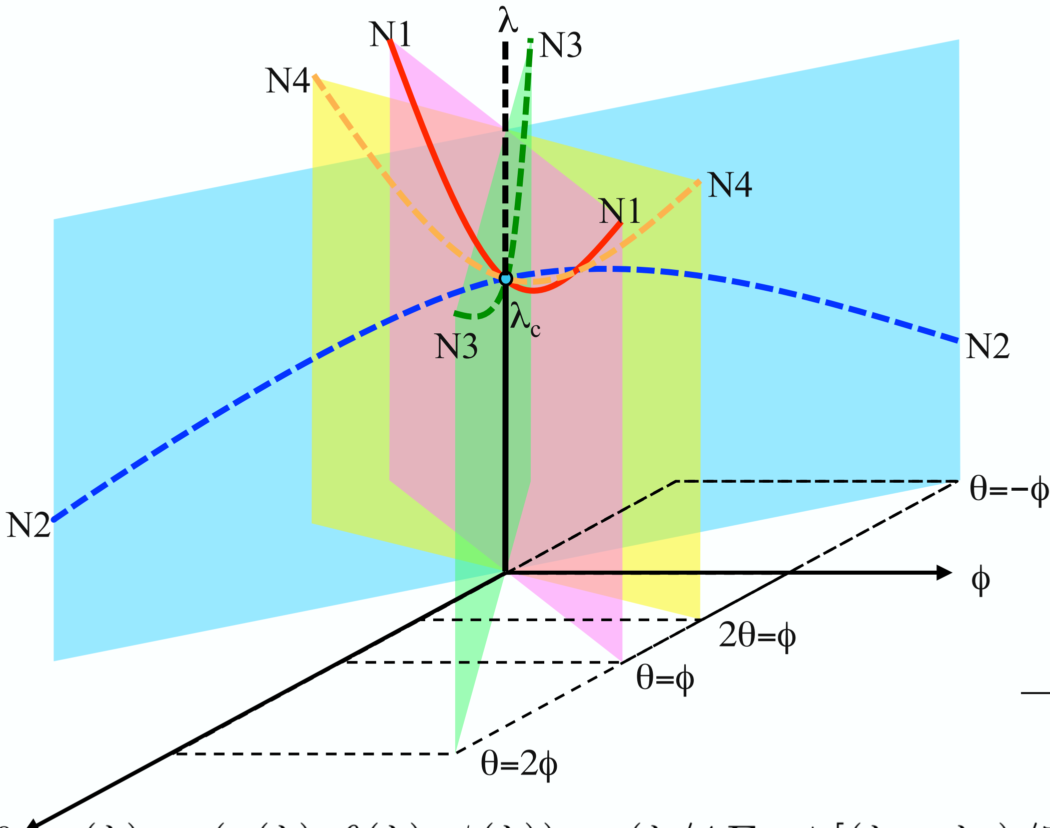
STABILITY OF ASYMMERIC BIFURCATED SOLUTIONS FOR PERFECT PLATE MODEL

$$M1: \quad \mathcal{E}_{,\mathbf{uu}} = \begin{bmatrix} 4E, & 0, & 0 \\ 0, & (\lambda-\lambda_c)L, & 0 \\ 0, & 0, & (\lambda-\lambda_c)L \end{bmatrix} \quad \text{Stable for } \lambda > \lambda_c$$

$$M2: \quad \mathcal{E}_{,\mathbf{uu}} = \begin{bmatrix} 4E, & 0, & 0 \\ 0, & 0, & (\lambda-\lambda_c)L \\ 0, & (\lambda-\lambda_c)L, & 0 \end{bmatrix} \quad \text{Unstable } \forall\lambda$$

$$M3: \quad \mathcal{E}_{,\mathbf{uu}} = \begin{bmatrix} 4E, & 0, & 0 \\ 0, & 0, & (\lambda_c-\lambda)L \\ 0, & (\lambda_c-\lambda)L, & 0 \end{bmatrix} \quad \text{Unstable } \forall\lambda$$

MULTIPLE BIFURCATION EXAMPLE (m=2)



SYMMETRIC PERFECT PLATE'S EQUILIBRIUM PATHS

____ stable path, _ _ _ unstable path

$$N1: \quad \mathbf{u}(\lambda) \equiv (v(\lambda), \theta(\lambda), \phi(\lambda)) = (\lambda/4E,\ \pm[(\lambda-\lambda_c)/7nL^3]^{1/2},\ \pm[(\lambda-\lambda_c)/7nL^3]^{1/2})$$

$$N2: \quad \mathbf{u}(\lambda) \equiv (v(\lambda), \theta(\lambda), \phi(\lambda)) = (\lambda/4E,\ \pm[(\lambda_c-\lambda)/9nL^3]^{1/2},\ \mp[(\lambda_c-\lambda)/9nL^3]^{1/2})$$

$$N3: \quad \mathbf{u}(\lambda) \equiv (v(\lambda), \theta(\lambda), \phi(\lambda)) = (\lambda/4E,\ \pm 2[(\lambda-\lambda_c)/18nL^3]^{1/2},\ \pm[(\lambda-\lambda_c)/18nL^3]^{1/2})$$

$$N4: \quad \mathbf{u}(\lambda) \equiv (v(\lambda), \theta(\lambda), \phi(\lambda)) = (\lambda/4E,\ \pm[(\lambda-\lambda_c)/18nL^3]^{1/2},\ \pm 2[(\lambda-\lambda_c)/18nL^3]^{1/2})$$

STABILITY OF SYMMETRIC BIFURCATED SOLUTIONS FOR PERFECT PLATE MODEL

$$N1: \quad \mathcal{E},_{\mathbf{uu}} = \begin{bmatrix} 4E, & 0, & 0 \\ 0, & (8/7)(\lambda - \lambda_c)L, & (6/7)(\lambda - \lambda_c)L \\ 0, & (6/7)(\lambda - \lambda_c)L, & (8/7)(\lambda - \lambda_c)L \end{bmatrix} \quad \text{stable for } \lambda > \lambda_c$$

$$N2: \quad \mathcal{E},_{\mathbf{uu}} = \begin{bmatrix} 4E, & 0, & 0 \\ 0, & 0, & 2(\lambda_c - \lambda)L \\ 0, & 2(\lambda_c - \lambda)L, & 0 \end{bmatrix} \quad \text{unstable } \forall\lambda$$

$$N3: \quad \mathcal{E},_{\mathbf{uu}} = \begin{bmatrix} 4E, & 0, & 0 \\ 0, & (3/2)(\lambda - \lambda_c)L, & (\lambda - \lambda_c)L \\ 0, & (\lambda - \lambda_c)L, & 0 \end{bmatrix} \quad \text{unstable } \forall\lambda$$

$$N4: \quad \mathcal{E},_{\mathbf{uu}} = \begin{bmatrix} 4E, & 0, & 0 \\ 0, & 0, & (\lambda - \lambda_c)L \\ 0, & (\lambda - \lambda_c)L, & (3/2)(\lambda - \lambda_c)L \end{bmatrix} \quad \text{unstable } \forall\lambda$$

REVIEW OF PERFECT RIGID PLATE MODEL

- **Model has a multiple** $(m = 2)$ **bifurcation at the critical load**
- **Principal branch changes stability at critical load**
- **Bifurcated branches emerging from critical load are:**

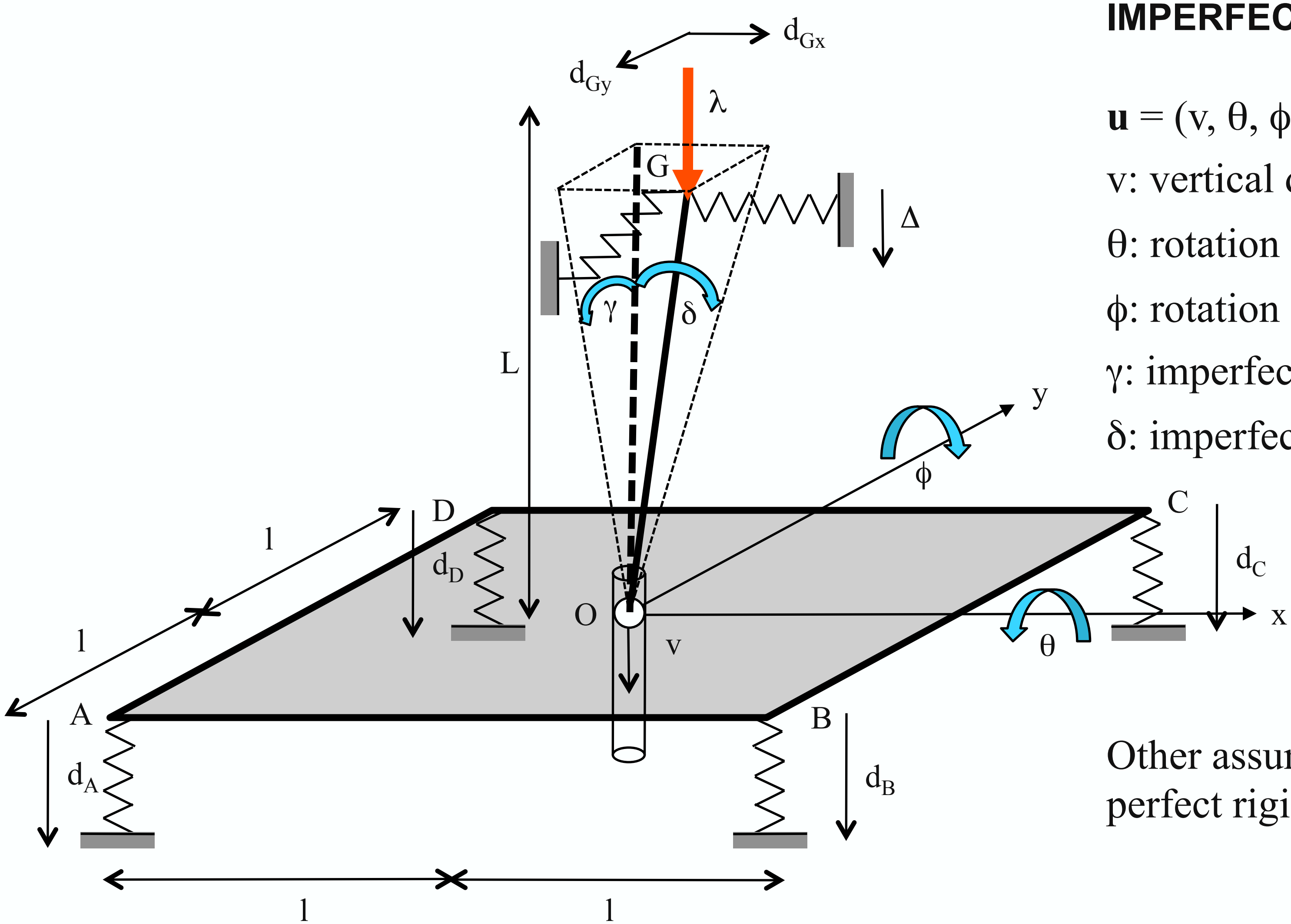
  a) **At most** $2^m - 1$ $(2^2 - 1 = 3)$ **in asymmetric systems**

  b) **At most** $(3^m - 1)/2$ $((3^2 - 1)/2 = 4)$ **in symmetric systems**

  c) **There is no general result about the stability of the bifurcated paths and unstable supercritical paths can be found for both the asymmetric and symmetric systems**

NOTE: Multiple bifurcations are motivated by applications in systems with a high degree of geometric symmetry (e.g. cylinders, crystals)

MULTIPLE BIFURCATION EXAMPLE (m=2)



IMPERFECT PLATE MODEL

$\mathbf{u} = (v, \theta, \phi)$, $\boldsymbol{\varepsilon} = (\gamma, \delta)$

v: vertical displacement

θ: rotation about x axis

ϕ: rotation about y axis

γ: imperfection about x axis

δ: imperfection about y axis

Other assumptions same as in perfect rigid plate model

KINEMATICS AND ENERGY OF THE IMPERFECT RIGID PLATE MODEL

KINEMATICS :

$$\Delta = v + L[(1 - \sin^2\gamma - \sin^2\delta)^{1/2} - (1 - \sin^2(\theta+\gamma) - \sin^2(\phi+\delta)^{1/2}]$$

$$\approx v + L\left(\frac{\theta^2+\phi^2}{2} + \theta\gamma + \phi\delta\right); \quad \text{New vertical displacement at G :}$$

ENERGY :

$$\overline{\mathcal{E}}(\mathbf{u},\lambda;\boldsymbol{\epsilon}) = 2E[v^2 + l^2(\theta^2+\phi^2)] + \frac{kL^2}{2}(\theta^2+\phi^2) + \frac{mL^3}{3}(3\theta^2\phi + \phi^3) +$$

$$+\frac{nL^4}{4}(2\theta^4 + 8\theta^3\phi - 6\theta^2\phi^2 + 8\theta\phi^3 + 2\phi^4) - \lambda[v + \frac{L}{2}(\theta^2 + \phi^2 + \boxed{2\gamma\theta + 2\delta\phi)}]$$

extra term added for the imperfect model

EQUILIBRIUM SOLUTIONS OF IMPERFECT RIGID PLATE MODEL

$$\mathbf{0} = \overline{\mathcal{E}}_{,\mathbf{u}} \equiv (\overline{\mathcal{E}}_{,v}, \overline{\mathcal{E}}_{,\theta}, \overline{\mathcal{E}}_{,\phi}) = (0,0,0): \quad \text{equilibrium}$$

$$\overline{\mathcal{E}}_{,v} = 4Ev - \lambda = 0$$

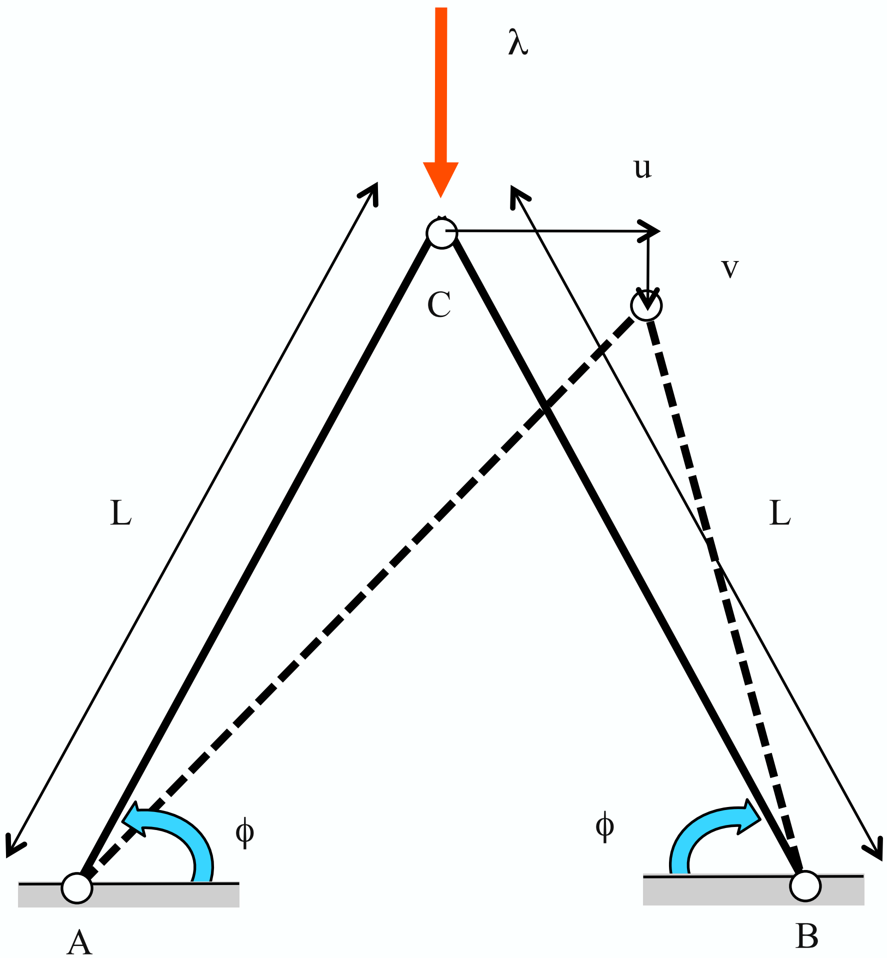
$$\overline{\mathcal{E}}_{,\theta} = -\lambda L\gamma + (\lambda_c - \lambda)L\theta + 2mL^3\theta\phi + nL^4(2\theta^3 + 6\theta^2\phi - 3\theta\phi^2 + 2\phi^3) = 0$$

$$\overline{\mathcal{E}}_{,\phi} = -\lambda L\delta + (\lambda_c - \lambda)L\phi + mL^3(\phi^2 + \theta^2) + nL^4(2\phi^3 + 6\phi^2\theta - 3\phi\theta^2 + 2\theta^3) = 0$$

•The physically relevant solution (i.e. the one that goes through zero load) of the imperfect structure skirts the principal path up to near critical load and then, depending on the shape (γ/δ) of the imperfection, it can skirt any of the bifurcated paths of its perfect counterpart.

• The worst imperfection shape senario is the one that follows the perfect system's bifurcated path with the steepest load decrease, i.e. M1 in the asymmetric case and N2 in the symmetric case.

LIMIT POINT EXAMPLE



PERFECT TWO BAR TRUSS MODEL

$\mathbf{u} = (u, v)$

u: horizontal displacement at C

v: vertical displacement at C

- Bars deform axially (no bending)
- Large displacements & rotations
- Small strains, elastic response
- Frictionless joints

LIMIT POINT EXAMPLE

KINEMATICS AND ENERGY OF THE PERFECT TWO BAR TRUSS MODEL

KINEMATICS :

$$e_i \equiv \frac{\ell_i - L}{L}, \ |e_i| \ll 1; \quad \text{engineering strain in bar } i$$

$$(\ell_1)^2 = (L\cos\phi + u)^2 + (L\sin\phi - v)^2, \quad \text{final length of bar 1}$$

$$(\ell_2)^2 = (L\cos\phi - u)^2 + (L\sin\phi - v)^2, \quad \text{final length of bar 2}$$

$$\varepsilon_i \equiv \frac{1}{2}\left[\left(\frac{\ell_i}{L}\right)^2 - 1\right] \approx \left(1 + \frac{e_i}{2}\right) e_i; \quad \text{strain measure in bar } i$$

ENERGY :

$$\mathcal{E}(\mathbf{u}, \lambda) = \frac{1}{2} EAL[(\varepsilon_1)^2 + (\varepsilon_2)^2] - \lambda v$$

LIMIT POINT EXAMPLE

EQUILIBRIUM SOLUTIONS OF PERFECT TRUSS MODEL

$$\mathcal{E}_{,u} = \frac{EA}{L}[\varepsilon_1(u + L\cos\phi) + \varepsilon_2(u - L\cos\phi)] = 0$$

$$\mathcal{E}_{,v} = \frac{EA}{L}[\varepsilon_1(v - L\sin\phi) + \varepsilon_2(v - L\sin\phi)] - \lambda = 0$$

$$\frac{EA}{L^3}\overset{0}{v}(\lambda)[\overset{0}{v}(\lambda) - 2L\sin\phi][\overset{0}{v}(\lambda) - L\sin\phi] - \lambda = 0,\ \overset{0}{u}(\lambda) = 0 : \quad \text{principal solution}$$

$$u^2 + (v - L\sin\phi)^2 = L^2(3\sin^2\phi - 2) :$$

bifurcated solution

$$\lambda = 2EA\cos^2\phi(\sin\phi - v/L) :$$

LIMIT POINT EXAMPLE

Stability of principal path (force control) :

$$\mathcal{E}_{,\mathbf{uu}} = \frac{EA}{L}\begin{bmatrix} (\overset{0}{v}/L)^2 - 2(\overset{0}{v}/L)\sin\phi + 2\cos^2\phi & 0 \\ 0 & 3[(\overset{0}{v}/L)^2 - 2(\overset{0}{v}/L)\sin\phi + \frac{2}{3}\sin^2\phi] \end{bmatrix}$$

$$\mathcal{E}_{,uu} = 0 \implies \overset{0}{v}/L = \sin\phi \pm (3\sin^2\phi - 2)^{1/2} : \text{ bifurcation points}$$

$$\mathcal{E}_{,vv} = 0 \implies \overset{0}{v}/L = \sin\phi(1 \pm 1/\sqrt{3}) : \text{ limit loads } (d\overset{0}{v}/d\lambda = 0)$$

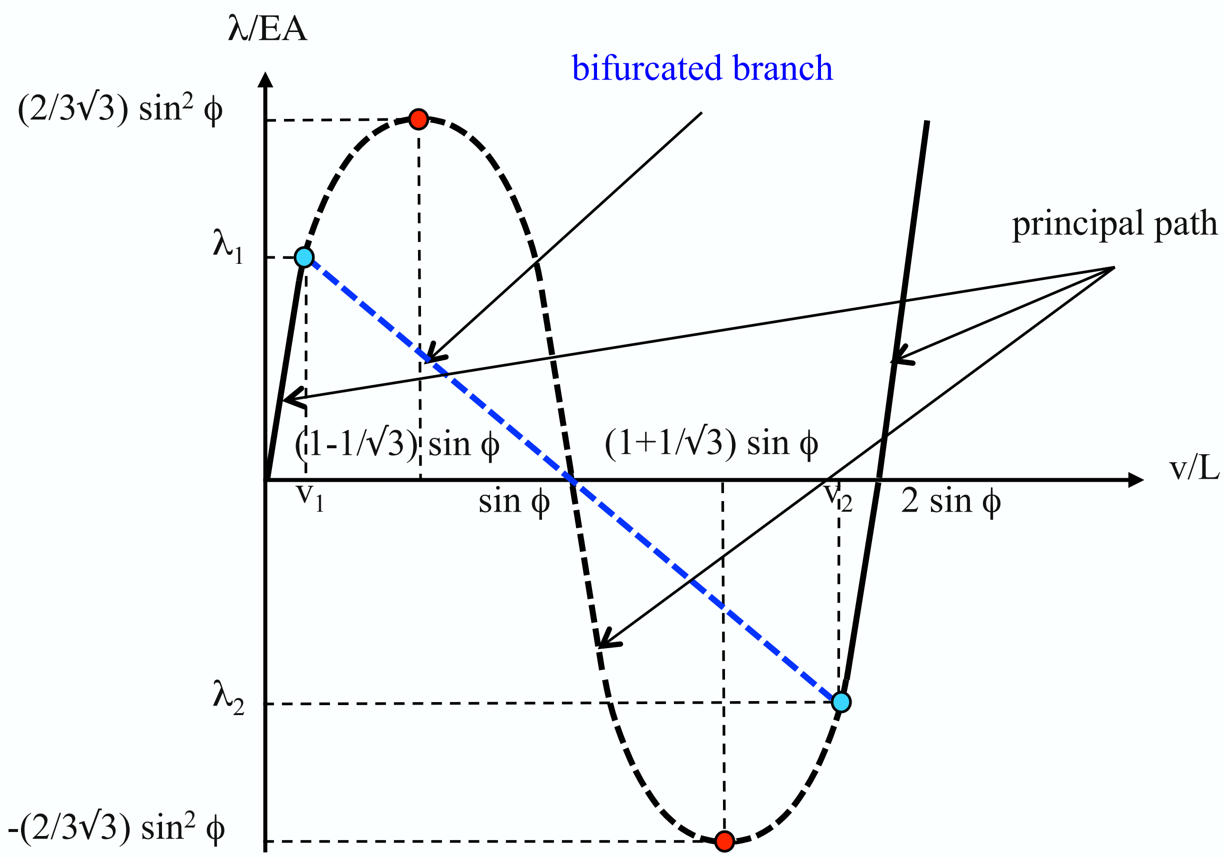
Stability of bifurcated path (force control) :

$$\mathcal{E}_{,\mathbf{uu}} = \frac{EA}{L}\begin{bmatrix} 2(u/L)^2 & 2(u/L)[(v/L) - \sin\phi] \\ 2(u/L)[(v/L) - \sin\phi] & 2[-(u/L)^2 + 4\sin^2\phi - 3] \end{bmatrix}$$

$$\text{Det}[\mathcal{E}_{,\mathbf{uu}}] = \frac{4EA}{L}\left(\frac{u}{L}\right)^2[\sin^2\phi - 1] < 0 \implies \text{ bifurcated path unstable}$$

LIMIT POINT EXAMPLE

PERFECT TRUSS RESULTS IN $\lambda - (v/L)$ **SPACE FOR:** $90^o > \phi > 60^o$ **(FORCE CTRL.)**

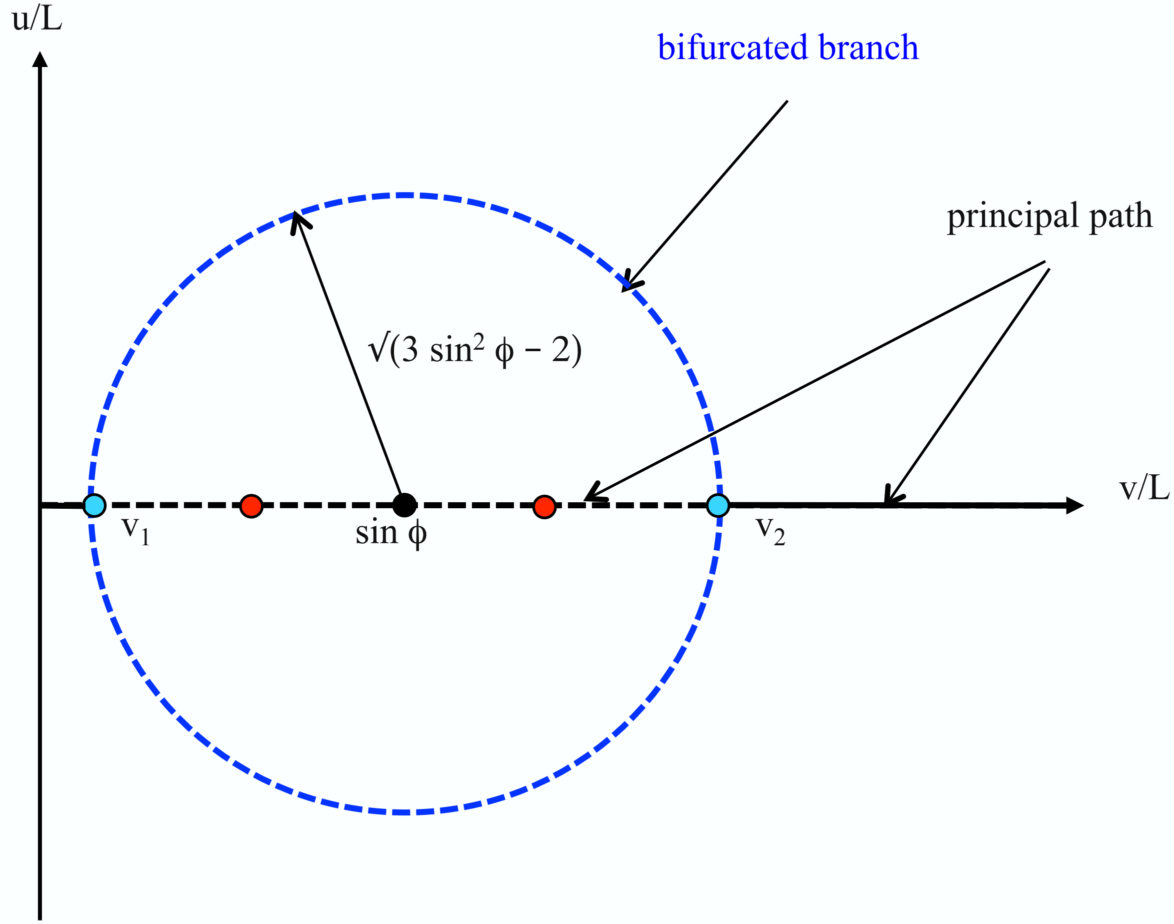


- Principal solution has limit loads and bifurcation points
- Bifurcated solution emerges from principal one before maximum and after minimum load
- Principal solution changes stability at bifurcation points
- Bifurcated solution is unstable for the case of load control

____ stable path, _ _ _ unstable path ● limit load ● bifurcation point

LIMIT POINT EXAMPLE

PERFECT TRUSS RESULTS IN (u/L) – (v/L) **SPACE FOR:** $90^o > \phi > 60^o$ **(FORCE CTRL.)**



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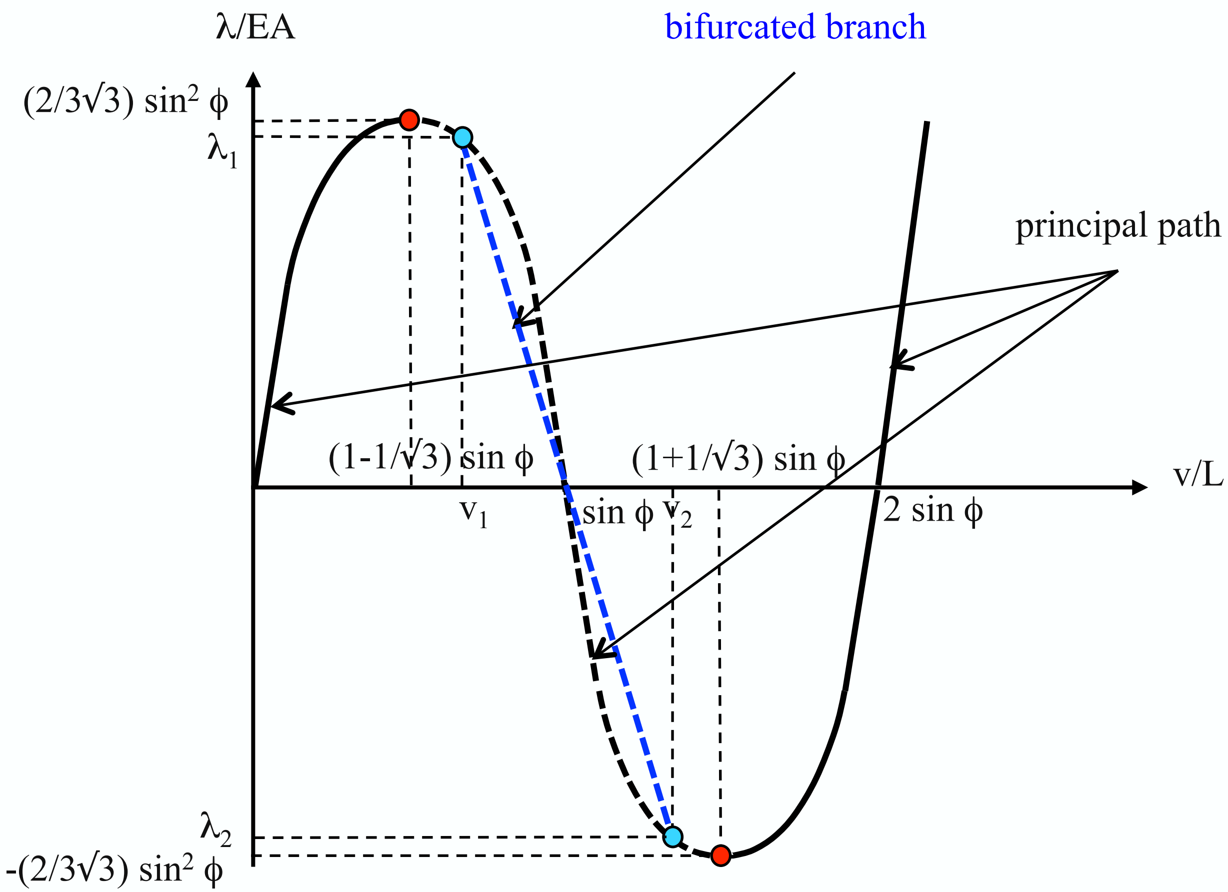
LIMIT POINT EXAMPLE

PERFECT TRUSS RESULTS IN $\lambda - (v/L)$ SPACE FOR: $60^o > \phi > 54.7^o$ (FORCE CTRL.)

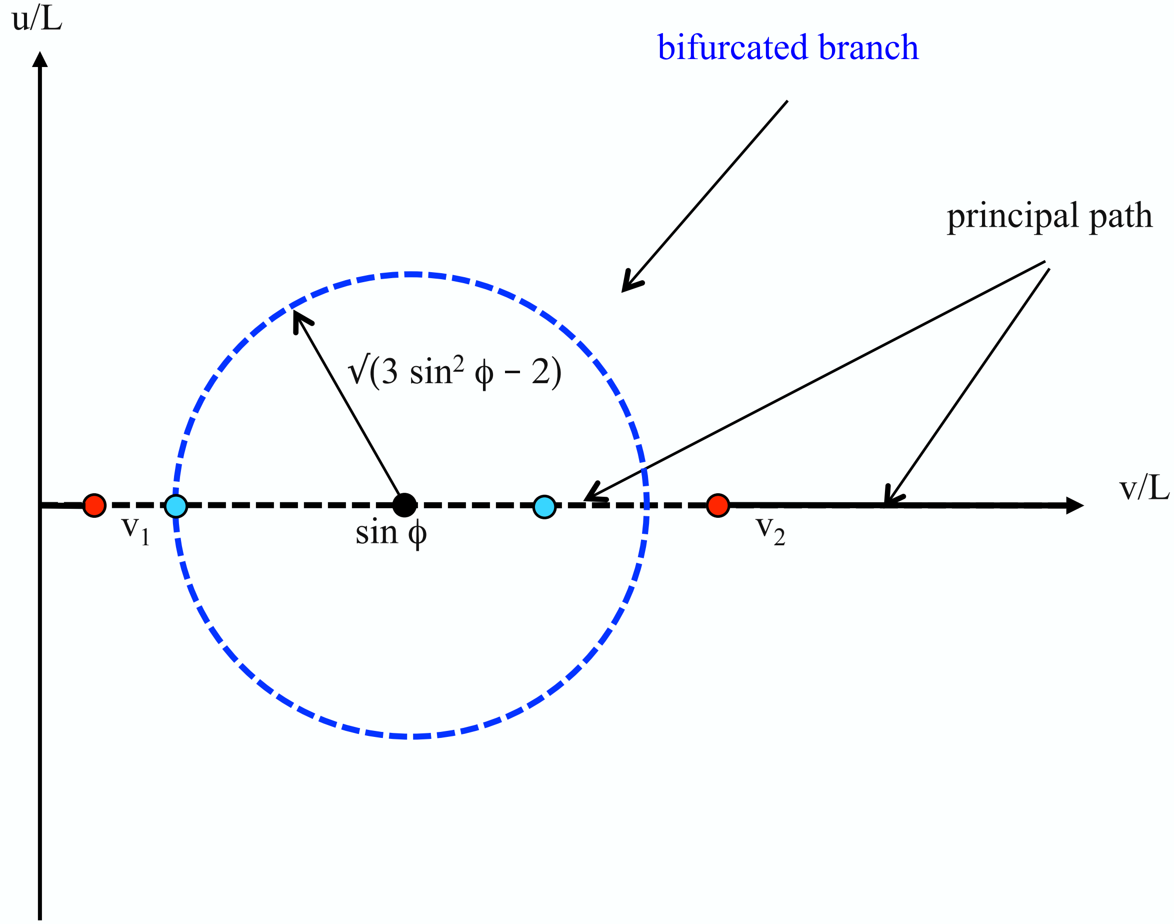


- Principal solution has limit loads and bifurcation points
- Bifurcated solution emerges from principal one after maximum and before minimum load
- Principal solution changes stability at limit points
- Bifurcated solution is unstable for the case of load control

____ stable path, _ _ _ unstable path ● limit load ● bifurcation point

LIMIT POINT EXAMPLE

PERFECT TRUSS RESULTS IN (u/L) – (v/L) **SPACE FOR:** $60^o > \phi > 54.7^o$ **(FORCE CTRL.)**

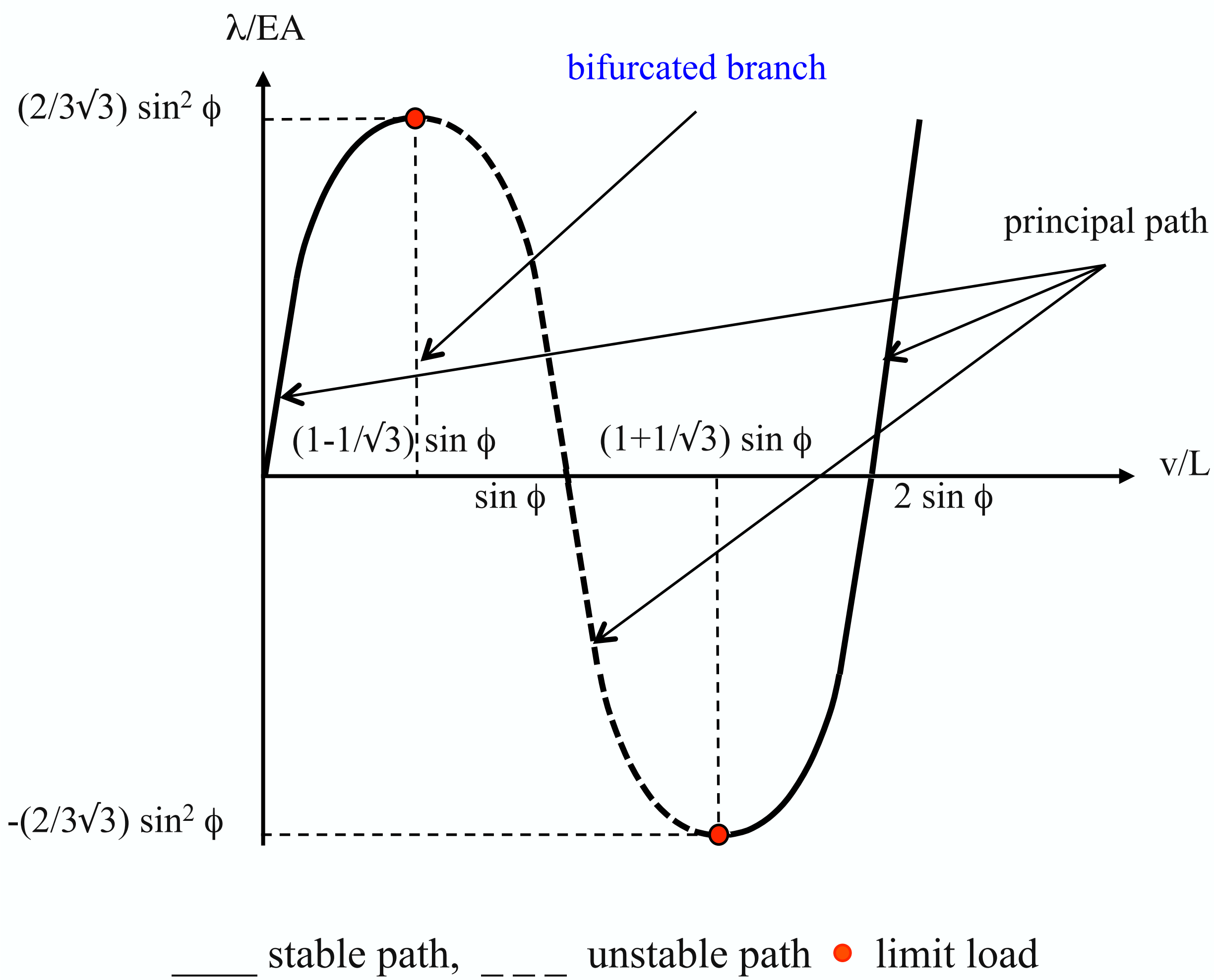


- Principal solution has limit loads and bifurcation points
- Bifurcated solution emerges from principal one after maximum and before minimum load
- Principal solution changes stability at limit points
- Bifurcated solution is unstable for the case of load control

____ stable path, _ _ _ unstable path ● limit load ● bifurcation point

LIMIT POINT EXAMPLE

PERFECT TRUSS RESULTS IN λ – (v/L) **SPACE FOR:** $54.7^o > \phi > 0^o$ **(FORCE CTRL.)**



- Principal solution has limit loads
- No bifurcated solution exists in this case
- Principal solution changes stability at limit loads

LIMIT POINT EXAMPLE

Energy for displacement control :

$$\mathcal{E}(u,\lambda) = \frac{1}{2}EAL[(\varepsilon_1)^2 + (\varepsilon_2)^2]; \quad \text{only one d.o.f. here : } u$$

Equilibrium solutions for displacement control :

$$\mathcal{E}_{,u} = (EA/L)[\varepsilon_1(u + L\cos\phi) + \varepsilon_2(u - L\cos\phi)] = 0$$

Principal solution : $\overset{0}{u} = 0$

Bifurcated solution : $u^2 + (v - L\sin\phi)^2 = L^2(3\sin^2\phi - 2)$

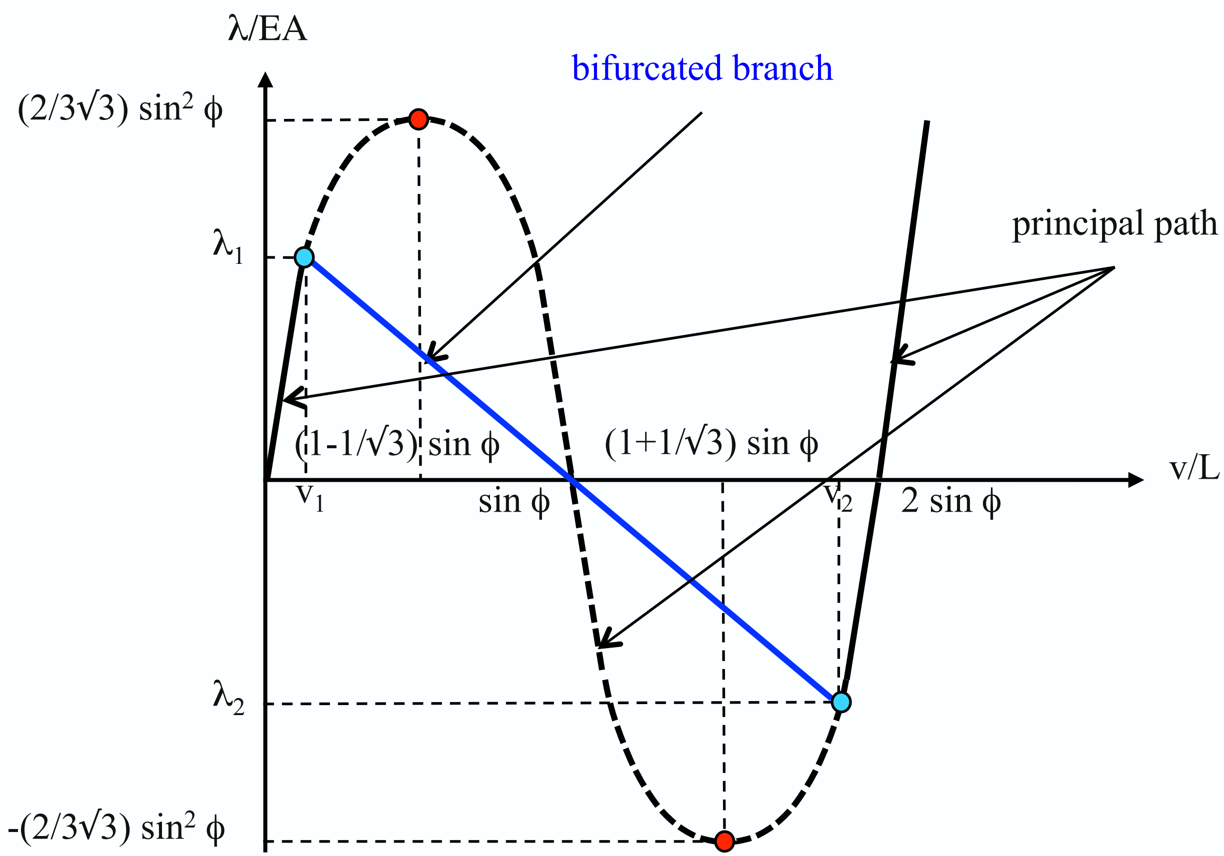
Stability for displacement control :

Principal : $\mathcal{E}_{,uu} = (EA/L)[(\overset{0}{v}/L)^2 - 2(\overset{0}{v}/L)\sin\phi + 2\cos^2\phi]$, changes at bifurcation

Bifurcated : $\mathcal{E}_{,uu} = 2(EA/L)[(u/L)^2] > 0$, always stable!

LIMIT POINT EXAMPLE

PERFECT TRUSS RESULTS IN λ – (v/L) SPACE FOR: $90^o > \phi > 60^o$ (DISPL. CTRL.)

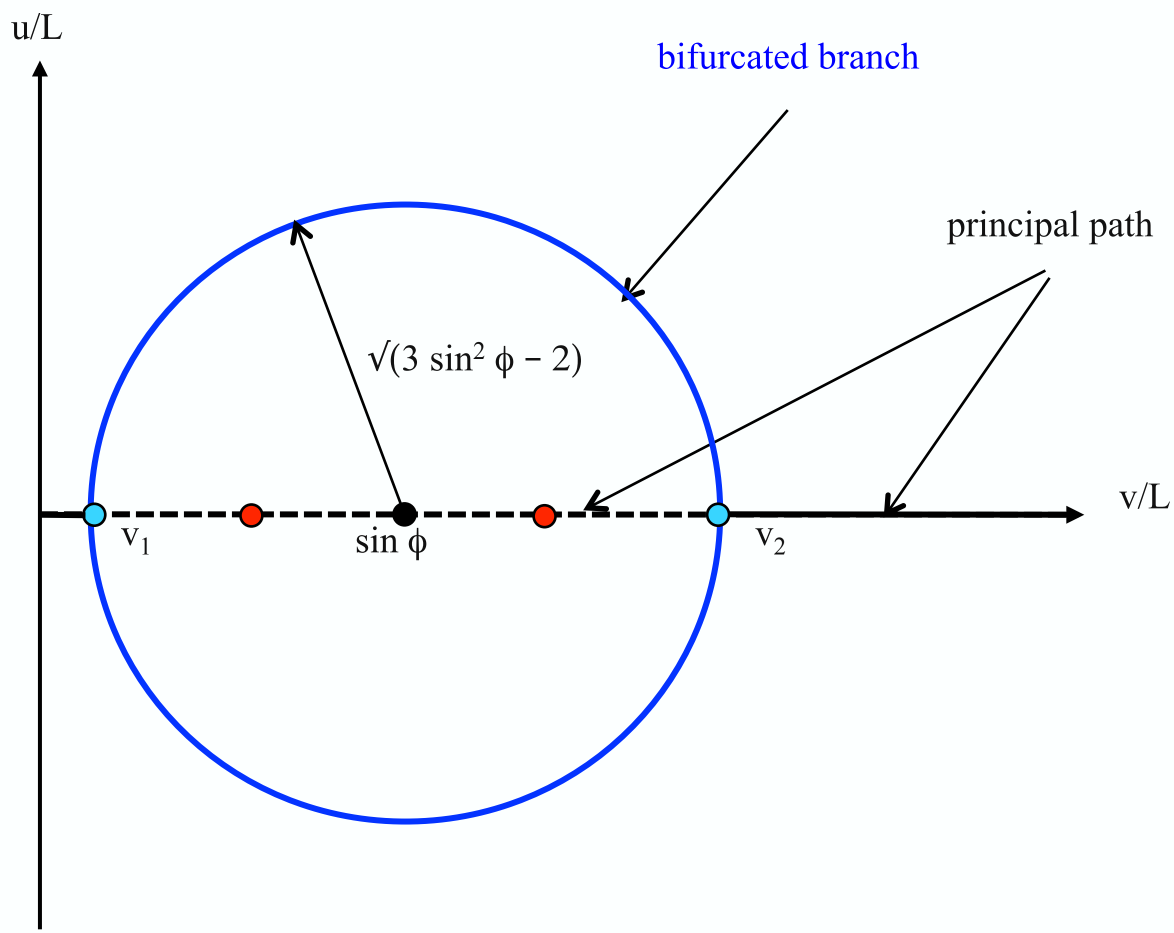


- Principal solution has limit loads and bifurcation points
- Bifurcated solution emerges from principal one before maximum and after minimum load
- Principal solution changes stability at bifurcation points
- Bifurcated solution is stable for the case of displacement control

____ stable path, _ _ _ unstable path ● limit load ● bifurcation point

LIMIT POINT EXAMPLE

PERFECT TRUSS RESULTS IN (u/L) – (v/L) **SPACE FOR:** $90^o > \phi > 60^o$ **(DISPL. CTRL.)**



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____ stable path, _ _ _ unstable path • limit load • bifurcation point