WHY STUDY STABILITY IN MECHANICS?

IN DESIGN WE GENERALLY ADDRESS TWO ISSUES:

- **CHECK OPERATING LOADS (STRESSES WITHIN ELASTIC LIMITS)**
- **DESIGN TO AVOID FAILURE (SAFETY AT EXTREME LOADS)**

FAILURE OF STRUCTURES FALLS INTO TWO BASIC TYPES:

- **FRACTURE (STRESS CONCENTRATION AT LOCAL FLAWS)**
- **BUCKLING (OVERALL STRUCTURAL FAILURE DUE TO INSTABILITY)**

REASON FOR BUCKLING INSTABILITY: NONLINEAR BEHAVIOR OF STRUCTURES

STUDY OF STABILITY IMPORTANT NOT ONLY FOR ENGINEERING STRUCTURES, BUT FOR A MUCH WIDER RANGE OF APPLICATIONS IN SOLIDS AND MATERIALS

COURSE OUTLINE

1. Concept of stability and examples of discrete systems
2. Concept of bifurcation and examples of discrete systems
3. General theory for continuum systems: applications to 1D structures (beams)
4. Continuum elastic systems: applications to 2D structures (plates, simple mode)
5. Continuum elastic systems: applications to 2D structures (plates, multiple mode)
6. FEM considerations & composite materials: applications to layered solids in 2D
7. Cellular solids: applications to honeycomb
8. Phase transformations in shape memory alloys: 1D continuum & 3D lattice models
9. REVIEW

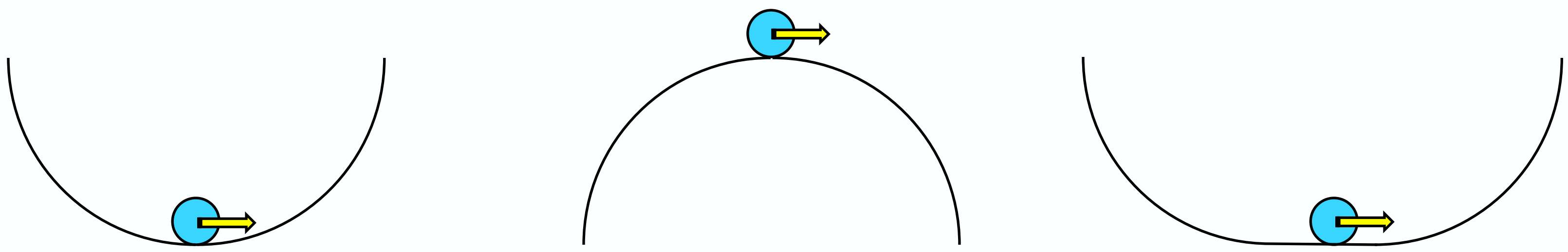
MOTIVATION

- STABILITY OF SOLIDS PLAYS IMPORTANT ROLE IN SOLID MECHANICS
- FIELD STARTS WITH EULER'S 1744 ELASTICA PAPER FOR COLUMN BUCKLING
- FIRST APLICATIONS IN CIVIL & MECHANICAL ENGINEERING INVOLVING THE BUCKLING OF VARIOUS TYPES OF STRUCTURES
- SUBSEQUENTLY, STRUCTURAL STABILITY OF PARAMOUNT IMPORTANCE IN AEROSPACE APPLICATIONS WHERE WEIGHT IS AT A PREMIUM (E.G. ROCKET FAILURES DUE TO CYLINDRICAL CASING BUCKLING)
- IN ADDITION TO STRUCTURAL SCALE, APPLICATIONS ALSO EXIST IN OTHER SCALES: GEOLOGICAL, E.G. LAYER FOLDING UNDER TECTONIC STRESSES, MATERIAL, E.G. FIBER KINKING IN COMPOSITES & LOCALIZATION OF DEFORMATION IN HONEYCOMB, EVEN AT ATOMISTIC, E.G. SHAPE MEMORY ALLOYS, SCALES.
- MANY EXCITING NEW APPLICATIONS OF SAME PRINCIPLES IN EXOTIC MATERIALS (E.G. PRINCIPLE OF TWISTED NEMATIC DEVICE THAT ALLOWS FOR LIQUID CRYSTAL DISPLAYS!)

CONCEPT OF STABILITY

STABILITY ACCORDING TO DICTIONARY: "THE STATE OR QUALITY OF BEING RESISTANT TO CHANGE, DETERIORATION OR DISPLACEMENT"

INTUITIVE IDEA OF STABILITY: BALL AT TOP OR BOTTOM OF HILL

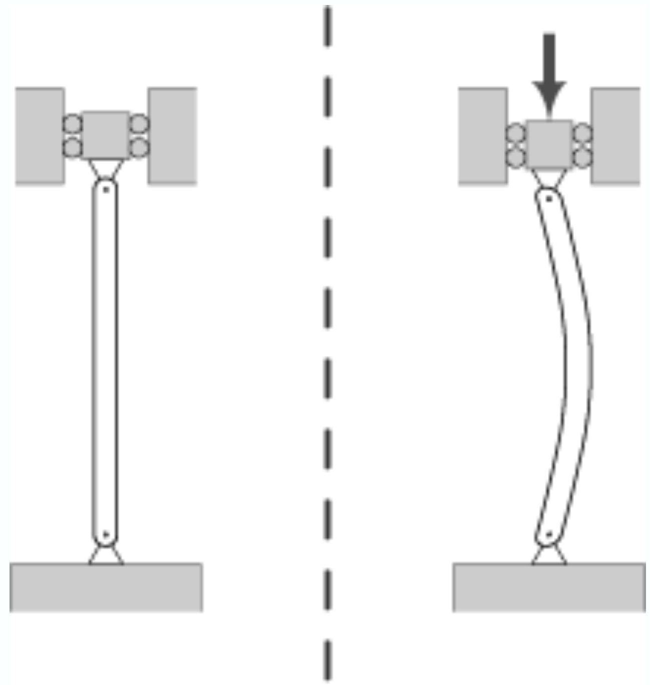


STABLE UNSTABLE NEUTRALLY STABLE

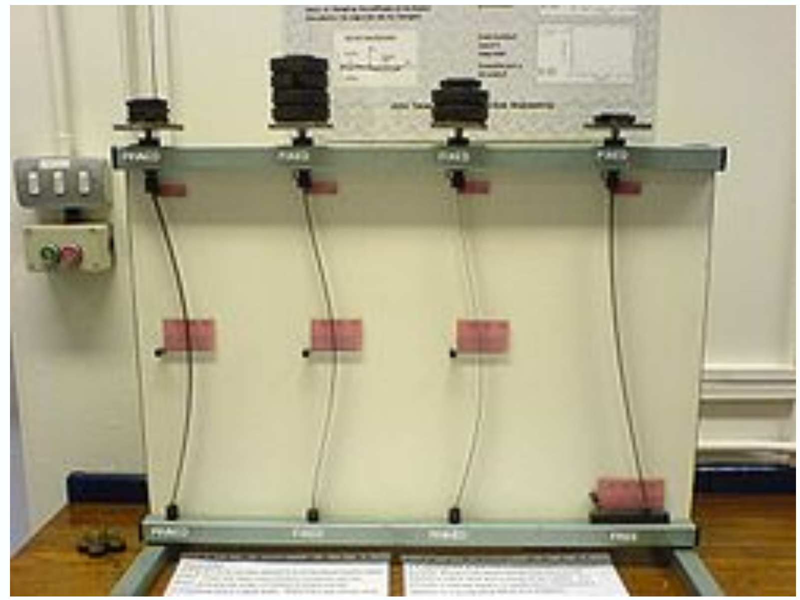
STABILITY PROBLEMS ONE CAN CONSIDER:

- STABILITY OF AN EQUILIBRIUM (e.g. loaded structures) – OBJECT OF THIS CLASS
- STABILITY OF A STEADY STATE (e.g. laminar flow)
- STABILITY OF A TIME-DEPENDENT PERIODIC SYSTEM (e.g. earth's orbit)
- STABILITY OF ARBITRARY TIME-DEPENDENT SYSTEM (e.g. acrobatic maneuver)

STRUCTURAL BUCKLING - BEAMS



SCHEMATICS OF THE EULER (1744) BUCKLING IN AXIALLY LOADED BEAMS (SIMPLE SUPPORT ON BOTH ENDS)



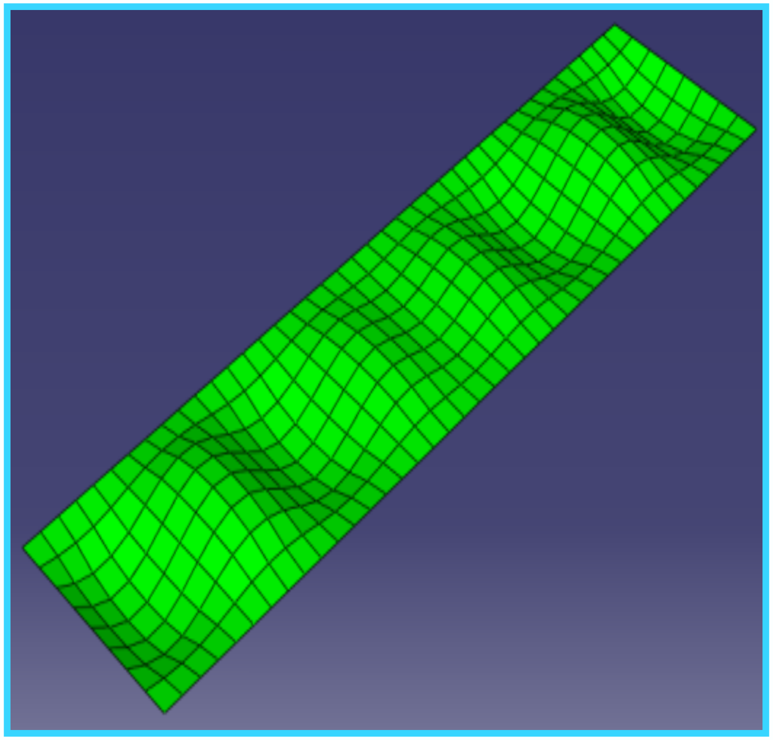
EXPERIMENTS IN THE EULER BUCKLING OF AXIALLY LOADED BEAMS UNDER DIFFERENT BOUNDARY CONDITIONS

STRUCTURAL BUCKLING - BEAMS

THERMAL BUCKLING OF RAIL TRACKS DUE TO HEATING BY SUN (SUN KINK)

ROAD BUCKLING DUE TO TECTONIC COMPRESSION OF SUBSTRATE

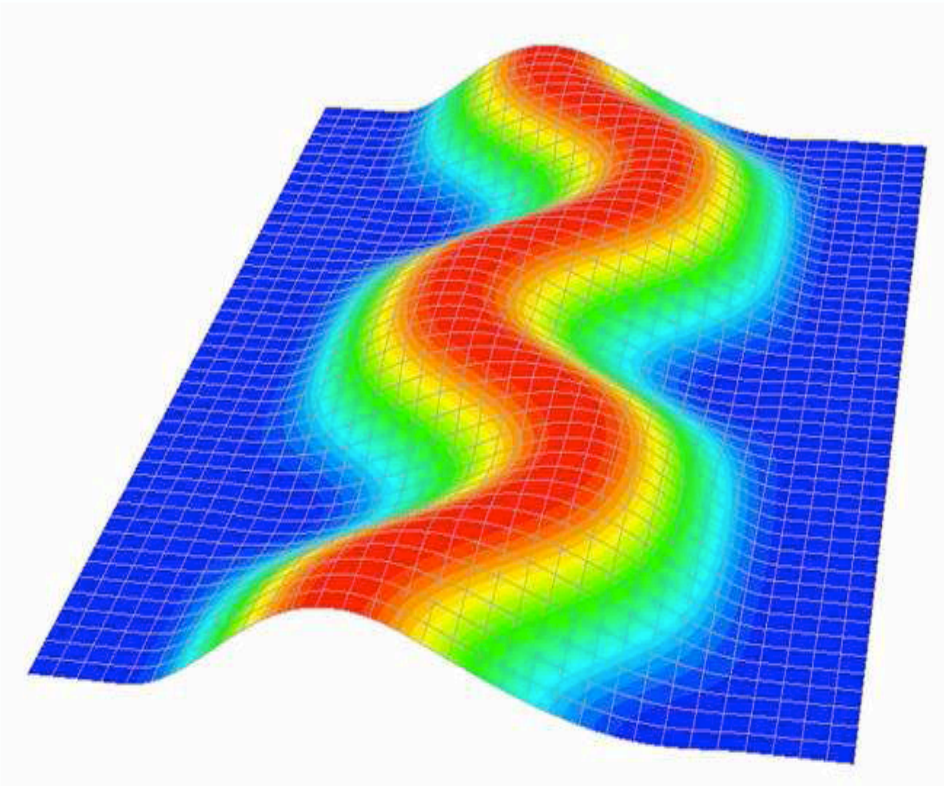
STRUCTURAL BUCKLING - PLATES



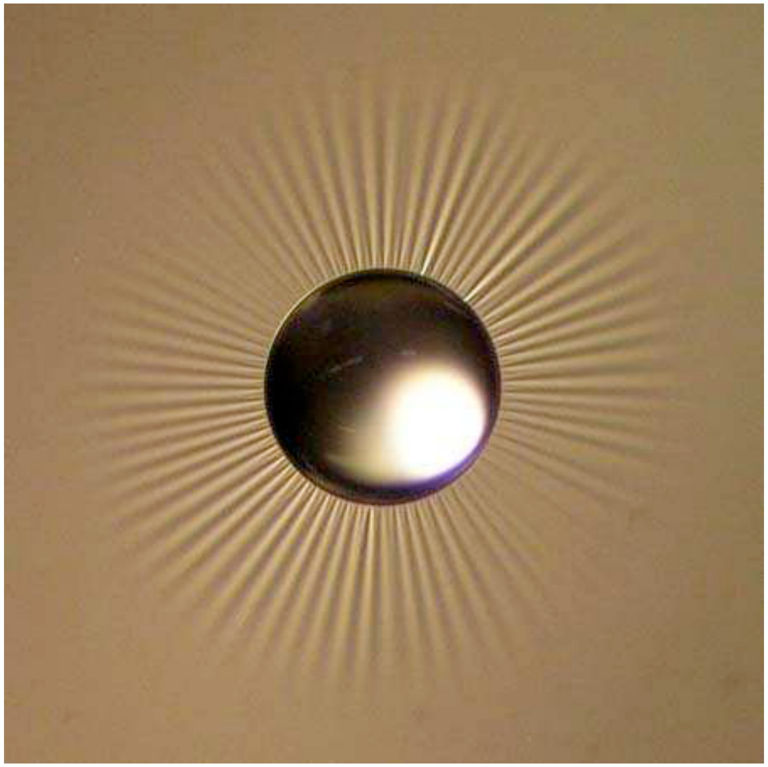
BUCKLING OF PLATE AXIALLY LOADED ALONG THE LONG SIDE AND WITH A SIMPLE SUPPORT ON ALL EDGES

BUCKLING OF SQUARE SECTION COLUMN USED IN AUTOMOTIVE APPLICATIONS TO ABSORB ENERGY (CRUMPLE ZONES)

BUCKLING OF THIN FILMS



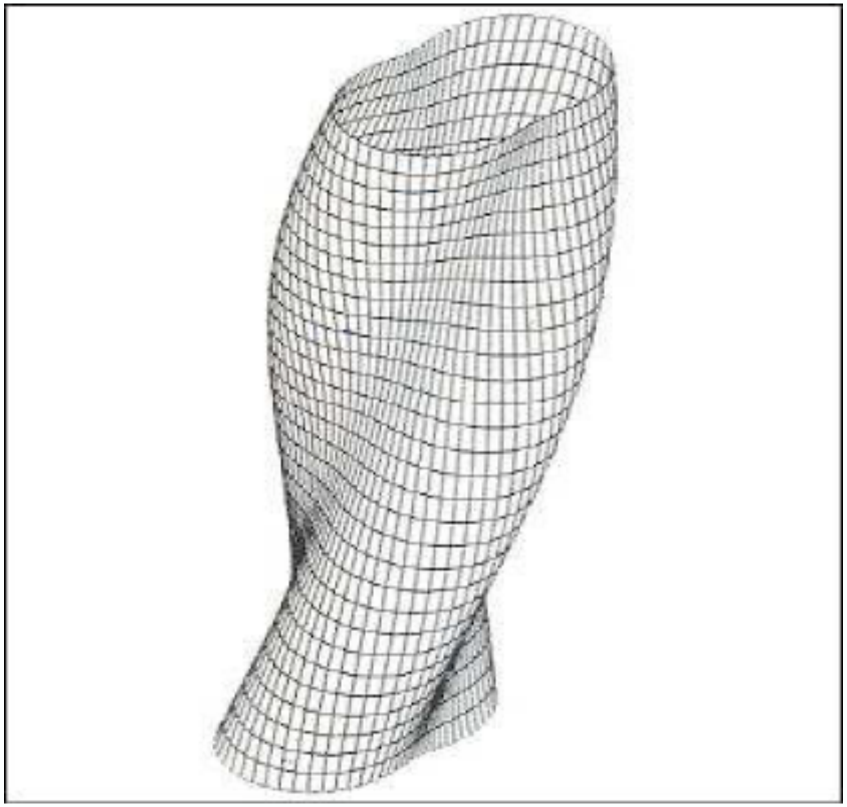
"TELEPHONE CORD" INSTABILITY



"BLISTER" INSTABILITY

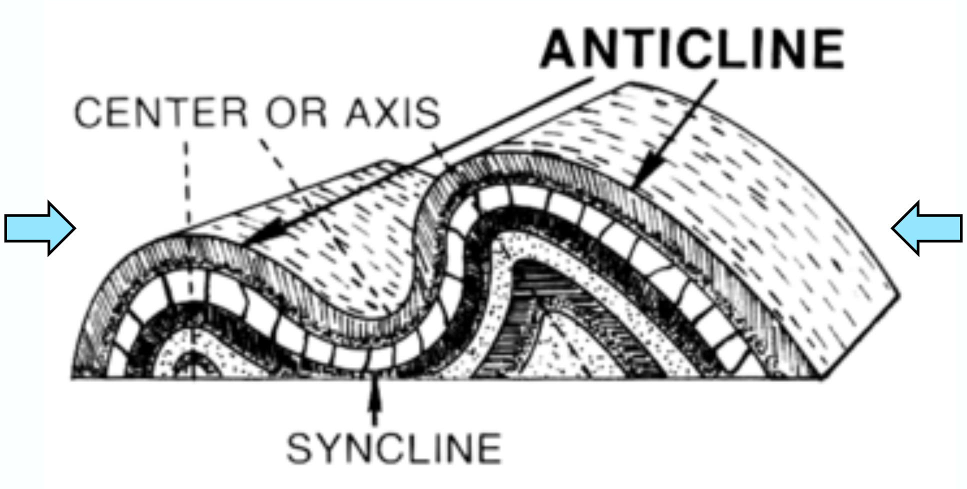
STRUCTURAL BUCKLING - CYLINDERS

BUCKLING OF CYLINDRICAL SHELL UNDER AXIAL COMPRESSION



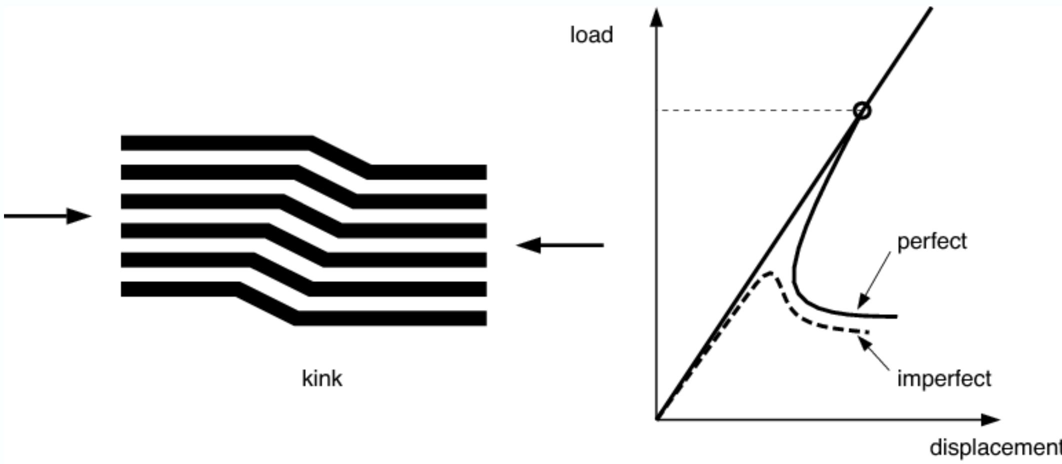
BUCKLING OF CYLINDRICAL SHELL UNDER TORSION

GEOLOGICAL BUCKLING

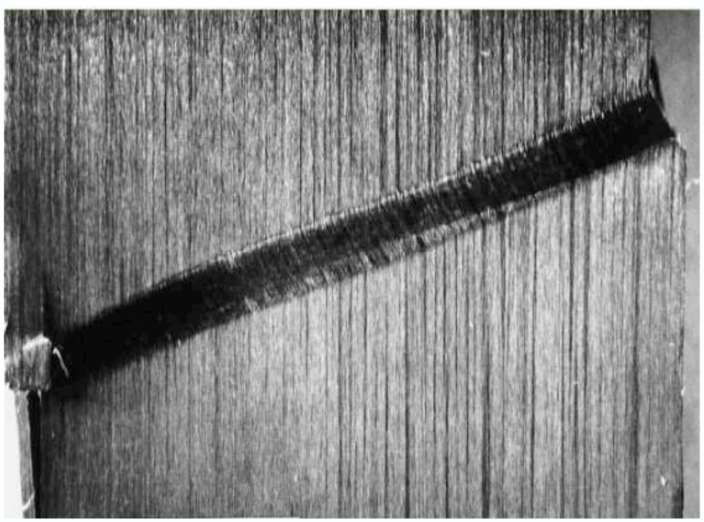


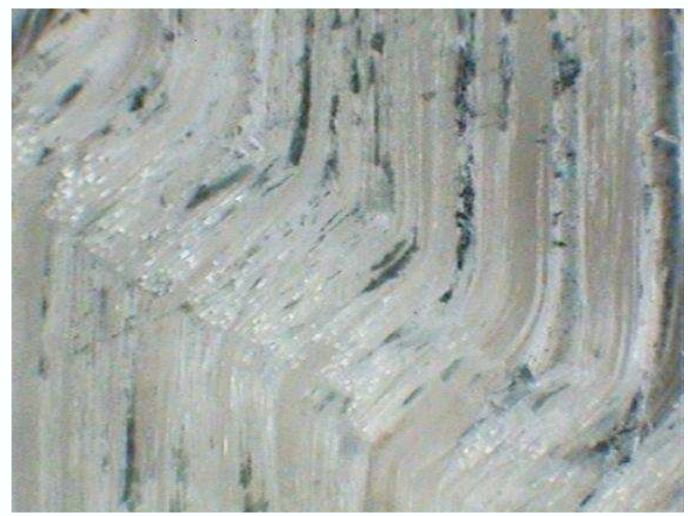
EXAMPLE OF GEOLOGICAL BUCKLING AT DIFFERENT SCALES

MICROSTRUCTURAL FAILURE MECHANISMS

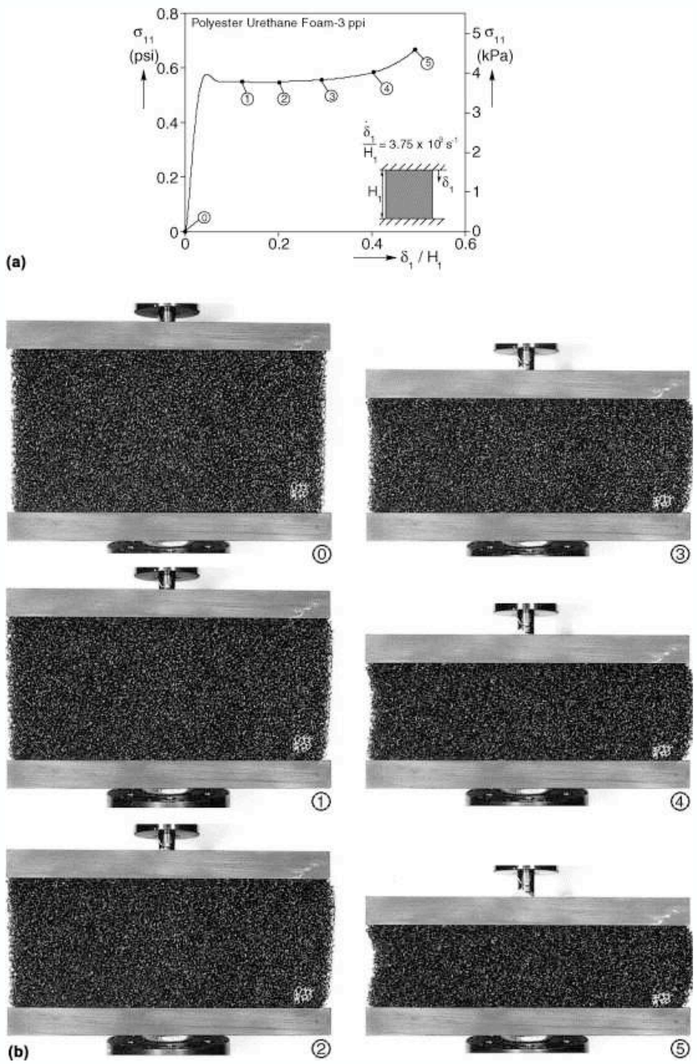


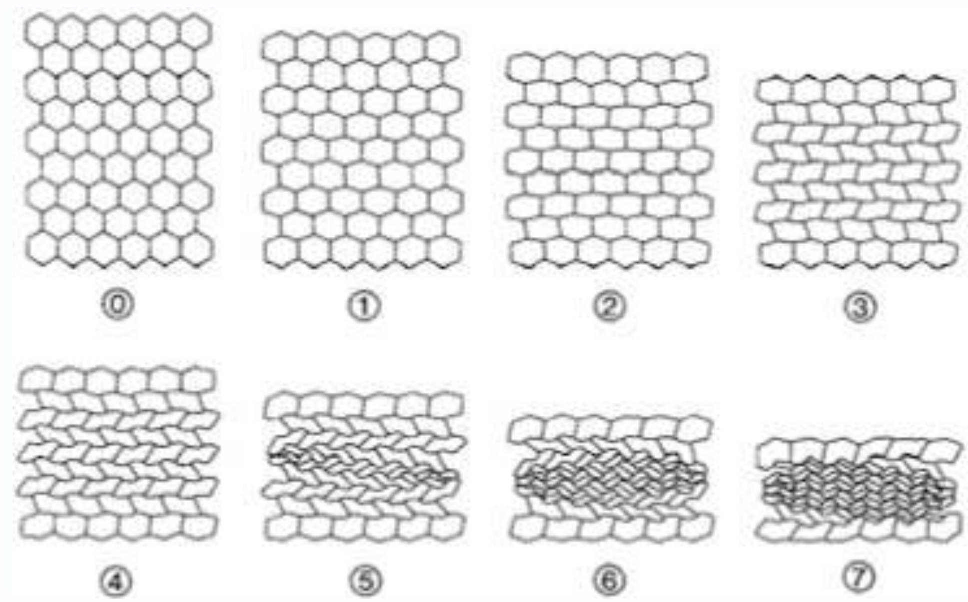
KINKBAND INSTABILITY IN AXIALLY LOADED, FIBER REINFORCED COMPOSITES





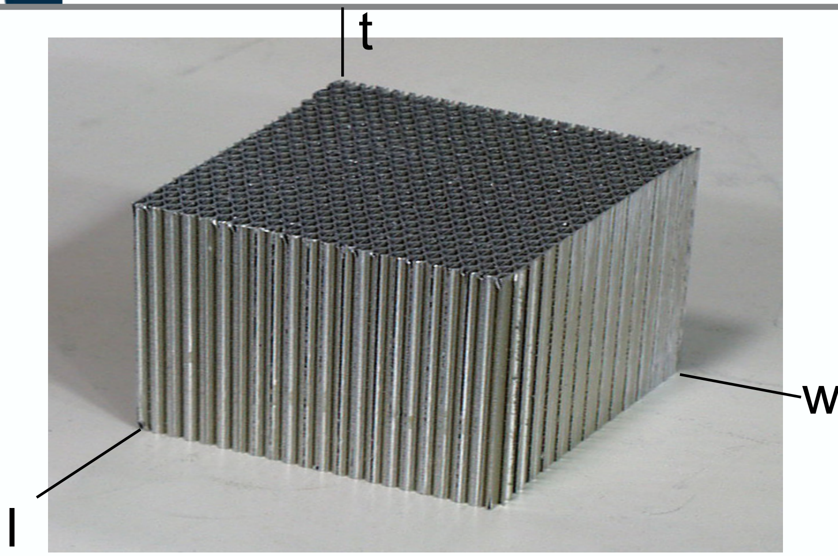
MICROSTRUCTURAL FAILURE MECHANISMS



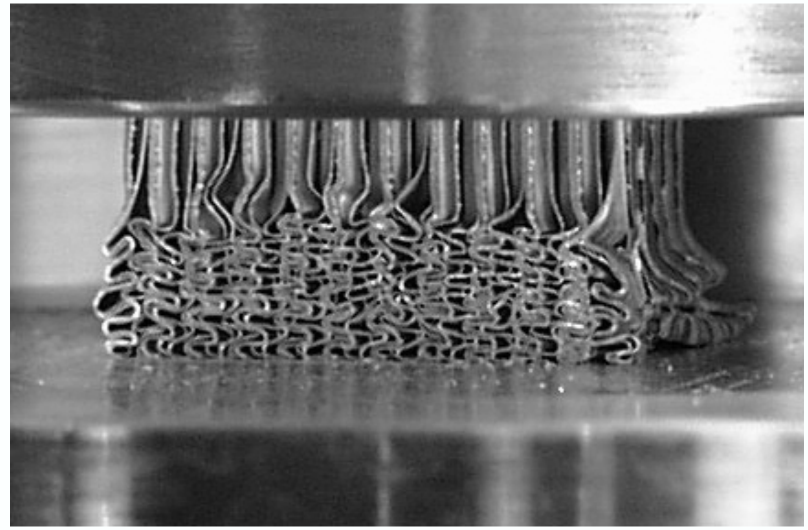


LOCALIZATION OF DEFORMATION INSTABILITY IN COMPRESSED FOAM AND HONEYCOMB

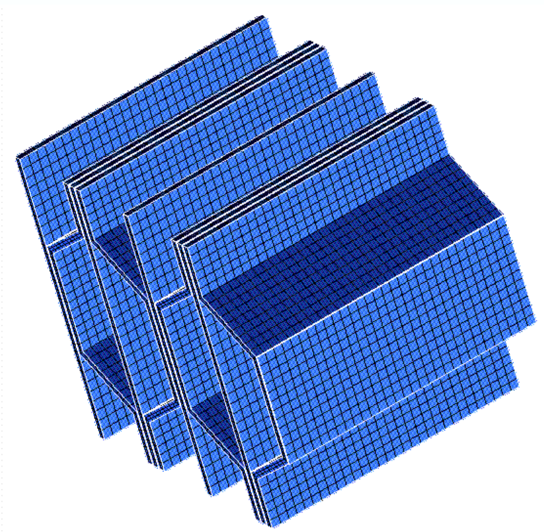
INSTABILITY EXAMPLES IN SOLIDS



REINFORCED COMPOSITE

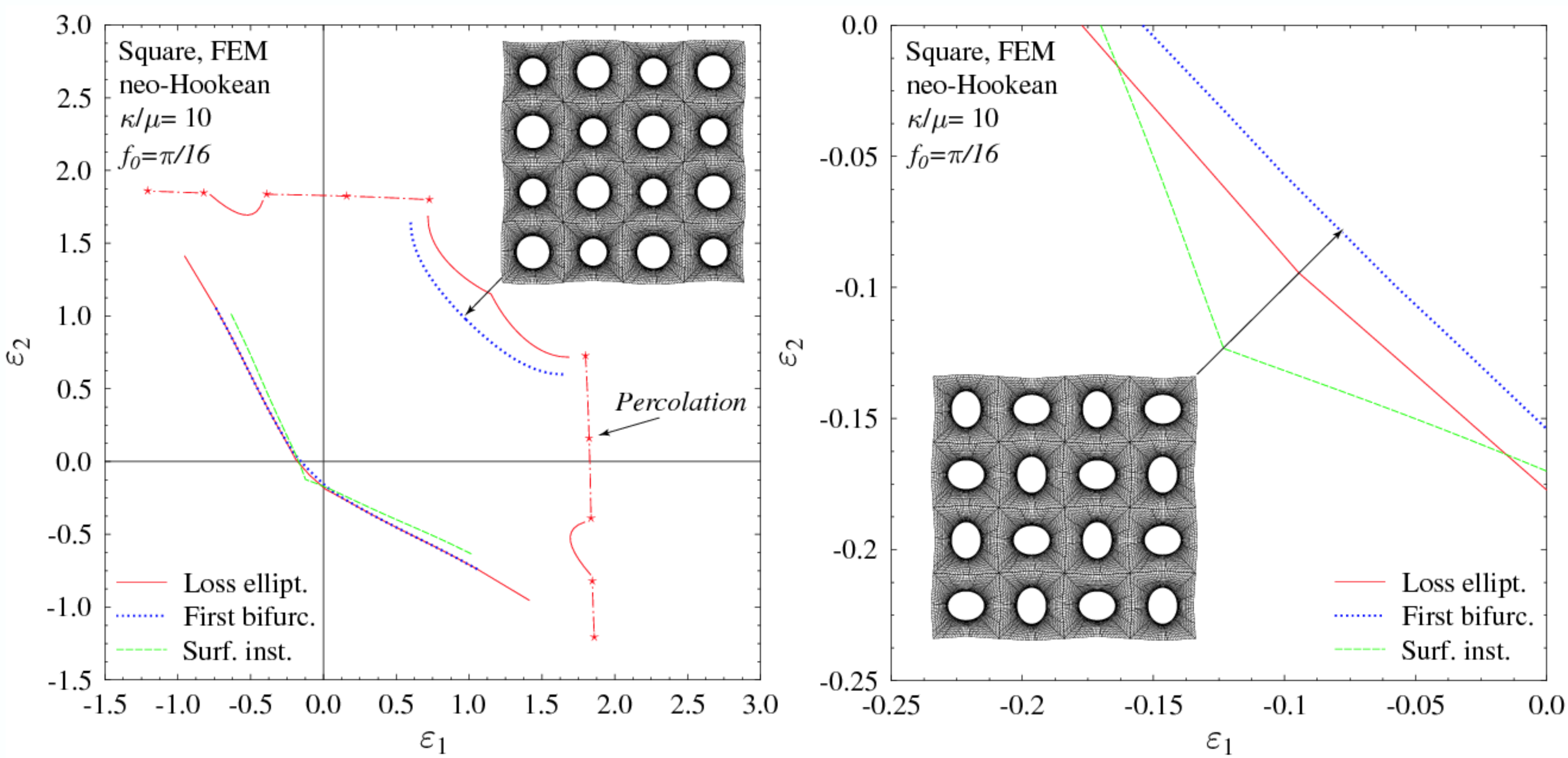


ROOM TEMPERATURE COLLAPSE

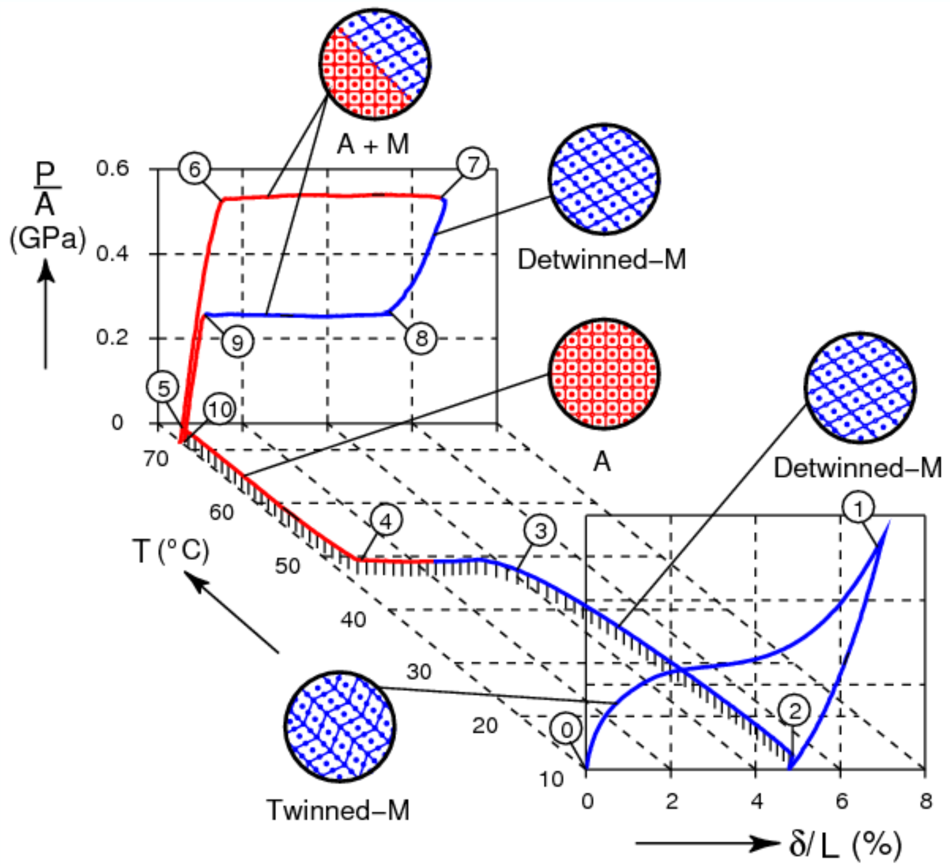


HIGH TEMPERATURE COLLAPSE

INSTABILITY IN PERIODIC POROUS ELASTOMERS



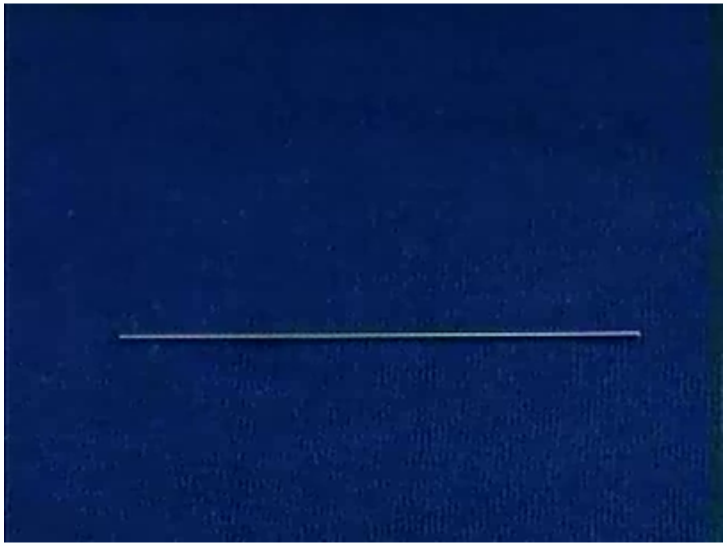
SHAPE MEMORY ALLOY (NiTi)



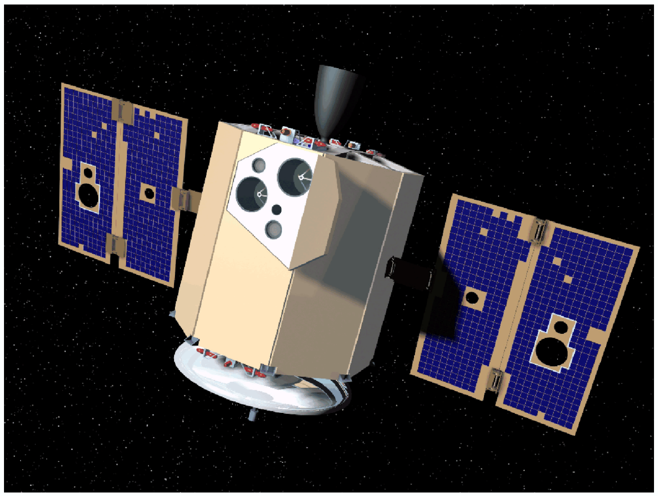
Tensile behavior of NiTi

(exhibiting the shape memory effect and pseudo–elasticity)

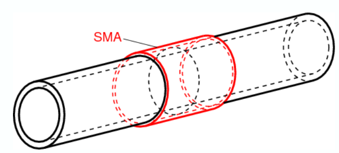
(J. Shaw 1997)



SHAPE MEMORY ALLOY (NiTi APPLICATIONS)



FRANGIBOLTS, SOLAR PANELS



SLEEVES, HYDRAULIC LINE JOINTS

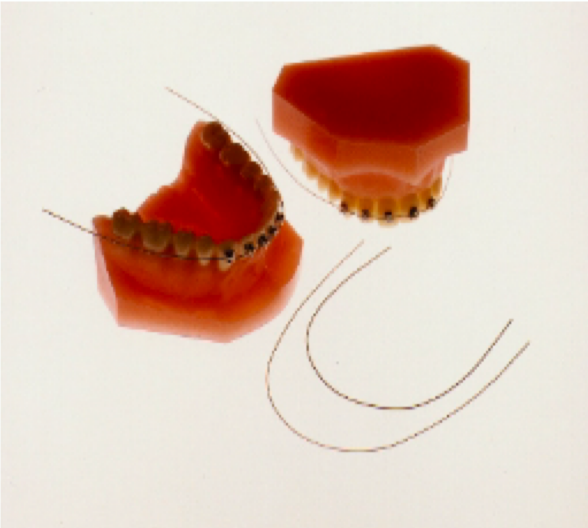
SHAPE MEMORY ALLOY (NiTi APPLICATIONS)

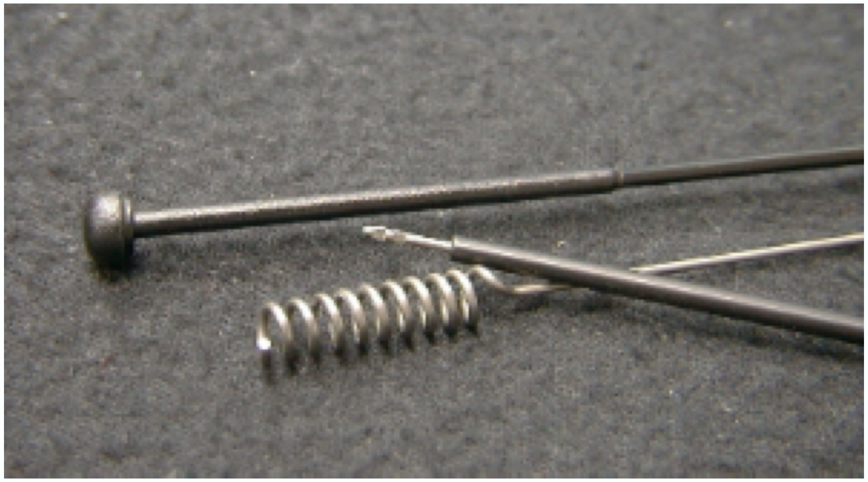
Stents

Dental wire

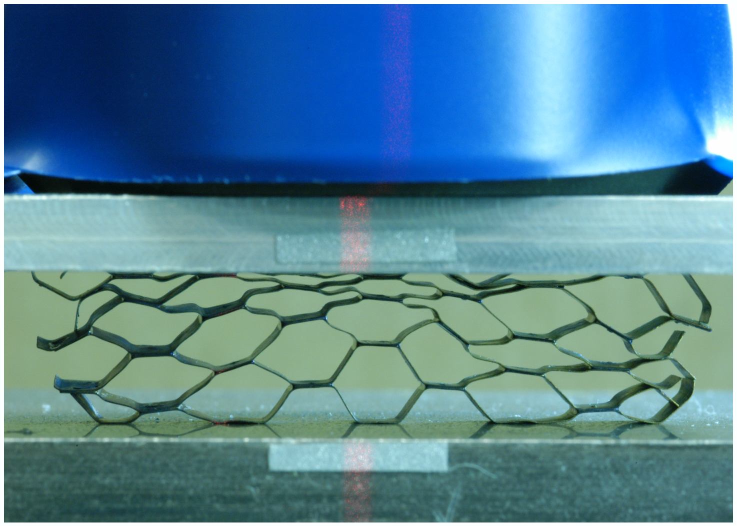
Glass frames

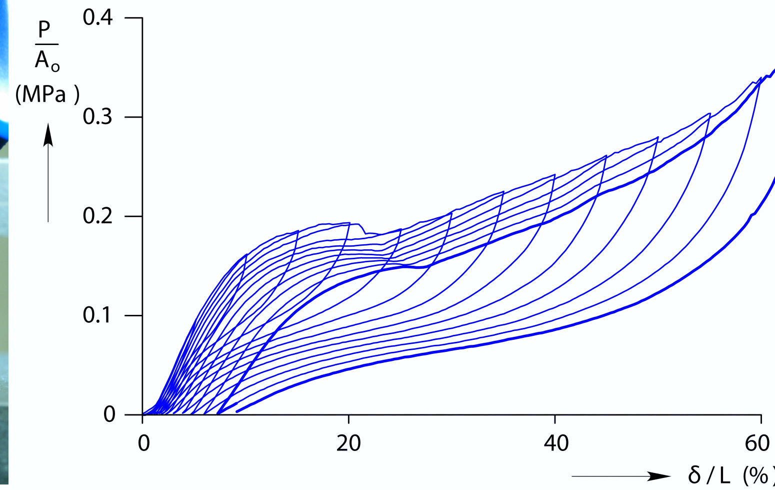
Cell phone antennas



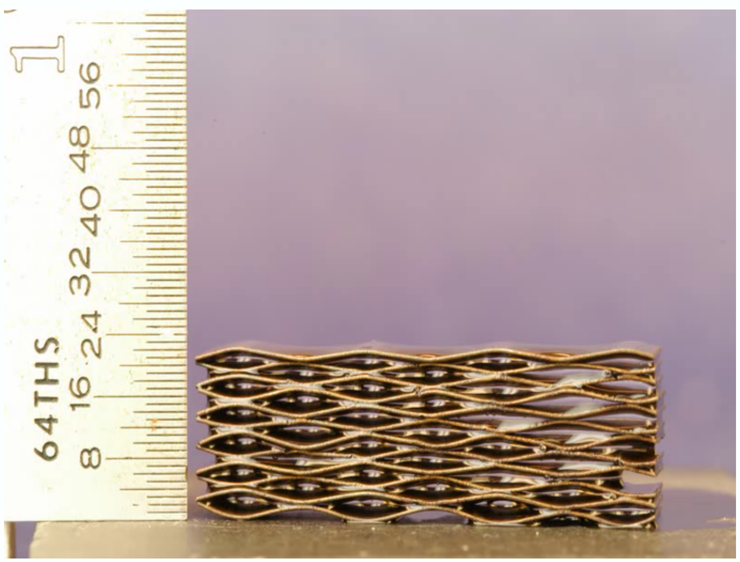


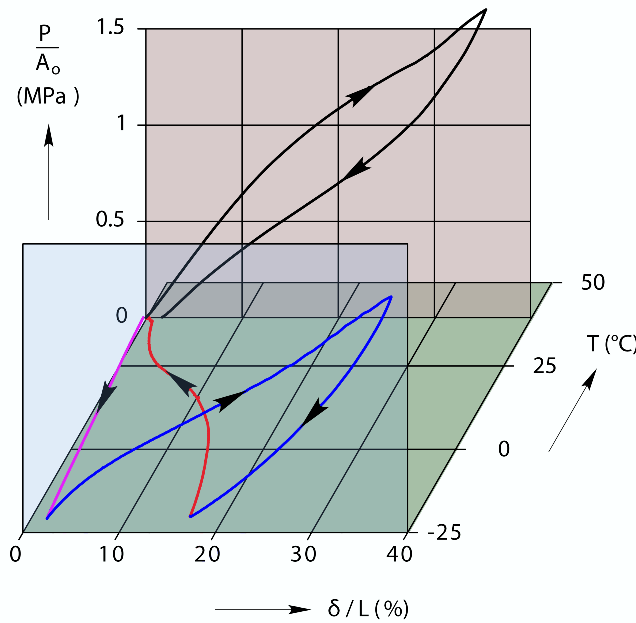
INSTABILITY EXAMPLES IN SOLIDS



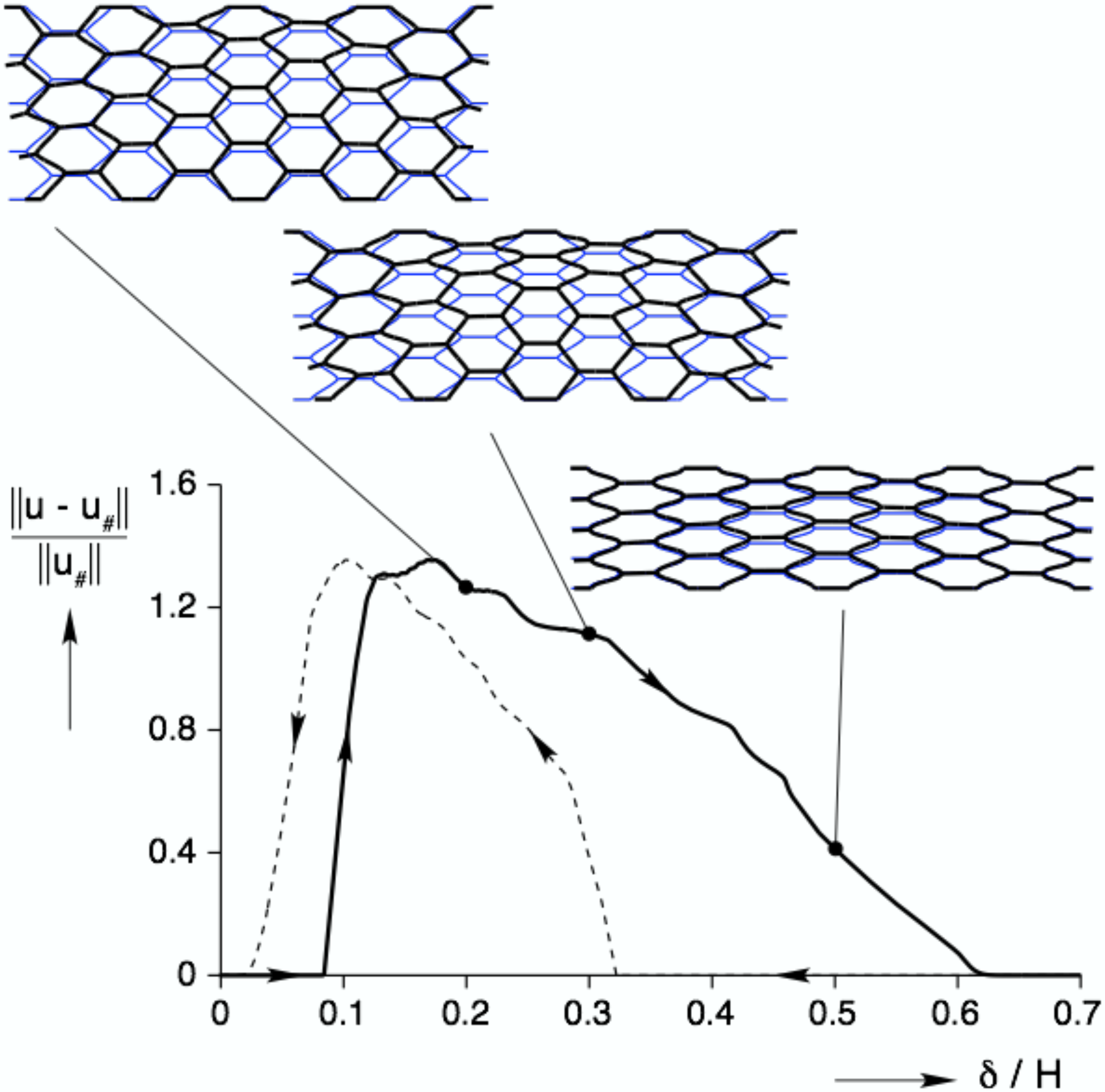


ISOTHERMAL RESPONSE





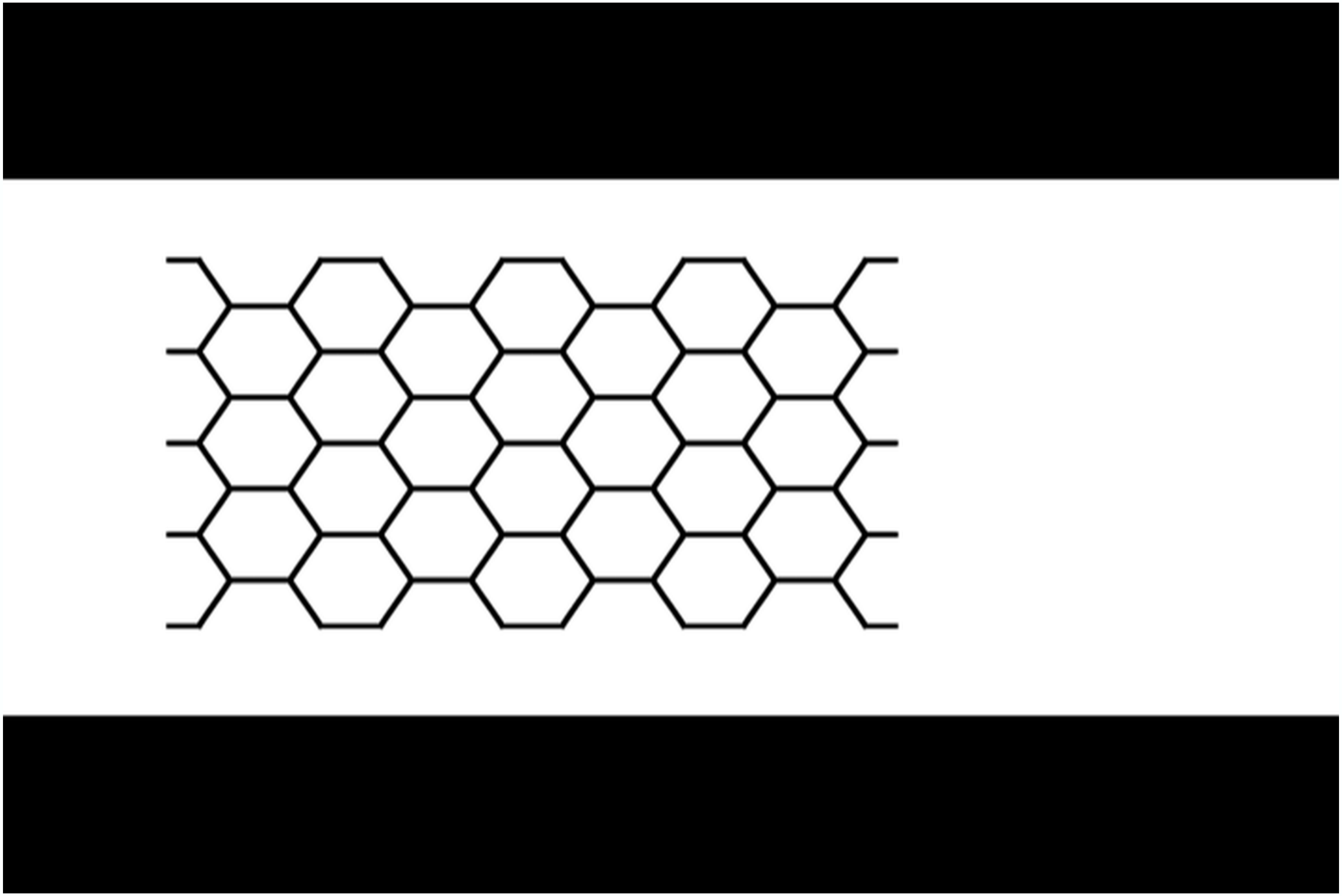
SHAPE MEMORY RESPONSE



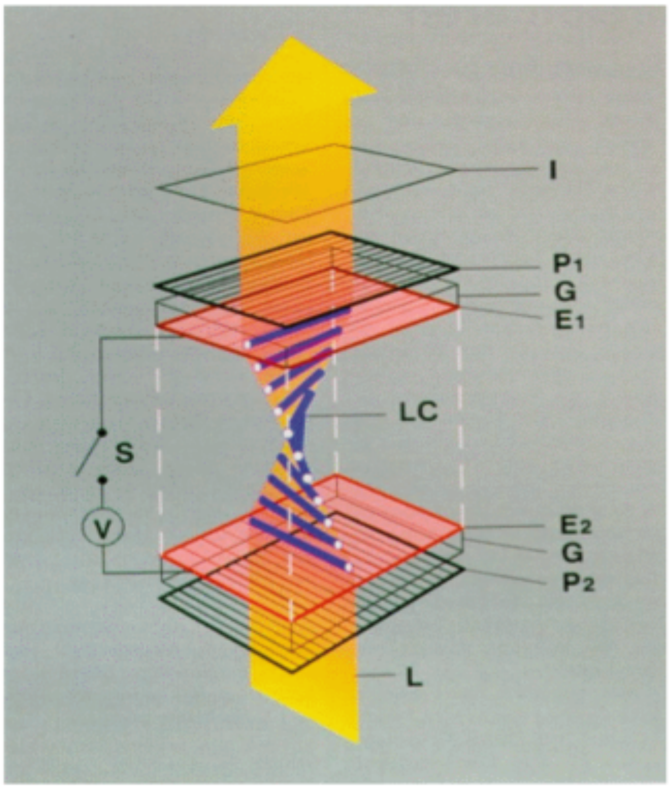
MORPHING EFFECT IN ISOTHERMAL RESPONSE (DUE TO STABILITY & PHASE TRANSFORMATION)

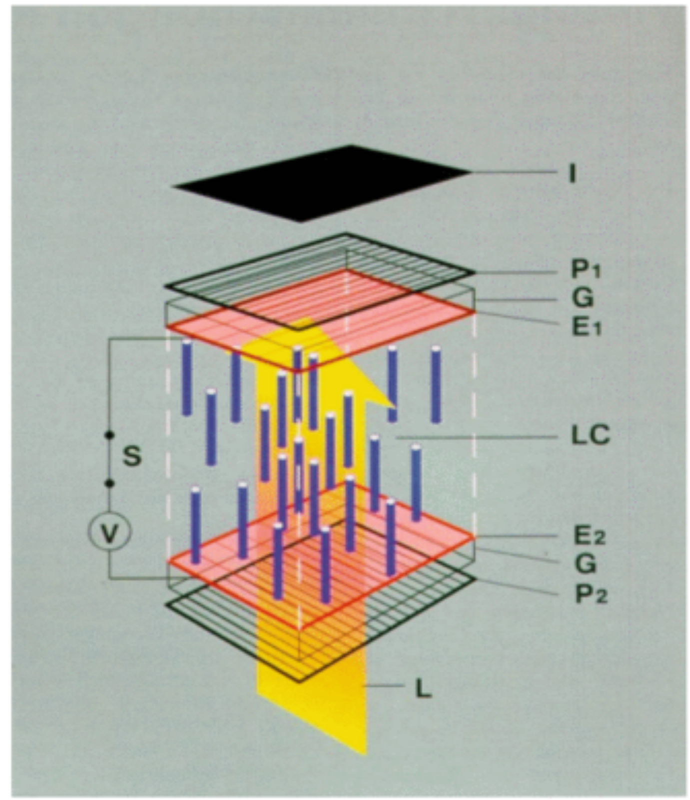
STABILITY OF INFINITE STRUCTURE EXPLAINS BEHAVIOR OF FINITE SPECIMENS

INSTABILITY EXAMPLES IN SOLIDS



TWISTED NEMATIC DEVICE – STABILITY LCD





UNDER NO ELECTRIC FIELD FILAMENTS LIE IN THE PLANE OF THE LCD AND ARE PARALLEL TO THE TWO GLASS PLATES

UNDER AN ELECTRIC FIELD FILAMENTS ROTATE OUT OF PLANE AND ARE NORMAL TO THE TWO GLASS PLATES

STABILITY OF AN EQUILIBRIUM

DEFINITION: A EQUILIBRIUM STATE IS STABLE IF A "SMALL" INITIAL PERTURBATION PRODUCES A SOLUTION THAT REMAINS "CLOSE" TO IT AT ALL SUBSEQUENT TIMES

$\dot{\mathbf{p}} = \mathbf{f}(\mathbf{p}, t), \quad \mathbf{p}(t) \in \mathbb{R}^n$ discrete system with n generalized d.o.f.

$\mathbf{p}_e(t) = \mathbf{p}_e, \quad \forall t \geq 0 \quad (\dot{\mathbf{p}}_e = 0),$ equilibrium state

$\mathbf{p}_0 \equiv \mathbf{p}(0),$ initial conditions at $t = 0$

THE SYSTEM IS STABLE WHEN :

$$\forall \varepsilon > 0, \quad \exists \, \{\mathbf{p}_0(\varepsilon), \; \eta(\varepsilon) > 0\} \text{ such } \parallel \mathbf{p}_0 - \mathbf{p}_e \parallel \leq \eta \quad \Longrightarrow \quad \parallel \mathbf{p}(t) - \mathbf{p}_e \parallel \leq \varepsilon, \quad \forall t > 0$$

OTHER STABILITY DEFINITIONS (ASYMPTOTIC STABILITY) :

$$\exists \eta > 0 \text{ such } \quad \parallel \mathbf{p}(0) - \mathbf{p}_e \parallel < \eta \quad \Longrightarrow \quad \lim_{t \longrightarrow \infty} \mathbf{p}(t) = \mathbf{p}_e$$

NOTE : $\parallel \cdot \parallel$ denotes Euclidean norm (all norms in $\mathbb{R}^n$ are equivalent)

TWO WIDELY USED METHODS TO CHECK STABILITY:

1. **LINEARIZATION METHOD**

   a) **Linearization of the equations of motion about equilibrium state**

   b) **Stability analysis of the linearized perturbed motions**

      STABILITY if all eigenvalues have negative real part

   c) **Justification of the results with respect to the actual motion of the system**

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2. **LYAPUNOV'S DIRECT METHOD**

   STABILITY guaranteed when a non-increasing funtional *L*(p*(t)*) can be found that satisfies certain bounding properties for the initial conditions and the current state (to be specified subsequently)

LINEARIZATION METHOD

$$\dot{\mathbf{p}} = \mathbf{f}(\mathbf{p}) = \mathbf{f}(\mathbf{p}_e) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{p}}\right]_e [\mathbf{p} - \mathbf{p}_e,] + \mathbf{o}(\| \mathbf{p} - \mathbf{p}_e \|), \quad \text{Taylor series expansion of } \mathbf{f}$$

$$0 = \mathbf{f}(\mathbf{p}_e), \quad \text{recall from equilibrium}$$

$$\Delta\mathbf{p} \equiv \mathbf{p} - \mathbf{p}_e, \quad \mathbf{A} \equiv \left[\frac{\partial \mathbf{f}}{\partial \mathbf{p}}\right]_e, \quad \text{definitions}$$

$$\Delta\dot{\mathbf{p}} = \mathbf{A}\Delta\mathbf{p}, \quad \text{LINEARIZED SYSTEM (approximates actual one)}$$

$$\Delta\mathbf{p}(t) = \exp[t\mathbf{A}]\Delta\mathbf{p}(0), \ \text{solution of linearized system}$$

$$\Delta\mathbf{p}(t) \text{ bounded } \forall t > 0 \text{ iff } \Re(a_i) < 0 \ \forall \text{ eigenvalues } \ a_i \text{ of } \mathbf{A}$$

STABILITY OF LINEARIZED SYSTEM

NOTE : for simplicity $\partial\mathbf{f}/\partial t = 0$, autonomous system $\implies$ $\mathbf{A}$ is a constant matrix

LINEARIZATION METHOD (LYAPUNOV'S THEOREM)

- **If the real part of all the eigenvalues a_i of the linearized system's matrix A are negative, (not necessarily strictly so) the system is stable**

- **If the real part of at least one eigenvalue a_i of the linearized system's matrix A is strictly positive, the system is unstable**

NOTE: Proof of stability for nonlinear system requires additional information about the growth of the difference between the linearized and nonlinear systems as a function of the independent variable p

LYAPUNOV'S DIRECT METHOD

A system is stable if a functional $L(\mathbf{p}(t))$ can be found with the following properties:

- $\dfrac{dL}{dt} \leq 0,$ (functional is nonincreasing)
- $L(\mathbf{p}(t)) \geq c \parallel \mathbf{p}(t) - \mathbf{p}_e \parallel^2,$ ($c > 0$; functional measures distance from equilibrium)
- $L(\mathbf{p}(0)) \leq d \parallel \mathbf{p}(0) - \mathbf{p}_e \parallel^2,$ ($d > 0$; functional measures initial perturbation)

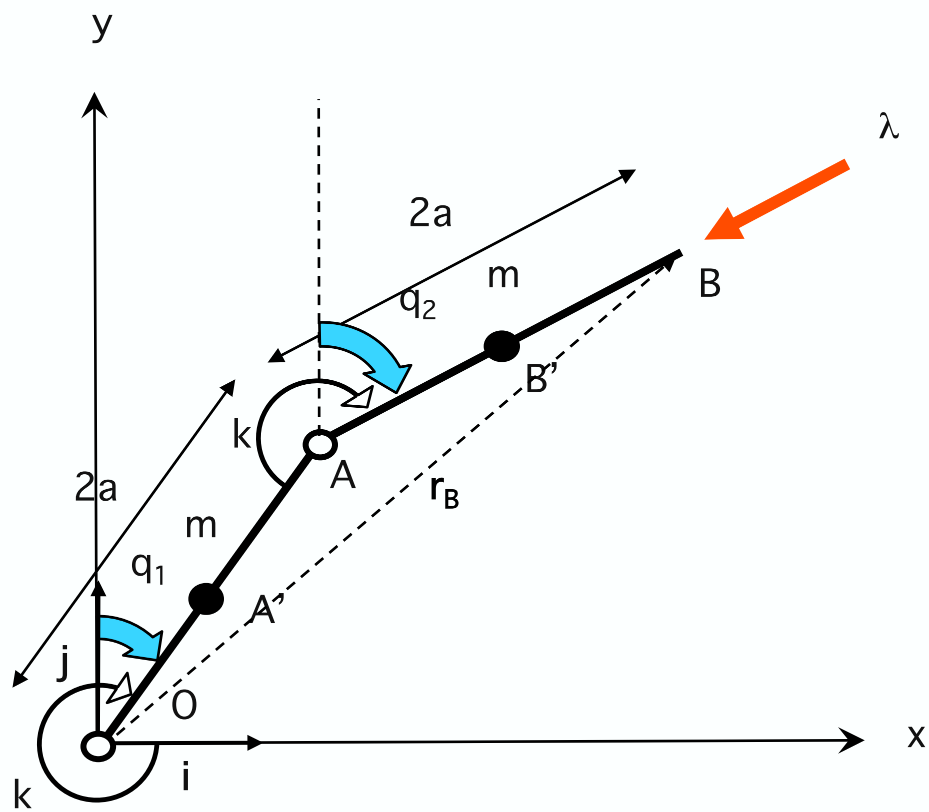
PROOF :

$$c \parallel \mathbf{p}(t) - \mathbf{p}_e \parallel^2 \leq L(\mathbf{p}(t)) \leq L(\mathbf{p}(0)) \leq d \parallel \mathbf{p}(0) - \mathbf{p}_e \parallel^2 \Longrightarrow \parallel \mathbf{p}(t) - \mathbf{p}_e \parallel \leq \varepsilon; \ (\eta \leq \varepsilon\sqrt{c/d})$$

NOTE: Finding a Lyapunov functional for a stable system is not always possible

FINITE D.O.F. SYSTEM: EXAMPLE – 1

STABILITY OF A TWO-BAR PLANAR, FRICTIONLESS MECHANISM SUBJECTED TO A FOLLOWER LOAD



FROM LAGRANGIAN DYNAMICS OF MECHANICAL SYSTEMS ONE HAS THE GENERALIZED EQUATIONS OF MOTION:

$$F_i^E = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i}, \qquad \mathcal{L} \equiv \mathcal{K} - \mathcal{E},$$

q_i ith generalized d.o.f., $i = 1, 2$

F_i^E generalized external force conjugate to q_i

$\mathcal{L}$ system's Lagrangian, $\mathcal{K}$ system's kinetic energy, $\mathcal{E}$ system's potential energy

FINITE D.O.F. SYSTEM: EXAMPLE – 1

CALCULATION OF KINETIC ENERGY OF MASSES AT A' & B'

$$\mathcal{K} = \frac{1}{2}\, m v_{A'}^2 + \frac{1}{2}\, m v_{B'}^2 = \frac{1}{2}\, m(\dot{\mathbf{r}}_{A'} \bullet \dot{\mathbf{r}}_{A'} + \dot{\mathbf{r}}_{B'} \bullet \dot{\mathbf{r}}_{B'})$$

$$\mathbf{r}_{A'} = (a \sin q_1)\mathbf{i} + (a \cos q_1)\mathbf{j}$$

$$\mathbf{r}_{B'} = (2a \sin q_1 + a \sin q_2)\mathbf{i} + (2a \cos q_1 + a \cos q_2)\mathbf{j}$$

$$\mathcal{K} = m a^2\, [\frac{5}{2}(\dot{q}_1)^2 + \frac{1}{2}(\dot{q}_2)^2 + 2\dot{q}_1 \dot{q}_2 \cos(q_1 - q_2)]$$

CALCULATION OF EXTERNAL FORCES (GENERALIZED VELOLCITIES ARE ARBITRARY)

$$-\lambda\, (\sin q_2\ \mathbf{i} + \cos q_2\ \mathbf{j}) \bullet \dot{\mathbf{r}}_B = F_1^E \dot{q}_1 + F_2^E \dot{q}_2$$

$$\mathbf{r}_B = 2a\, [(\sin\ q_1 + \sin\ q_2)\mathbf{i} + (\cos\ q_1 + \cos\ q_2)\mathbf{j}]$$

$$F_1^E = 2a\lambda \sin(q_1 - q_2)\ , \quad F_2^E = 0$$

FINITE D.O.F. SYSTEM: EXAMPLE – 1

CALCULATION OF POTENTIAL ENERGY OF SPRINGS AT A & B

$$\mathcal{E} = \frac{1}{2}[k(q_1)^2 + k(q_1 - q_2)^2]$$

BY SUBSTITUTING IN GENERAL EQUATIONS, NONLINEAR SYSTEM EQUILIBRIUM IS:

$$2kq_1 - kq_2 - 2a\lambda \sin(q_1 - q_2) + ma^2[5\ddot{q}_1 + 2\ddot{q}_2 \cos(q_1 - q_2)$$
$$-2\dot{q}_2(\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2) + 2\dot{q}_1\dot{q}_2 \sin(q_1 - q_2)] = 0 \;, \quad (q_1 - \text{equation})$$

$$-kq_1 + kq_2 + ma^2[\ddot{q}_2 + 2\ddot{q}_1 \cos(q_1 - q_2) - 2\dot{q}_1(\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2)$$
$$-2\dot{q}_1\dot{q}_2 \sin(q_1 - q_2)] = 0 \;. \quad (q_2 - \text{equation})$$

NOTICE THAT STRAIGHT CONFIGURATION ($q_1 = q_2 = 0$) IS AN EQUILIBRIUM SOLUTION

FINITE D.O.F. SYSTEM: EXAMPLE – 1

THE LINEARIZED SYSTEM ABOUT THE $q_1 = q_2 = 0$ EQUILIBRIUM STATE IS:

$$ma^2 \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \Delta\ddot{q}_1 \\ \Delta\ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 2k - 2a\lambda & 2a\lambda - k \\ -k & k \end{bmatrix} \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{in compact form}: \ \mathbf{M}\Delta\ddot{q} + \mathbf{K}\Delta\mathbf{q} = \mathbf{0}, \quad \mathbf{M}: \ \text{mass matrix}, \quad \mathbf{K}: \ \text{stiffness matrix}$$

THE LINEARIZED SYSTEM HAS THE FOLLOWING SOLUTION:

$$\Delta\mathbf{q}(t) = \textstyle\sum_{I=1}^{2}\Big(\mathbf{X}_I \exp(s_{_I} t) + \overline{\mathbf{X}}_I \exp(\overline{s}_{_I} t)\Big), \quad \text{solution of linear system}$$

$$\text{Det}[\mathbf{K} + (s_I)^2\mathbf{M}] = 0, \quad s_I = s_1, \ s_2 : \text{eigenvalues of linear system}$$

$$[\mathbf{K} + (s_I)^2\mathbf{M}] \ \mathbf{X}_I = 0, \quad \mathbf{X}_I = \mathbf{X}_1, \ \mathbf{X}_2 : \text{eigenvectors of linear system}$$

FINITE D.O.F. SYSTEM: EXAMPLE – 1

THE SYSTEM'S CHARACTERISTIC EQUATION AND ITS DISCRIMINANT ARE:

$$m^2 a^4 (s_I)^4 + m a^2 (11k - 6a\lambda)(s_I)^2 + k^2 = 0$$

$$\Delta = 3m^2 a^4 (3k - 2a\lambda)(13k - 6a\lambda), \quad \Delta: \text{ discriminant of biquadratic}$$

THE LINEARIZED SYSTEM'S EIGENVALUES DEPEND ON THE LOAD AS FOLLOWS:

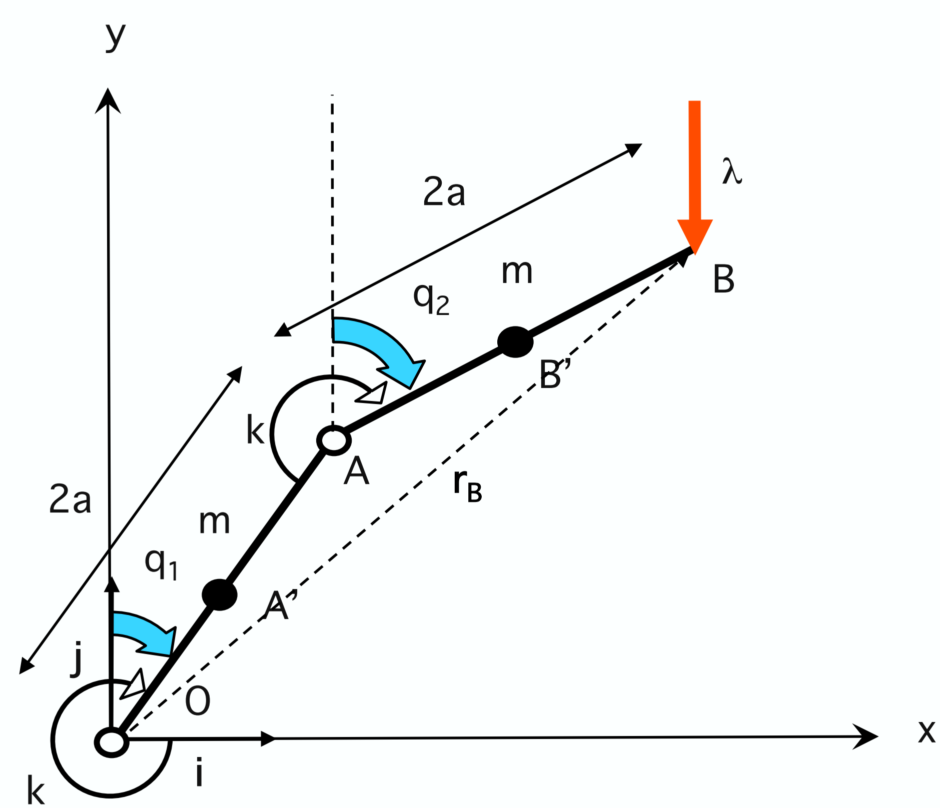
$$0 < \lambda < 3k/2a, \quad \text{all roots purely imaginary} \quad \Longrightarrow \quad \text{stable}$$

$$3k/2a < \lambda < 13k/6a, \quad \text{two roots with positive real part} \quad \Longrightarrow \quad \text{unstable}$$

$$13k/6a < \lambda, \quad \text{two real positive roots} \quad \Longrightarrow \quad \text{unstable}$$

ABOVE SYSTEM IS FRICTIONLESS, THIS IS WHY FOR LOW LOADS ($0 < \lambda < 3k/2a$) THE AMPLITUDE OF ITS OSCILLATIONS WILL NOT DECAY. FOR REALISTIC CASE, WHEN A SMALL DISSIPATION IS PRESENT, SYSTEM IS ASYMPTOTICALLY STABLE FOR LOADS $0 < \lambda < 3k/2a$.

STABILITY OF A TWO-BAR PLANAR, FRICTIONLESS MECHANISM SUBJECTED TO A LOAD AT A FIXED DIRECTION



FROM LAGRANGIAN DYNAMICS OF MECHANICAL SYSTEMS ONE HAS THE GENERALIZED EQUATIONS OF MOTION:

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i}, \quad \mathcal{L} \equiv \mathcal{K} - \mathcal{E},$$

q_i ith generalized d.o.f., $i = 1, 2$

conservative system, NO generalized external forces q_i

$\mathcal{L}$ system's Lagrangian, $\mathcal{K}$ system's kinetic energy, $\mathcal{E}$ system's potential energy

FINITE D.O.F. SYSTEM: EXAMPLE – 2

CALCULATION OF KINETIC ENERGY OF MASSES AT A' & B' SAME AS BEFORE

$$\mathcal{K} = ma^2 \, [\frac{5}{2}(\dot{q}_1)^2 + \frac{1}{2}(\dot{q}_2)^2 + 2\dot{q}_1\dot{q}_2 \cos(q_1 - q_2)]$$

CALCULATION OF POTENTIAL ENERGY OF SPRINGS AND APPLIED LOAD

$$\mathcal{E} = \frac{1}{2}[k(q_1)^2 + k(q_1 - q_2)^2] + [2\lambda a \, (\cos q_1 + \cos q_2)]$$

BY SUBSTITUTING IN GENERAL EQUATIONS, NONLINEAR SYSTEM EQUILIBRIUM IS:

$$2kq_1 - kq_2 - 2a\lambda \sin q_1 + ma^2[5\ddot{q}_1 + 2\ddot{q}_2 \cos(q_1 - q_2)$$

$$-2\dot{q}_2(\dot{q}_1 - \dot{q}_2) \sin(q_1 - q_2) + 2\dot{q}_1\dot{q}_2 \sin(q_1 - q_2)] = 0 \; , \quad (q_1 - \text{equation})$$

$$-kq_1 + kq_2 - 2a\lambda \sin q_2 + ma^2[\ddot{q}_2 + 2\ddot{q}_1(q_1 - q_2) \cos(q_1 - q_2)$$

$$-2\dot{q}_1(\dot{q}_1 - \dot{q}_2) \sin(q_1 - q_2) - 2\dot{q}_1\dot{q}_2 \sin(q_1 - q_2)] = 0 \; . \quad (q_2 - \text{equation})$$

FINITE D.O.F. SYSTEM: EXAMPLE – 2

THE LINEARIZED SYSTEM ABOUT THE $q_1 = q_2 = 0$ EQUILIBRIUM STATE IS:

$$ma^2 \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \Delta\ddot{q}_1 \\ \Delta\ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 2k - 2a\lambda & -k \\ -k & k - 2\alpha\lambda \end{bmatrix} \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

THE SYSTEM'S CHARACTERISTIC EQUATION IS:

$$m^2a^4(s_I)^4 + ma^2(11k - 12a\lambda)(s_I)^2 + k^2 - 6a\lambda k + 4a^2$$

THE LINEARIZED SYSTEM'S EIGENVALUES DEPEND ON THE LOAD AS FOLLOWS:

$0 < \lambda < (3 - \sqrt{5}k)/4a$, all roots purely imaginary $\Longrightarrow$ stable

$(3 - \sqrt{5}k)/4a < \lambda < (3 + \sqrt{5}k)/4a$, two roots with positive real part $\Longrightarrow$ unstable

$(3 + \sqrt{5}k)/4a < \lambda$, two real positive roots $\Longrightarrow$ unstable

ABOVE SYSTEM IS FRICTIONLESS, THIS IS WHY FOR LOW LOADS ($0 < \lambda < (3-\sqrt{5}k)/4a$) THE AMPLITUDE OF ITS OSCILLATIONS WILL NOT DECAY. FOR REALISTIC CASE, WHEN A SMALL DISSIPATION IS PRESENT, SYSTEM IS ASYMPTOTICALLY STABLE FOR LOADS $0 < \lambda < (3-\sqrt{5}k)/4a$.

FINITE D.O.F. SYSTEM: EXAMPLE – 2

SINCE THE SYSTEM IS CONSERVATIVE, CHECK MINIMUM POTENTIAL ENERGY

$$\mathcal{E} = \left[\frac{1}{2}\, k(q_1)^2 + \frac{1}{2}\, k(q_1 - q_2)^2\right] + \left[2\lambda a\, (cos\; q_1 + cos\; q_2)\right]$$

$$\left[\frac{\partial^2 \mathcal{E}(\mathbf{q})}{\partial q_i \partial q_j}\right]_{\mathbf{q}_e} = \begin{bmatrix} 2k - 2\lambda a & -k \\ -k & k - 2\lambda a \end{bmatrix}$$

$\mathcal{E}$ looses positive definitness at : $\lambda = (3 - \sqrt{5})k/4a$

IMPORTANT NOTE: IN CONSERVATIVE SYSTEMS, THE MATRIX A GOVERNING THE LINEARIZED PROBLEM IS SYMMETRIC ($A = A^T$)

STAIBLITY OF CONSERVATIVE SYSTEMS (LEJEUNE-DIRICHLET THEOREM)

$\mathbf{p} \equiv (\mathbf{q}, \dot{\mathbf{q}})$; $\mathbf{q}$: generalized displ., $\mathcal{K}(\mathbf{q}, \dot{\mathbf{q}})$: (kinetic energy), $\mathcal{E}(\mathbf{q})$: (potential energy)

- $\dot{\mathcal{K}} + \dot{\mathcal{E}} = 0$, (**CONSERVATIVE SYSTEM**)
- $\mathcal{E}(\mathbf{q}) \geq \mathcal{E}(\mathbf{q}_e)$

$\Longrightarrow L(\mathbf{p}(t)) \equiv \mathcal{E}(\mathbf{q}) + \mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) - \mathcal{E}(\mathbf{q}_e)$ **IS SYSTEM'S LYAPUNOV FUNCTIONAL**

PROOF :

$\exists\ c_1 > 0,\ c_2 > 0$ such : $\mathcal{E}(\mathbf{q}) - \mathcal{E}(\mathbf{q}_e) \geq c_1 \parallel \mathbf{q} - \mathbf{q}_e \parallel^2, \quad \mathcal{K} \geq c_2 \parallel \dot{\mathbf{q}} \parallel^2$

$$\mathcal{E}(\mathbf{q}) - \mathcal{E}(\mathbf{q}_e) + \mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) \geq c \parallel (\mathbf{q} - \mathbf{q}_e, \dot{\mathbf{q}} - \dot{\mathbf{q}}_e) \parallel^2 = c \parallel \mathbf{p} - \mathbf{p}_e \parallel^2$$

$\exists\ d_1 > 0,\ d_2 > 0$ such : $\mathcal{E}(\mathbf{q}(0)) - \mathcal{E}(\mathbf{q}_e) \leq d_1 \parallel \mathbf{q}(0) - \mathbf{q}_e \parallel^2, \quad \mathcal{K} \leq d_2 \parallel \dot{\mathbf{q}}(0) \parallel^2$

$$\mathcal{E}(\mathbf{q}(0)) - \mathcal{E}(\mathbf{q}_e) + \mathcal{K}(\mathbf{q}(0), \dot{\mathbf{q}}(0)) \leq d \parallel \mathbf{q}(0) - \mathbf{q}_e, \dot{\mathbf{q}}(0) - \dot{\mathbf{q}}_e \parallel^2 = d \parallel \mathbf{p}(0) - \mathbf{p}_e \parallel^2$$

CONSERVATIVE SYSTEM IS STABLE IFF POTENTIAL ENERGY MINIMIZED AT EQUILIBRIUM