REVIEW – IMPORTANT POINTS TO REMEMBER

WHAT IS THE FINITE ELEMENT METHOD?

RALEIGH-RITZ NUMERICAL SOLUTION TECHNIQUE IN APPLIED MATHEMATICS:

- IDEA STARTED WITH VIBRATION THEORY: FOR CONTINUUM PROBLEMS WITH AN ENERGY, USE SHAPE FUNCTIONS TO CONVERT INFINITE DIMENSIONAL PROBLEM TO A DISCRETE ONE THAT CAN BE SOLVED WITH MATRIX ALGEBRA (1909)

- BY ABOUT 1970’s PEOPLE REALIZED THAT THE APPROXIMATE ENGINEERING F.E.M. TECHNIQUE WAS A RALEIGH-RITZ METHOD WITH INGENIOUS SHAPE FUNCTIONS OF COMPACT SUPPORT

THE REST IS THE HISTORY OF ONE OF THE GREATEST CONTRIBUTIONS OF MECHANICS AND APPLIED MATHEMATICS TO MODERN EGINEERING TECHNOLOGY

- APPROACH THAT STARTED WITH LINEAR ELASTICITY WAS EXTENDED TO THE MOST GENERAL TYPE OF NONLINEAR, INELASTIC SOLIDS & STRUCTURES

- METHOD IS APPLICABLE TO A WIDE CLASS OF BOUNDARY PROBLEMS BUT IS BEST SUITED FOR ELLIPTIC PROBLEMS

- FINITE ELEMENTS TECHNOLOGY IS ONE OF THE MOST IMPORTANT CONTRIBUTIONS OF MECHANICS THAT REVOLUTIONIZED ENGINEERING TECHNOLOGY

TOPICS COVERED

1. INTRODUCTION TO THE FINITE ELEMENT METHOD USING 1-D MODELS.
2. CHOLESKY METHOD FOR SOLVING LINEAR SYSTEMS.
3. TRUSSES AND FRAMES IN 2D AND 3D.
4. ASSEMBLY OF STIFFNESS MATRIX & FORCE VECTOR, CONNECTIVITY
5. VARIATIONAL FORMULATION FOR LINEAR ELASTICITY B.V.P.
6. PLANE STRESS/STRAIN PROBLEMS USING CONSTANT STRAIN TRIANGLES.
7. ISOPARAMETRIC ELEMENTS FOR 2D PROBLEMS.
8. NUMERICAL INTEGRATION, GENERALIZATION TO 3D PROBLEMS.
9. HIGHER ORDER GRADIENT ENERGIES: BEAMS (1D) AND PLATES (2D).
10. LOCKING PHENOMENA DUE TO CONSTRAINTS.
11. TIME-DEPENDENT ANALYSES, EIGENMODES.
12. NON-LINEAR PROBLEMS – INCREMENTAL NEWTON-RAPHSON.
13. NON-LINEAR BEAMS (1D) & FINITE STRAIN ELASTICITY (2D).
14. NOTIONS OF FRACTURE IN 2D (CRACK-TIP SINGULARITIES)

SIMPLEST CASE: 1D ELASTIC BAR EXAMPLE

TO ILLUSTRATE ENERGY, RALEIGH-RITZ & FEM

ONE DIMENSIONAL EXAMPLE – ENERGY METHOD

STARTING POINT FOR FEM: use potential energy minimization (variational method) for the case of a non-disspative mechanics problem – all problems in elasticity (linear or nonlinear fall in this category)

$$\text{Potential} \;:\; \mathcal{P} = \mathcal{P}_{int} + \mathcal{P}_{ext}$$

$$\text{Internal} \;:\; \mathcal{P}_{int} = \int_V \frac{1}{2}\sigma(x)\,\epsilon(x)\,dV = \frac{1}{2}\int_0^L E\,A(x)\left(\frac{du}{dx}\right)^2 dx$$

$$\text{External} \;:\; \mathcal{P}_{ext} = -\int_V \rho\,g\,u(x)\,dV = -\int_0^L \rho\,g\,A(x)\,u(x)\,dx$$

CLAIM: of all admissible displacement fields *u(x)*, i.e. continuous functions that satisfy the essential boundary condition: *u(0)* = *0*, the actual equilibrium solution minimizes the potential energy functional $\mathcal{P}(u(x))$

VARIATIONAL FORMULATION: We must minimize potential energy functional $\mathcal{P}(u(x))$, to find equilibrium $u_{eq}(x)$

$$\mathcal{P}(u_{eq} + \epsilon\delta u) \geq \mathcal{P}(u_{eq}) \; ; \qquad u(x) \equiv u_{eq}(x) + \epsilon\delta u(x) \; , \; \epsilon \in \mathbb{R}$$

$$\frac{d}{d\epsilon}\left[\mathcal{P}(u_{eq} + \epsilon\delta u)\right]_{\epsilon=0} = 0 \; ; \qquad \text{extremum } (1)$$

$$\frac{d^2}{d\epsilon^2}\left[\mathcal{P}(u_{eq} + \epsilon\delta u)\right]_{\epsilon=0} > 0 \; ; \qquad \text{minimum } (2)$$

Raleigh-Ritz method: instead of minimizing energy in an infinite dimensional space, we minimize in a finite dimensional space. We use an approximate displacement $u^{app}(x)$ – which involves a finite number of variables Q_i $(i=1, \ldots n)$ – and minimize $\mathcal{P}(\mathbf{Q})$ with respect to **Q.**

$$\mathcal{P}(u^{app}(x)) = \mathcal{P}(\mathbf{Q}); \quad u^{app} = \sum_{i=1}^{i=n} Q_i N_i(x), \quad \mathbf{Q} \equiv [Q_1, Q_2, \cdots Q_n]$$

ONE DIMENSIONAL EXAMPLE – RALEIGH-RITZ METHOD

$$\frac{\partial \mathcal{P}(\mathbf{Q})}{\partial Q_i} = \int_0^L \left[EA \frac{du^{app}}{dx} \frac{\partial}{\partial Q_i} \left(\frac{du^{app}}{dx} \right) - \rho g A \frac{\partial u^{app}}{\partial Q_i} \right] dx = 0$$

$$\sum_{j=1}^{j=n} \left\{ \int_0^L \left[EA \frac{dN_i}{dx} \frac{dN_j}{dx} \right] dx \right\} Q_j - \int_0^L \left[\rho g A N_i \right] dx = 0$$

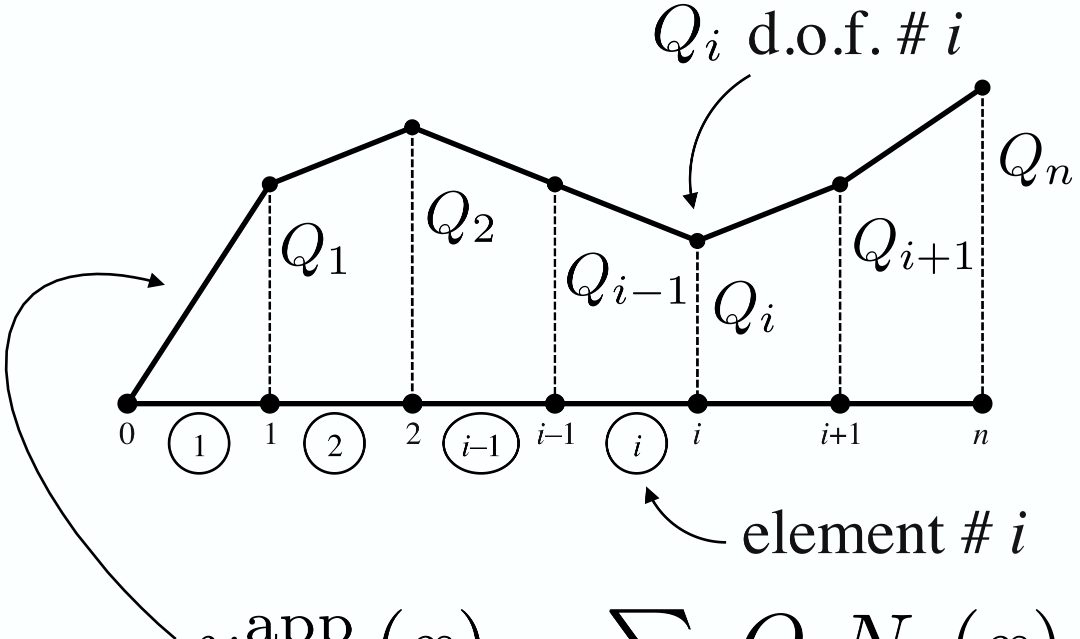
$$\sum_{j=1}^{j=n} K_{ij} Q_j - F_i = 0 \; ; \quad (\text{in compact form} : \mathbf{KQ} = \mathbf{F})$$

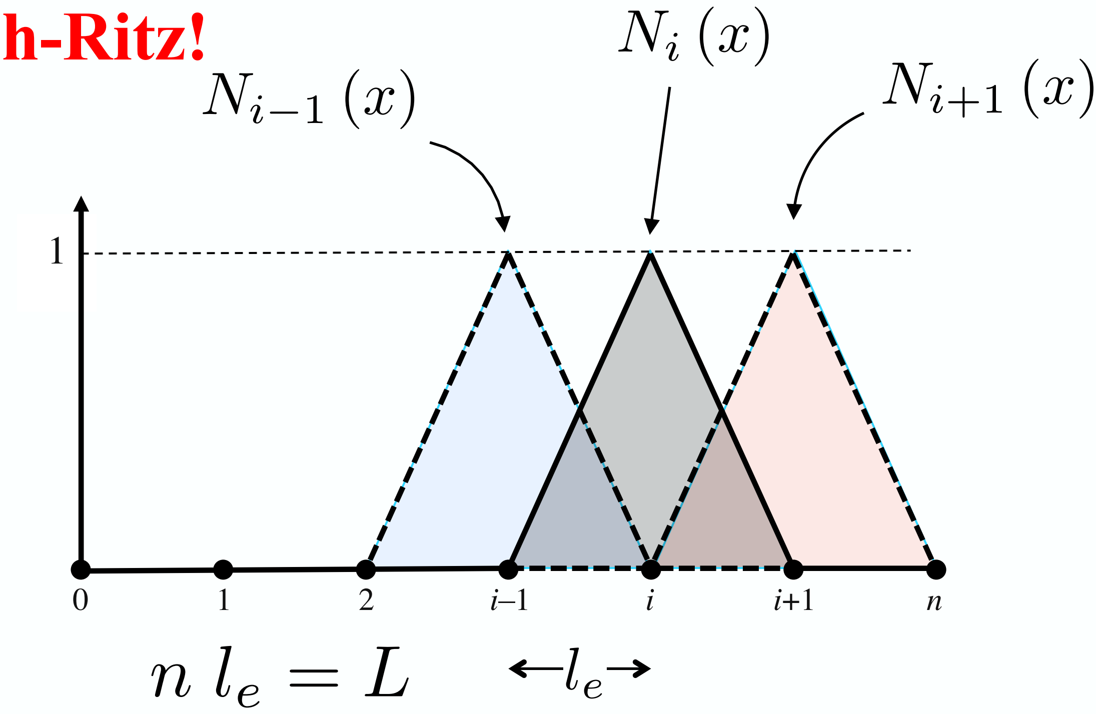
$$K_{ij} \equiv \int_0^L \left[EA \frac{dN_i}{dx} \frac{dN_j}{dx} \right] dx \; , F_i \equiv \int_0^L \left[\rho g A N_i \right] dx$$

Stiffness matrix: **K**, Force vector: **F**, Degrees of Freedom: **Q**

ONE DIMENSIONAL EXAMPLE – FINITE ELEMENT METHOD

FEM method: **Special case of Rayleigh-Ritz!**





$$u^{\text{app}}(x) = \sum_i Q_i N_i(x)$$

Shape function $N_i(x)$

Easy physical interpretation of d.o.f. (degree of freedom) Q_i at node x_i: due to its construction, $u^{app}(x_i) = Q_i$

Shape functions $N_i(x)$ have compact support: $N_i(x_i) = 1$, $N_i(x_{i-1}) = N_i(x_{i+1}) = 0$.
Compactness of support of shape function great advantage of FEM

ONE DIMENSIONAL EXAMPLE – SPARSE STIFFNESS MATRIX

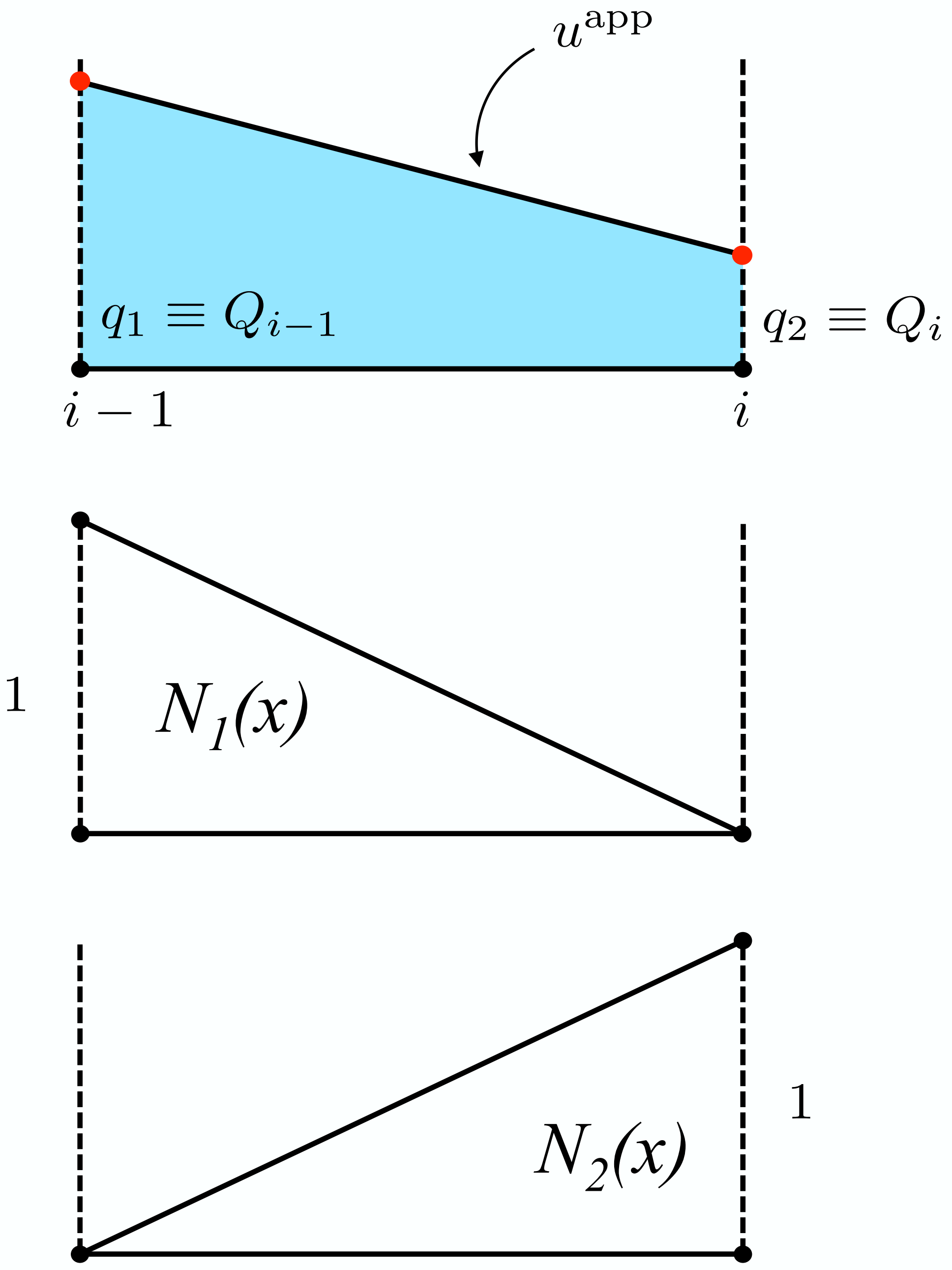
$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & 0 \\ 0 & K_{32} & K_{33} & K_{34} & 0 \\ 0 & 0 & K_{43} & K_{44} & K_{45} \\ 0 & 0 & 0 & K_{54} & K_{55} \end{bmatrix}$$

$$K_{ii} = \int_{x_{i-1}}^{x_{i+1}} E\ A(x) \left(\frac{1}{l_e}\right)^2 dx$$

$$K_{i\ i+1} = -\int_{x_i}^{x_{i+1}} E\ A(x) \left(\frac{1}{l_e}\right)^2 dx$$

Stiffness matrix **K** is banded, i.e. populated about the diagonal. This structure, due to the compactness of shape functions, has great advantages in both solution time and storage requirements. For reasonable systems, we use Cholesky (André-Louis Cholesky X-1895) decomposition, for very large systems, iterative methods that take advantage of the sparse structure of **K**.

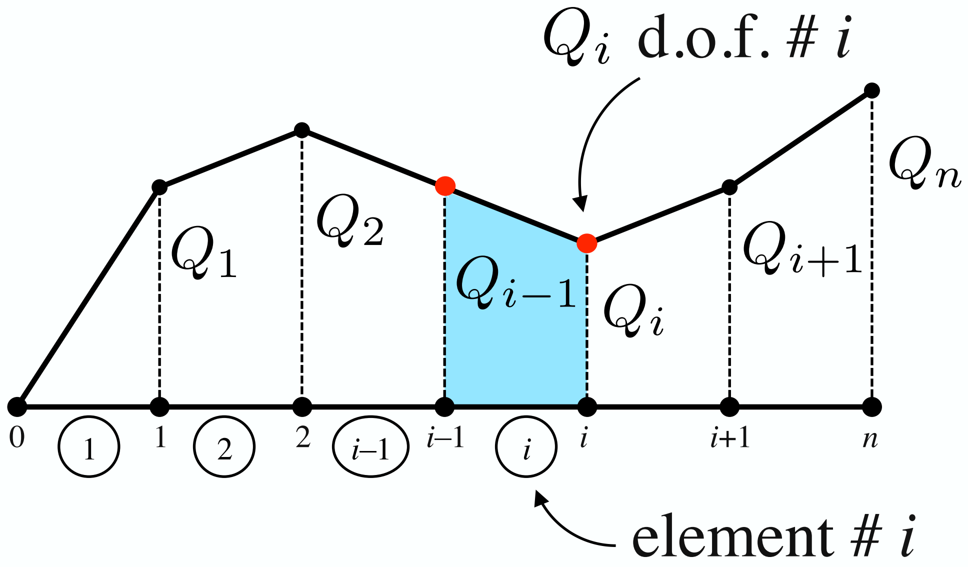
ONE DIMENSIONAL EXAMPLE – ELEMENT STIFFNESS, FORCE



In element i: $u^{app}(x) = q_1 N_1(x) + q_2 N_2(x)$

Local degree of freedom $\mathbf{q}^T_e = [q_1, q_2]$

We find element contribution to global stiffness matrix **K** and force vector **F**



ONE DIMENSIONAL EXAMPLE – ELEMENT STIFFNESS, FORCE

$$\mathcal{P}(\mathbf{Q}) = \mathcal{P}_{int}(\mathbf{Q}) + \mathcal{P}_{ext}(\mathbf{Q})$$

$$\mathcal{P}_{int}(\mathbf{Q}) = \sum_e \mathcal{P}^e_{int} \; ; \quad \mathcal{P}^e_{int} = \int_{l_e} \left[\frac{1}{2} E \; A(x) \left(q_1 \frac{dN_1}{dx} + q_2 \frac{dN_2}{dx} \right)^2 \right] dx = \frac{1}{2} \mathbf{q}_e^T \mathbf{k}_e \mathbf{q}_e$$

$$\mathcal{P}_{ext}(\mathbf{Q}) = \sum_e \mathcal{P}^e_{ext} \; ; \quad \mathcal{P}^e_{ext} = - \int_{l_e} \left[\rho g \; A(x) \left(q_1 N_1(x) + q_2 N_2(x) \right) \right] dx = -\mathbf{q}_e^T \mathbf{f}_e$$

$$\boxed{[\mathbf{k}_e]_{ij} \equiv \int_{l_e} \left[E \; A(x) \left(\frac{dN_i}{dx} \right) \left(\frac{dN_j}{dx} \right) \right] dx \; ;}$$

$\mathbf{k}_e$ element stiffness matrix

$$\boxed{[\mathbf{f}_e]_i \equiv - \int_{l_e} \left[\rho g \; A(x) N_i(x) \right] dx \; ;}$$

$\mathbf{f}_e$ element force vector

Finding element stiffness matrix $\mathbf{k}_e$ and element force vector $\mathbf{f}_e$ in the structure

ONE DIMENSIONAL EXAMPLE – ASSEMBLE STIFFNESS, FORCE

$$\begin{array}{c|ccccc|} & & i & & j & \\ & & \vdots & & \vdots & \\ i & \cdots & k^e_{11} & \cdots & k^e_{12} & \cdots \\ & & \vdots & & \vdots & \\ j & \cdots & k^e_{21} & \cdots & k^e_{22} & \cdots \\ & & \vdots & & \vdots & \end{array}$$

$$\left[\begin{array}{ccccc} \cdots & f^e_1 & \cdots & f^e_2 & \cdots \end{array}\right]$$

Assembling global stiffness matrix **K** and global force vector **F** from element stiffness matrix $\mathbf{k}_e$ and element force vector $\mathbf{f}_e$

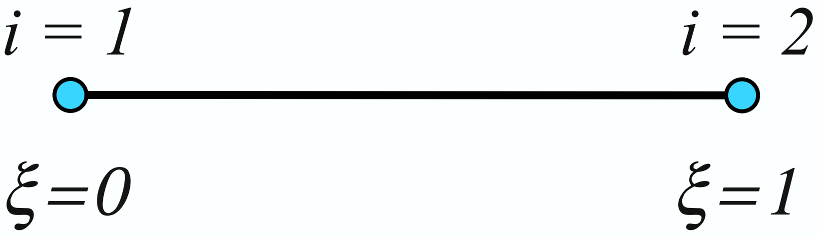
RULE: for each element *e* add to global stiffness matrix & force vector the components in the appropriate places recalling local to global numbering

1→i, 2→j for this 2-node element

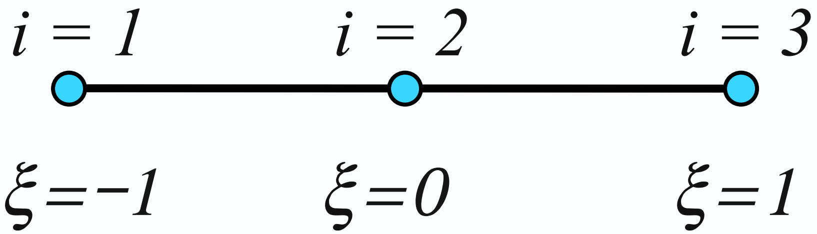
SIMPLEST CASE: 1D ELASTIC BAR EXAMPLE TO ILLUSTRATE ISOPARAMETRIC MASTER ELEMENT

ONE DIMENSIONAL EXAMPLE – MASTER ELEMENT

It is very convenient to write shape functions in a master element with respect to a normalized coordinate (ξ)



$$m = 2\,; \quad N_1(\xi) = 1 - \xi$$
$$N_2(\xi) = \xi$$



$$m = 3\,; \quad N_1(\xi) = \xi(\xi - 1)/2$$
$$N_2(\xi) = (1 - \xi)(\xi + 1)/2$$
$$N_3(\xi) = \xi(\xi + 1)/2$$

$$[\mathbf{k}_e]_{ij} \equiv \int_\xi \left[E\ A(x(\xi))\,(dN_i/d\xi)\,(dN_j/d\xi)\,(dx/d\xi)^{-1} \right] d\xi$$

$$[\mathbf{f}_e]_i \equiv -\int_\xi \left[\rho g\ A(x(\xi)) N_i(\xi)\,(dx/d\xi)^{-1} \right] d\xi$$

ONE DIMENSIONAL EXAMPLE – ISOPARAMETRIC CASE

Question: what do we choose for *x(ξ)*?

Answer: (easy) same representation as for displacement!

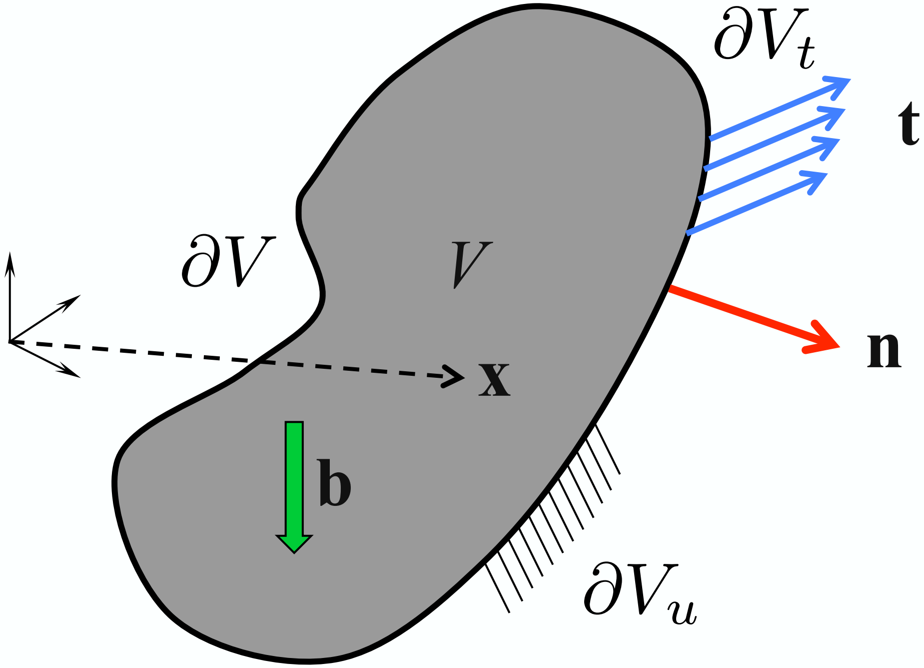
This type of parametrization that uses the same interpolation scheme for both the displacement and the geometric coordinates is called isoparametric representation and is widely used in F.E.M.

$$u(\xi) = \sum_{i=1}^{i=m} u_i N_i(\xi) \ , \quad u(\xi_i) = u_i \ ; \qquad \text{d.o.f. at node i}$$

$$x(\xi) = \sum_{i=1}^{i=m} x_i N_i(\xi) \ , \quad x(\xi_i) = x_i \ ; \qquad \text{coordinate at node i}$$

NEXT CASE: 2D ELASTICITY TO ILLUSTRATE THE FEM METHOD IN HIGHER DIMENSION PROBLEMS

REVIEW OF SMALL STRAIN LINEAR ELASTICITY



Solid occupies domain: V

Domain boundary: ∂V

Body forces: $\mathbf{b}$

Surface traction: $\mathbf{t}$

Surface normal (outward): $\mathbf{n}$

Traction prescribed on: ∂V_t

Displacement prescribed on: ∂V_u

Position vector: $\mathbf{x}$

Energy density: $W(\boldsymbol{\varepsilon})$

Stress-strain: $\sigma_{ij} = \dfrac{\partial W}{\partial \epsilon_{ij}}$

(general nonlinear elastic material)

REVIEW OF SMALL STRAIN LINEAR ELASTICITY

$$\text{Potential} \ : \quad \mathcal{P} = \mathcal{P}_{int} + \mathcal{P}_{ext}$$

$$\text{Internal} \ : \quad \mathcal{P}_{int} = \int_V W(\epsilon_{ij}) \, dV \ ; \quad \sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$$

$$\text{External} \ : \quad \mathcal{P}_{ext} = - \int_V b_i u_i \, dV - \int_{\partial V_t} t_i u_i \, dS$$

$$\mathcal{P}(\mathbf{u} + \epsilon \delta \mathbf{u}) \geq \mathcal{P}(\mathbf{u}) \ ; \qquad \delta \mathbf{u}(\mathbf{x}) = 0 \ \forall \ \mathbf{x} \in \partial V_u \ , \ \epsilon \in \mathbb{R}$$

$$\frac{d}{d\epsilon} \left[\mathcal{P}(\mathbf{u} + \epsilon \delta \mathbf{u}) \right]_{\epsilon=0} = 0 \ ; \qquad \text{extremum (1)}$$

$$\frac{d^2}{d\epsilon^2} \left[\mathcal{P}(\mathbf{u} + \epsilon \delta \mathbf{u}) \right]_{\epsilon=0} > 0 \ ; \qquad \text{minimum (2)}$$

REVIEW OF SMALL STRAIN LINEAR ELASTICITY

$$\text{Linearized kinematics} \ : \quad \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\text{Boundary traction} \ : \quad t_j = n_i \, \sigma_{ij} \ ; \qquad \text{(Cauchy tetrahedron)}$$

$$\text{Linear elasticity} \ : \quad \sigma_{ij} = L_{ijkl} \, \epsilon_{kl}$$

$$\text{Major symmetry} \ : \quad L_{ijkl} = L_{klij} \ ; \qquad \text{(due to energy existence)}$$

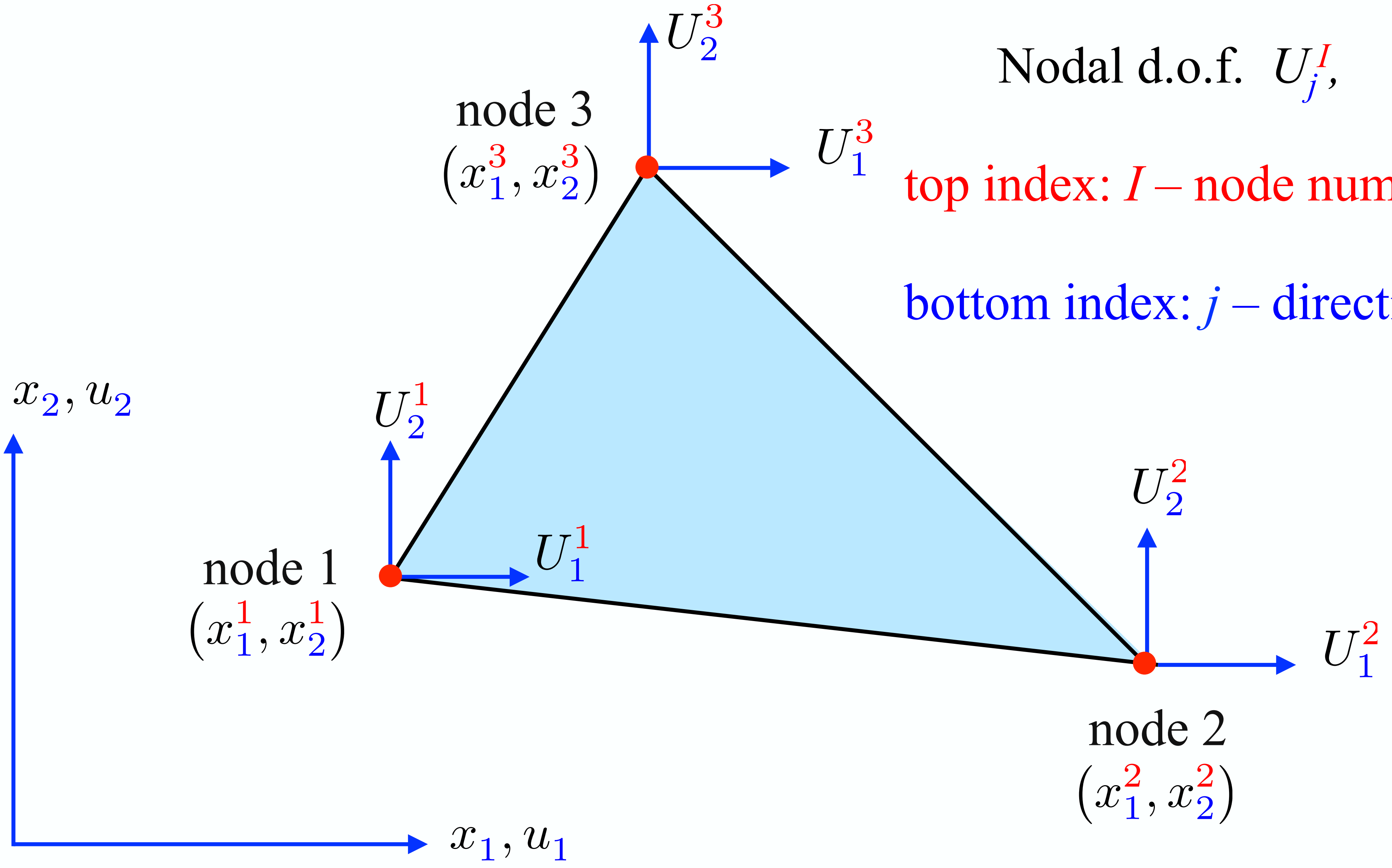
$$\text{Minor symmetries} \ : \quad L_{ijkl} = L_{jikl} = L_{ijlk} \ ; \ (\sigma_{ij} = \sigma_{ji} \ , \ \ \epsilon_{ij} = \epsilon_{ji})$$

$$\text{Energy density} \ : \quad W = \int_0^{\epsilon} \sigma_{ij} \, \epsilon_{ij} \, d\epsilon = \frac{1}{2} L_{ijkl} \, \epsilon_{ij} \, \epsilon_{kl}$$

Linearized strain: ε_{ij} Cauchy stress: σ_{ij} Elastic moduli tensor: L_{ijkl}

SIMPLEST 2D ELEMENT: CONSTANT STRAIN TRIANGLE

FEM IN 2D: CONSTANT STRAIN TRIANGLES



Nodal d.o.f. U_j^I,

top index: I – node number

bottom index: j – direction

Element d.o.f. $\mathbf{q}_e^T = [U_1^1, U_2^1, U_1^2, U_2^2, U_1^3, U_2^3]$

FEM IN 2D: CONSTANT STRAIN TRIANGLES

$$u_i\left(x_1,x_2\right)=N_1\left(x_1,x_2\right)U_i^1+N_2\left(x_1,x_2\right)U_i^2+N_3\left(x_1,x_2\right)U_i^3$$

$$\boxed{u_i\left(x_1,x_2\right)=\sum_{I=1}^{3}N_I\left(x_1,x_2\right)U_i^I}$$

displacement interpolation

$$u_i\left(x_1^I,x_2^I\right)=U_i^I\ ,\quad \text{nodal value requirement}$$

$$N_1\left(x_1^1,x_2^1\right)=1\ ,\quad N_1\left(x_1^2,x_2^2\right)=0\ ,\quad N_1\left(x_1^3,x_2^3\right)=0$$

$$N_2\left(x_1^1,x_2^1\right)=0\ ,\quad N_2\left(x_1^2,x_2^2\right)=1\ ,\quad N_2\left(x_1^3,x_2^3\right)=0$$

$$N_3\left(x_1^1,x_2^1\right)=0\ ,\quad N_3\left(x_1^2,x_2^2\right)=0\ ,\quad N_3\left(x_1^3,x_2^3\right)=1$$

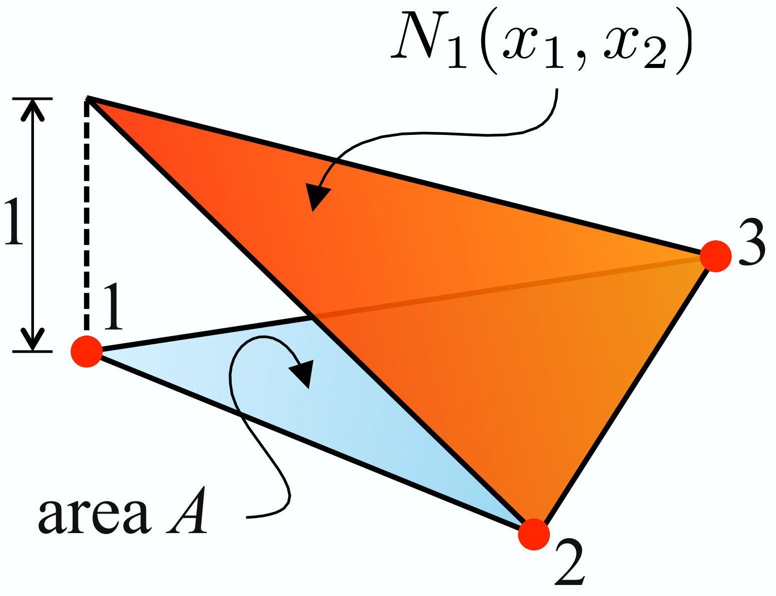
FEM IN 2D: CONSTANT STRAIN TRIANGLES

Shape functions $N_I(x_1, x_2)$ are bilinear in terms of coordinates

(3 constants found from the 3 nodal conditions – example N_1

$$N_1\left(x_1, x_2\right) = ax_1 + bx_2 + c$$

$$\begin{cases} 1 = ax_1^1 + bx_2^1 + c \\ 0 = ax_1^2 + bx_2^2 + c \\ 0 = ax_1^3 + bx_2^3 + c \end{cases}$$



FEM IN 2D: CONSTANT STRAIN TRIANGLES

The three bilinear shape functions $N_I(x_1, x_2)$; $(I=1, 2, 3)$

$$N_1(x_1, x_2) = \frac{1}{2A}\left[x_1^2 x_2^3 - x_1^3 x_2^2 + \left(x_2^2 - x_2^3\right) x_1 - \left(x_1^2 - x_1^3\right) x_2\right]$$

$$N_2(x_1, x_2) = \frac{1}{2A}\left[x_1^3 x_2^1 - x_1^1 x_2^3 + \left(x_2^3 - x_2^1\right) x_1 - \left(x_1^3 - x_1^1\right) x_2\right]$$

$$N_3(x_1, x_2) = \frac{1}{2A}\left[x_1^1 x_2^2 - x_1^2 x_2^1 + \left(x_2^1 - x_2^2\right) x_1 - \left(x_1^1 - x_1^2\right) x_2\right]$$

FEM IN 2D: CONSTANT STRAIN TRIANGLES

Displacement discretization is conveniently written in matrix form: $\mathbf{u} = \mathbf{N}\mathbf{q}_e$

$$\underbrace{\begin{bmatrix} u_1(x_1,x_2) \\ u_2(x_1,x_2) \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} N_1(x_1,x_2) & 0 & N_2(x_1,x_2) & 0 & N_3(x_1,x_2) & 0 \\ 0 & N_1(x_1,x_2) & 0 & N_2(x_1,x_2) & 0 & N_3(x_1,x_2) \end{bmatrix}}_{\mathbf{N}} \underbrace{\begin{bmatrix} U_1^1 \\ U_2^1 \\ U_1^2 \\ U_2^2 \\ U_1^3 \\ U_2^3 \end{bmatrix}}_{\mathbf{q}_e}$$

Kinematic discretization is also written in matrix form: $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{q}_e$

FEM IN 2D: CONSTANT STRAIN TRIANGLES

Kinematics discretization is conveniently written in matrix form: $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{q}_e$

$$\mathbf{B} \equiv \frac{1}{2A}\begin{bmatrix} x_2^2 - x_2^3 & 0 & x_2^3 - x_2^1 & 0 & x_2^1 - x_2^2 & 0 \\ 0 & -\left(x_1^2 - x_1^3\right) & 0 & -\left(x_1^3 - x_1^1\right) & 0 & -\left(x_1^1 - x_1^2\right) \\ -\left(x_1^2 - x_1^3\right) & x_2^2 - x_2^3 & -\left(x_1^3 - x_1^1\right) & x_2^3 - x_2^1 & -\left(x_1^1 - x_1^2\right) & x_2^1 - x_2^2 \end{bmatrix}$$

NOTE: **B** matrix is constant (constant strain triangle!)

$\varepsilon^T = [\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}]$; where $\gamma_{12} = 2\,\varepsilon_{12}$

Recall: $\mathbf{q}_e{}^T = [U_1{}^1, U_2{}^1, U_1{}^2, U_2{}^2, U_1{}^3, U_2{}^3]$

FEM IN 2D: CONSTANT STRAIN TRIANGLES

Constitutive equation also written in matrix form: $\boldsymbol{\sigma} = \mathbf{L}\boldsymbol{\varepsilon}$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{bmatrix} \longleftarrow 2\epsilon_{12}$$

$$\mathbf{L} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}, \quad \mathbf{L} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

plane strain

plane stress

FEM IN 2D: CONSTANT STRAIN TRIANGLES

$$\mathcal{P}^e_{int} = \int_A \frac{1}{2}[\boldsymbol{\epsilon}^T \boldsymbol{\sigma}(x_1, x_2)]\, dA = \frac{1}{2}\mathbf{q}_e^T \left[\int_A [\mathbf{B}^T \mathbf{L}(x_1, x_2)\mathbf{B}]\, dA\right] \mathbf{q}_e$$

$$= \frac{1}{2}\mathbf{q}_e^T \mathbf{k}_e \mathbf{q}_e$$

Element stiffness matrix: $\mathbf{k}_\mathrm{e}$

$$-\mathcal{P}^e_{ext} = \int_A [\mathbf{u}^T(x_1, x_2)\mathbf{b}(x_1, x_2)]\, dA + \int_{\partial A} [\mathbf{u}^T(x_1, x_2)\mathbf{t}(x_1, x_2)]\, dl$$

$$= \mathbf{q}_e^T \left[\int_A [\mathbf{N}^T(x_1, x_2)\ \mathbf{b}(x_1, x_2)]\, dA + \int_{\partial A} [\mathbf{N}^T(x_1, x_2)\ \mathbf{t}(x_1, x_2)]\, dl\right]$$

$$= \mathbf{q}_e^T \mathbf{f}_e$$

Element force vector: $\mathbf{f}_\mathrm{e}$

FEM IN 2D: CONSTANT STRAIN TRIANGLES

Element stiffness matrix: $\mathbf{k}_e$ for constant moduli **L**

$$\mathbf{k}_e = \int_A \left[\mathbf{B}^T \mathbf{L} \mathbf{B}\right] dA$$

$$\mathbf{k}_e = A\,\mathbf{B}^T \mathbf{L} \mathbf{B}$$

Element force vector: $\mathbf{f}_e$ for constant body forces **b** & traction **t**

$$\mathbf{f}_e = \int_A [\mathbf{N}^T(x_1, x_2)\ \mathbf{b}]\, dA + \int_{\partial A} [\mathbf{N}^T(x_1(l), x_2(l))\ \mathbf{t}]\, dl$$

$$\mathbf{f}_e^T = \left[\frac{b_1}{3} + \frac{t_1}{2}, \ \frac{b_2}{3} + \frac{t_2}{2}, \ \frac{b_1}{3} + \frac{t_1}{2}, \ \frac{b_2}{3} + \frac{t_2}{2}, \ \frac{b_1}{3}, \ \frac{b_2}{3}\right]$$

NOTE: element has traction applied on the side defined by nodes 1 & 2

ISOPARAMETRIC CONSIDERATIONS FOR 2D FEM

FEM IN 2D: ISOPARAMETRIC CONSTANT STRAIN TRIANGLES

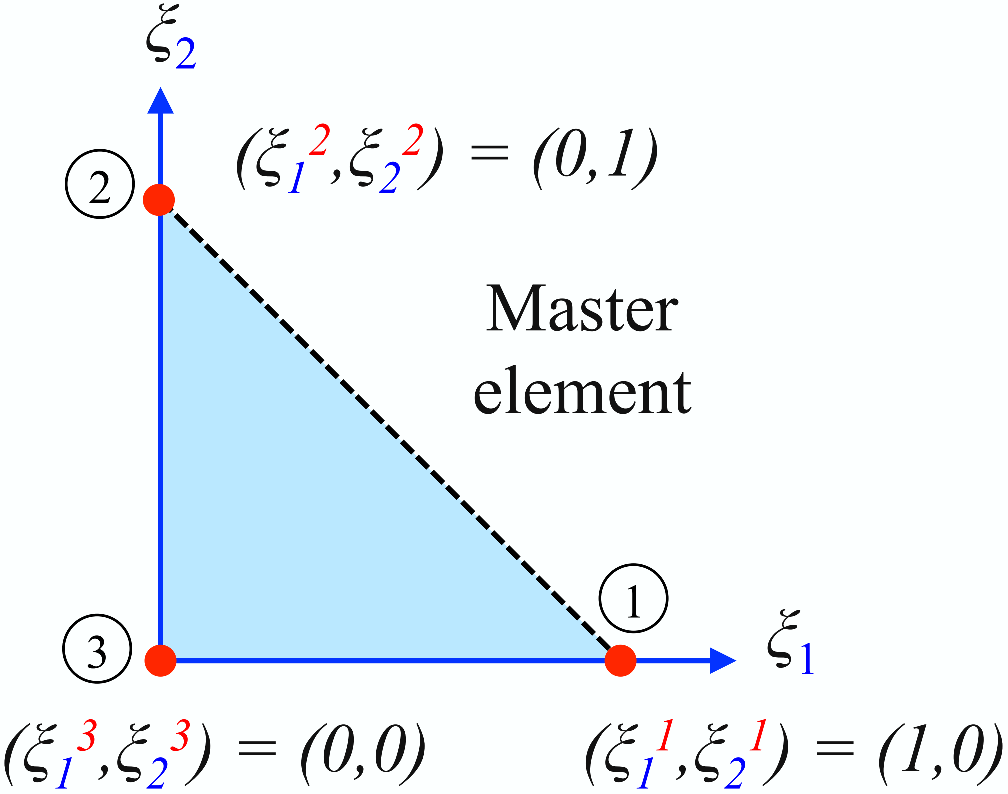
$$u_j\left(\xi_1,\xi_2\right)=\sum_{I=1}^{3}N_I\left(\xi_1,\xi_2\right)U_j^I$$

Parameters: ξ_1, ξ_2

$$u_j\left(\xi_1^I,\xi_2^I\right)=U_j^I$$

$$x_j\left(\xi_1,\xi_2\right)=\sum_{I=1}^{3}N_I\left(\xi_1,\xi_2\right)x_j^I$$

$$x_j\left(\xi_1^I,\xi_2^I\right)=x_j^I$$



FEM IN 2D: ISOPARAMETRIC CONSTANT STRAIN TRIANGLES

Master element shape functions $N_I(x_1, x_2)$ are found to be:

$$N_1\left(\xi_1^1, \xi_2^1\right) = 1, \quad N_1\left(\xi_1^2, \xi_2^2\right) = 0, \quad N_1\left(\xi_1^3, \xi_2^3\right) = 0$$

$$\Longrightarrow \quad N_1\left(\xi_1, \xi_2\right) = \xi_1$$

$$N_2\left(\xi_1^1, \xi_2^1\right) = 0, \quad N_2\left(\xi_1^2, \xi_2^2\right) = 1, \quad N_2\left(\xi_1^3, \xi_2^3\right) = 0$$

$$\Longrightarrow \quad N_2\left(\xi_1, \xi_2\right) = \xi_2$$

$$N_3\left(\xi_1^1, \xi_2^1\right) = 0, \quad N_3\left(\xi_1^2, \xi_2^2\right) = 0, \quad N_3\left(\xi_1^3, \xi_2^3\right) = 1$$

$$\Longrightarrow \quad N_3\left(\xi_1, \xi_2\right) = 1 - \xi_1 - \xi_2$$

FEM IN 2D: ISOPARAMETRIC CONSTANT STRAIN TRIANGLES

For strains we need the transformation (Hessian) matrix **J** and its inverse **J**$^{-1}$

$$\begin{bmatrix} \dfrac{\partial u_i}{\partial \xi_1} \\[2ex] \dfrac{\partial u_i}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x_1}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_1} \\[2ex] \dfrac{\partial x_1}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_2} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u_i}{\partial x_1} \\[2ex] \dfrac{\partial u_i}{\partial x_2} \end{bmatrix} \quad , \quad \mathbf{J} = \begin{bmatrix} x_1^1 - x_1^3 & x_2^1 - x_2^3 \\[2ex] x_1^2 - x_1^3 & x_2^2 - x_2^3 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial u_i}{\partial x_1} \\[2ex] \dfrac{\partial u_i}{\partial x_2} \end{bmatrix} = \mathbf{J}^{-\mathbf{1}} \begin{bmatrix} \dfrac{\partial u_i}{\partial \xi_1} \\[2ex] \dfrac{\partial u_i}{\partial \xi_2} \end{bmatrix} \quad , \quad \text{recall} : x_j(\xi_1, \xi_2) = \sum_{I=1}^{3} N_I(\xi_1, \xi_2) x_j^I$$

FEM IN 2D: ISOPARAMETRIC CONSTANT STRAIN TRIANGLES

Definition of matrix **A**

$$\boldsymbol{\epsilon} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_2} \\ \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} \end{bmatrix} = \overbrace{\begin{bmatrix} \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_2}{\partial x_1} & 0 & 0 \\ 0 & 0 & \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_2} \\ \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_2} & \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_2}{\partial x_1} \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} \dfrac{\partial u_1}{\partial \xi_1} \\ \dfrac{\partial u_1}{\partial \xi_2} \\ \dfrac{\partial u_2}{\partial \xi_1} \\ \dfrac{\partial u_2}{\partial \xi_2} \end{bmatrix}$$

Definition of matrix **G**

$$\begin{bmatrix} \dfrac{\partial u_1}{\partial \xi_1} \\ \dfrac{\partial u_1}{\partial \xi_2} \\ \dfrac{\partial u_2}{\partial \xi_1} \\ \dfrac{\partial u_2}{\partial \xi_2} \end{bmatrix} = \overbrace{\begin{bmatrix} \dfrac{\partial N_1}{\partial \xi_1} & 0 & \dfrac{\partial N_2}{\partial \xi_1} & 0 & \dfrac{\partial N_3}{\partial \xi_1} & 0 \\ \dfrac{\partial N_1}{\partial \xi_2} & 0 & \dfrac{\partial N_2}{\partial \xi_2} & 0 & \dfrac{\partial N_3}{\partial \xi_2} & 0 \\ 0 & \dfrac{\partial N_1}{\partial \xi_1} & 0 & \dfrac{\partial N_2}{\partial \xi_1} & 0 & \dfrac{\partial N_3}{\partial \xi_1} \\ 0 & \dfrac{\partial N_1}{\partial \xi_2} & 0 & \dfrac{\partial N_2}{\partial \xi_2} & 0 & \dfrac{\partial N_3}{\partial \xi_2} \end{bmatrix}}^{\mathbf{G}} \begin{bmatrix} U_1^1 \\ U_2^1 \\ U_1^2 \\ U_2^2 \\ U_1^3 \\ U_2^3 \end{bmatrix}$$

FEM IN 2D: ISOPARAMETRIC CONSTANT STRAIN TRIANGLES

Finding element stiffness using master element

$$
\begin{aligned}
\mathcal{P}^{e}_{int} &= \int_A \frac{1}{2}[\boldsymbol{\epsilon}^T \boldsymbol{\sigma}(x_1, x_2)]\, dA \\
&= \tfrac{1}{2}\mathbf{q}_e^T \int_{\boldsymbol{\xi}} [\mathbf{G}^T\mathbf{A}^T\mathbf{L}(\xi_1, \xi_2)\mathbf{A}\mathbf{G}(\det \mathbf{J})]\, d\boldsymbol{\xi}\Bigg]\, \mathbf{q}_e \\
&= \frac{1}{2}\mathbf{q}_e^T \mathbf{k}_e \mathbf{q}_e
\end{aligned}
$$

ISOPARAMETRIC QUADS & HIGHER ORDER 2D ELEMENTS

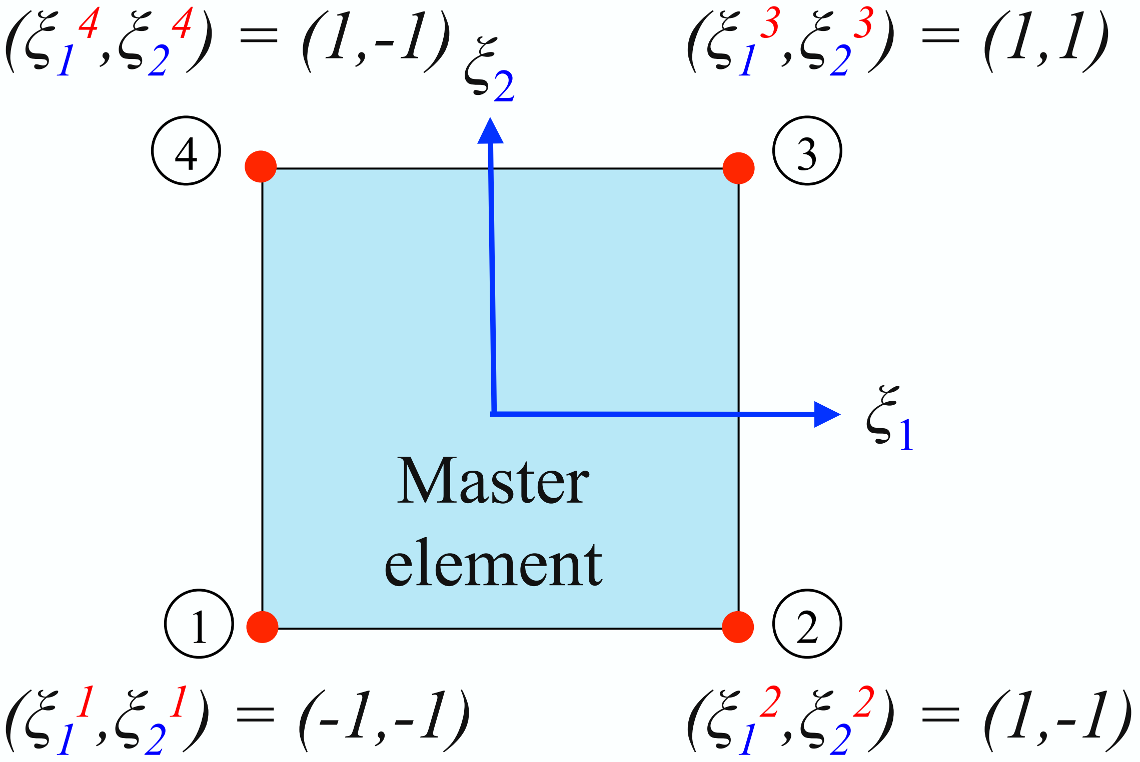
Isoparametric quadrilateral element (quad)

$$u_j\left(\xi_1,\xi_2\right)=\sum_{I=1}^{4}N_I\left(\xi_1,\xi_2\right)U_j^I \qquad N_I(\xi_1^J,\xi_2^J)=\delta_{IJ};\;\;(I,J=1,\dots 4)$$

$$u_j\left(\xi_1^I,\xi_2^I\right)=U_j^I$$

$$x_j\left(\xi_1,\xi_2\right)=\sum_{I=1}^{4}N_I\left(\xi_1,\xi_2\right)x_j^I$$

$$x_j\left(\xi_1^I,\xi_2^I\right)=x_j^I$$



FEM IN 2D: ISOPARAMETRIC 4-NODE QUAD ELEMENTS

$$N_I(\xi_1,\xi_2) = \frac{1}{4}(1+\xi_1^I\xi_1)(1+\xi_2^I\xi_2)$$

Shape functions $N_I(\xi_1, \xi_2)$ and coordinate transformation matrix **J** for 4-node isoparametric quads

$$\mathbf{J} \equiv \left[\frac{\partial x_j}{\partial \xi_i}\right] = \left[\sum_{I=1}^{4} \frac{\partial N_I}{\partial \xi_i}(\xi_1,\xi_2) x_j^I\right]$$

$$\mathbf{J} = \begin{bmatrix} \frac{1}{4}\sum_{I=1}^{4} \xi_1^I(1+\xi_2^I\xi_2)x_1^I & \frac{1}{4}\sum_{I=1}^{4} \xi_1^I(1+\xi_2^I\xi_2)x_2^I \\ \frac{1}{4}\sum_{I=1}^{4} \xi_2^I(1+\xi_1^I\xi_1)x_1^I & \frac{1}{4}\sum_{I=1}^{4} \xi_2^I(1+\xi_1^I\xi_1)x_2^I \end{bmatrix}$$

FEM IN 2D: ISOPARAMETRIC 4-NODE QUAD ELEMENTS

$$\boldsymbol{\epsilon} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} \\[2ex] \dfrac{\partial u_2}{\partial x_2} \\[2ex] \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_2}{\partial x_1} & 0 & 0 \\[2ex] 0 & 0 & \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_2} \\[2ex] \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_2} & \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_2}{\partial x_1} \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \dfrac{\partial u_1}{\partial \xi_1} \\[2ex] \dfrac{\partial u_1}{\partial \xi_2} \\[2ex] \dfrac{\partial u_2}{\partial \xi_1} \\[2ex] \dfrac{\partial u_2}{\partial \xi_2} \end{bmatrix}$$

Recall definition of matrix $\mathbf{A} = \dfrac{\mathbf{1}}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$

Recall definition of matrix $\mathbf{G}$

$$\begin{bmatrix} \frac{\partial u_1}{\partial \xi_1} \\ \frac{\partial u_1}{\partial \xi_2} \\ \frac{\partial u_2}{\partial \xi_1} \\ \frac{\partial u_2}{\partial \xi_2} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial \xi_1} & 0 & \frac{\partial N_2}{\partial \xi_1} & 0 & \frac{\partial N_3}{\partial \xi_1} & 0 & \frac{\partial N_4}{\partial \xi_1} & 0 \\ \frac{\partial N_1}{\partial \xi_2} & 0 & \frac{\partial N_2}{\partial \xi_2} & 0 & \frac{\partial N_3}{\partial \xi_2} & 0 & \frac{\partial N_4}{\partial \xi_2} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi_1} & 0 & \frac{\partial N_2}{\partial \xi_1} & 0 & \frac{\partial N_3}{\partial \xi_1} & 0 & \frac{\partial N_4}{\partial \xi_1} \\ 0 & \frac{\partial N_1}{\partial \xi_2} & 0 & \frac{\partial N_2}{\partial \xi_2} & 0 & \frac{\partial N_3}{\partial \xi_2} & 0 & \frac{\partial N_4}{\partial \xi_2} \end{bmatrix}}_{\mathbf{G}} \overbrace{\begin{bmatrix} U_1^1 \\ U_2^1 \\ U_1^2 \\ U_2^2 \\ U_1^3 \\ U_2^3 \\ U_1^4 \\ U_2^4 \end{bmatrix}}^{\mathbf{q}_e}$$

FEM IN 2D: ISOPARAMETRIC 4-NODE QUAD ELEMENTS

$$\mathcal{P}^e_{int} = \int_{A_e} \frac{1}{2}[\boldsymbol{\epsilon}^T \boldsymbol{\sigma}(x_1, x_2)]\, dA = \frac{1}{2}\mathbf{q}_e^T \left. \int_{\boldsymbol{\xi}} [\underbrace{\mathbf{G}^T \mathbf{A}^T}_{\boldsymbol{\varepsilon}^T} \underbrace{\mathbf{L}\ \mathbf{A}\ \mathbf{G}}_{\boldsymbol{\sigma}}]\, \underbrace{\det(\mathbf{J})\, d\boldsymbol{\xi}}_{dA} \right] \mathbf{q}_e$$

$$= \frac{1}{2}\mathbf{q}_e^T \mathbf{k}_e \mathbf{q}_e$$

Element stiffness matrix: $\mathbf{k}_\mathrm{e}$

$$-\mathcal{P}^e_{ext} = \int_{A_e} [\mathbf{u}^T(x_1, x_2)\mathbf{b}(x_1, x_2)]\, dA + \int_{\partial A_e} [\mathbf{u}^T(x_1, x_2)\mathbf{t}(x_1, x_2)]\, dl$$

$$= \mathbf{q}_e^T \left[\int_{\boldsymbol{\xi}} [\mathbf{N}^T\ \mathbf{b}]\, \det(\mathbf{J})\, d\boldsymbol{\xi} + \int_{\partial \boldsymbol{\xi}} [\mathbf{N}^T\ \mathbf{t}]\, dl(\xi) \right]$$

$$= \mathbf{q}_e^T \mathbf{f}_e$$

Element force vector: $\mathbf{f}_\mathrm{e}$

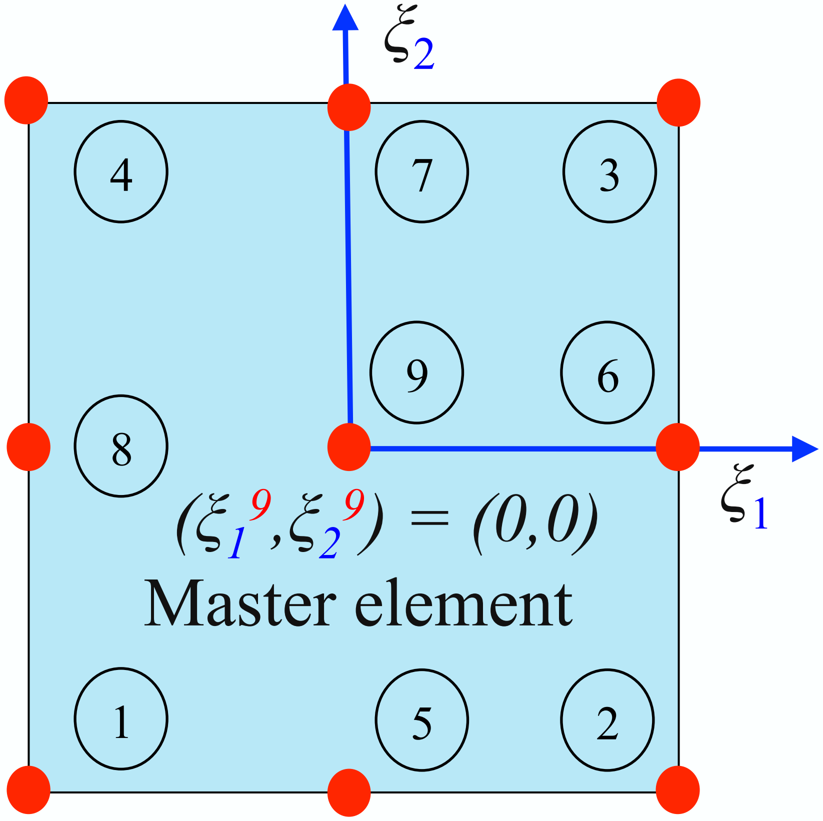
FEM IN 2D: ISOPARAMETRIC 9-NODE QUAD ELEMENTS

Shape functions must satisfy: $N_I(\xi_1^J, \xi_2^J) = \delta_{IJ}$

$(\xi_1^4,\xi_2^4) = (1,-1)$

$(\xi_1^7,\xi_2^7) = (0,-1)$

$(\xi_1^3,\xi_2^3) = (1,1)$



$(\xi_1^8,\xi_2^8) = (-1,0)$

$(\xi_1^6,\xi_2^6) = (1,0)$

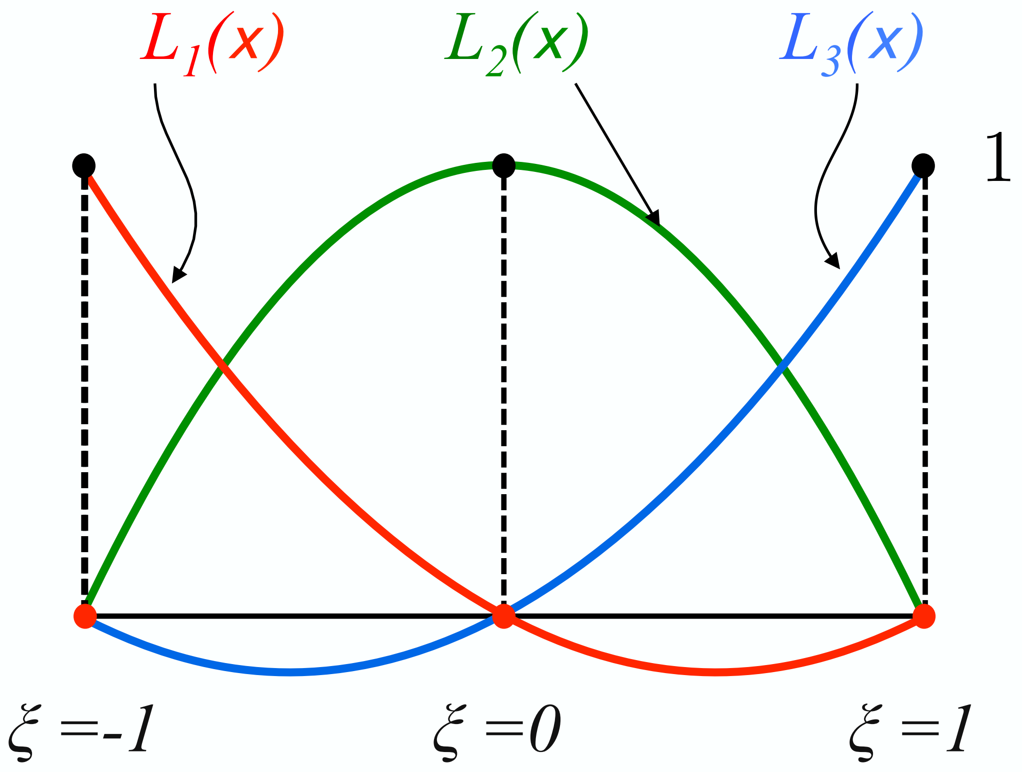
$(\xi_1^9,\xi_2^9) = (0,0)$

$(\xi_1^1,\xi_2^1) = (-1,-1)$

$(\xi_1^5,\xi_2^5) = (0,-1)$

$(\xi_1^2,\xi_2^2) = (1,-1)$

FEM IN 2D: ISOPARAMETRIC 9-NODE QUAD ELEMENTS



Recall quadratic Lagrangian functions $L_i(\xi)$ in interval [-1,1]

Shape functions are products:
$N_I(\xi_1, \xi_2)=L_i(\xi_i)\, L_j(\xi_2)$

$$u_j\left(\xi_1,\xi_2\right)=\sum_{I=1}^{9}N_I\left(\xi_1,\xi_2\right)U_j^I$$

$$u_j\left(\xi_1^I,\xi_2^I\right)=U_j^I$$

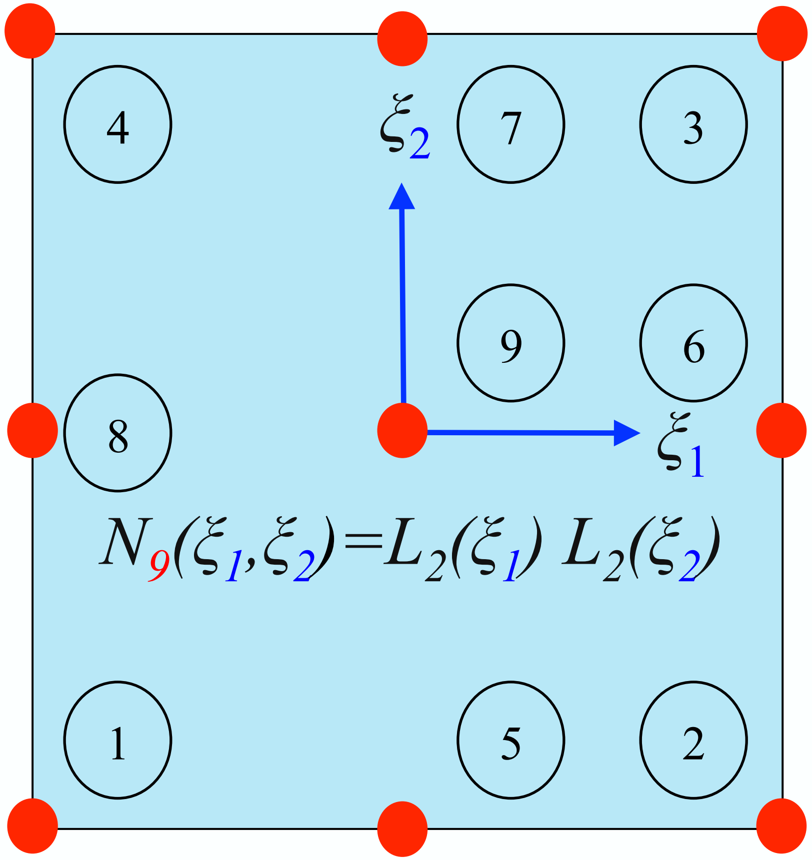
$$x_j\left(\xi_1,\xi_2\right)=\sum_{I=1}^{9}N_I\left(\xi_1,\xi_2\right)x_j^I$$

$$x_j\left(\xi_1^I,\xi_2^I\right)=x_j^I$$

$$N_I(\xi_1^J,\xi_2^J)=\delta_{IJ};\quad (I,J=1,\ldots 9)$$

FEM IN 2D: ISOPARAMETRIC 9-NODE QUAD ELEMENTS

$N_4(\xi_1,\xi_2)=L_1(\xi_1)\,L_3(\xi_2)$ $\quad N_7(\xi_1,\xi_2)=L_2(\xi_1)\,L_3(\xi_2)$ $\quad N_3(\xi_1,\xi_2)=L_3(\xi_1)\,L_3(\xi_2)$



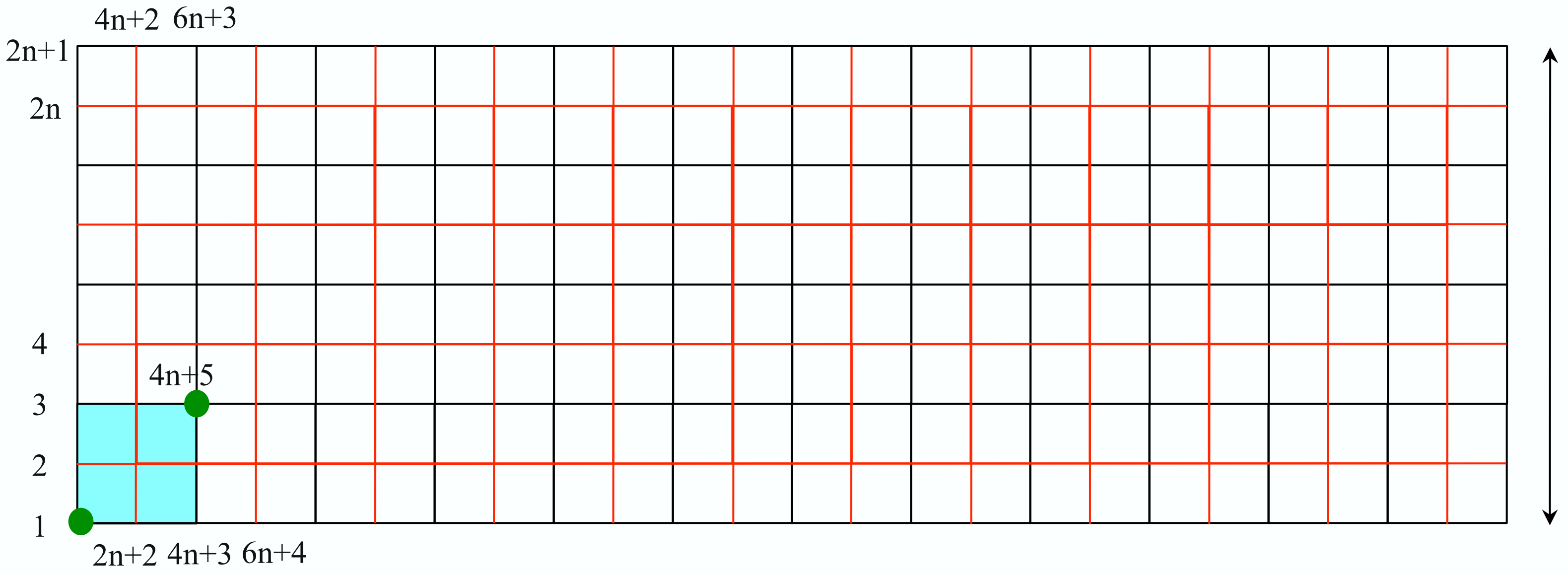
$N_8(\xi_1,\xi_2)=L_1(\xi_1)\,L_2(\xi_2)$ $\quad N_6(\xi_1,\xi_2)=L_3(\xi_1)\,L_2(\xi_2)$

$N_9(\xi_1,\xi_2)=L_2(\xi_1)\,L_2(\xi_2)$

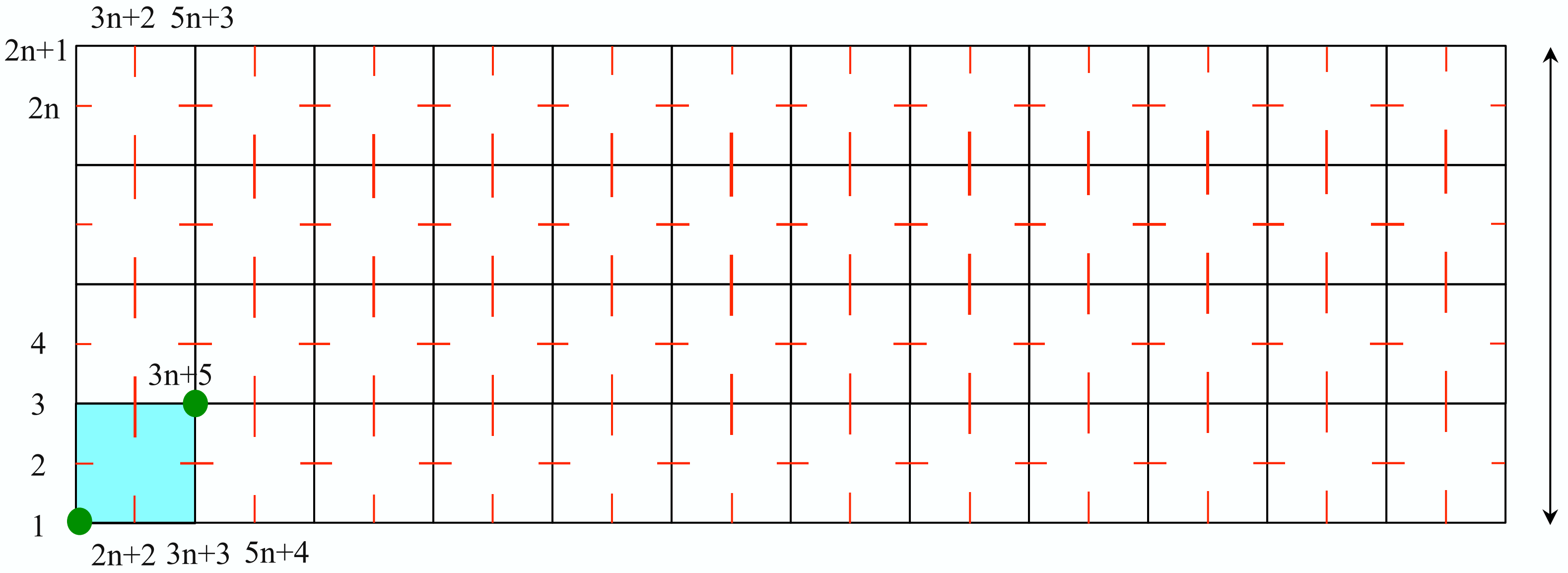
$N_1(\xi_1,\xi_2)=L_1(\xi_1)\,L_1(\xi_2)$ $\quad N_5(\xi_1,\xi_2)=L_2(\xi_1)\,L_1(\xi_2)$ $\quad N_2(\xi_1,\xi_2)=L_3(\xi_1)\,L_1(\xi_2)$

FEM IN 2D: ISOPARAMETRIC 9-NODE QUAD DISADVANTAGE

Disadvantage of 9-node quad elements: needless increase of bandwidth



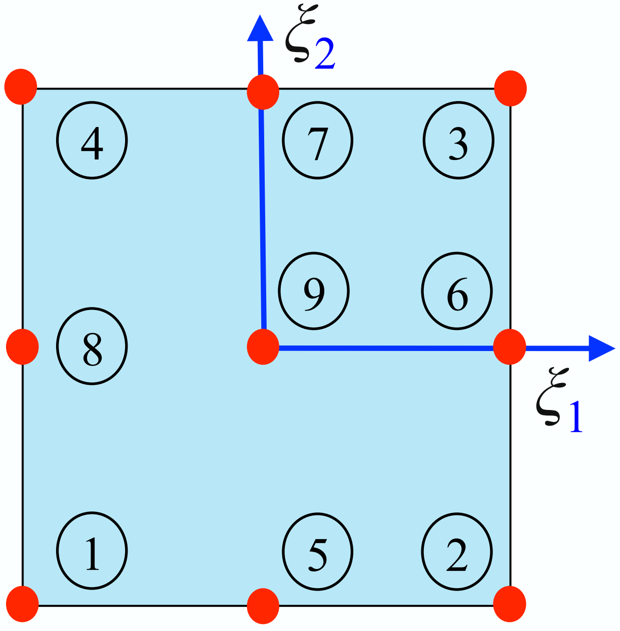
Bandwidth: 2 × (4n+5) for 9-node quads



Bandwidth: 2 × (3n+5) for 8-node quads

FEM IN 2D: STATIC CONDENSATION 9-NODE TO 8-NODE

Way to eliminate internal nodes: static condensation



$$
\begin{bmatrix}
\mathbf{K}^G_{11} & \mathbf{K}^G_{12} & \mathbf{K}^G_{13} & \mathbf{K}^G_{14} & \mathbf{K}^G_{15} & \mathbf{K}^G_{16} & \mathbf{K}^G_{17} & \mathbf{K}^G_{18} & \mathbf{k}^e_{19} \\
\mathbf{K}^G_{21} & \mathbf{K}^G_{22} & \mathbf{K}^G_{23} & \mathbf{K}^G_{24} & \mathbf{K}^G_{25} & \mathbf{K}^G_{26} & \mathbf{K}^G_{27} & \mathbf{K}^G_{28} & \mathbf{k}^e_{29} \\
\mathbf{K}^G_{31} & \mathbf{K}^G_{32} & \mathbf{K}^G_{33} & \mathbf{K}^G_{34} & \mathbf{K}^G_{35} & \mathbf{K}^G_{36} & \mathbf{K}^G_{37} & \mathbf{K}^G_{38} & \mathbf{k}^e_{39} \\
\mathbf{K}^G_{41} & \mathbf{K}^G_{42} & \mathbf{K}^G_{43} & \mathbf{K}^G_{44} & \mathbf{K}^G_{45} & \mathbf{K}^G_{46} & \mathbf{K}^G_{47} & \mathbf{K}^G_{48} & \mathbf{k}^e_{39} \\
\mathbf{K}^G_{51} & \mathbf{K}^G_{52} & \mathbf{K}^G_{53} & \mathbf{K}^G_{54} & \mathbf{K}^G_{55} & \mathbf{K}^G_{56} & \mathbf{K}^G_{57} & \mathbf{K}^G_{58} & \mathbf{k}^e_{59} \\
\mathbf{K}^G_{61} & \mathbf{K}^G_{62} & \mathbf{K}^G_{63} & \mathbf{K}^G_{64} & \mathbf{K}^G_{65} & \mathbf{K}^G_{66} & \mathbf{K}^G_{67} & \mathbf{K}^G_{68} & \mathbf{k}^e_{69} \\
\mathbf{K}^G_{71} & \mathbf{K}^G_{72} & \mathbf{K}^G_{73} & \mathbf{K}^G_{74} & \mathbf{K}^G_{75} & \mathbf{K}^G_{76} & \mathbf{K}^G_{77} & \mathbf{K}^G_{78} & \mathbf{k}^e_{79} \\
\mathbf{K}^G_{81} & \mathbf{K}^G_{82} & \mathbf{K}^G_{83} & \mathbf{K}^G_{84} & \mathbf{K}^G_{85} & \mathbf{K}^G_{86} & \mathbf{K}^G_{87} & \mathbf{K}^G_{88} & \mathbf{k}^e_{89} \\
\mathbf{k}^e_{91} & \mathbf{k}^e_{92} & \mathbf{k}^e_{93} & \mathbf{k}^e_{94} & \mathbf{k}^e_{95} & \mathbf{k}^e_{96} & \mathbf{k}^e_{97} & \mathbf{k}^e_{98} & \mathbf{k}^e_{99}
\end{bmatrix}
\begin{bmatrix}
\mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \\ \mathbf{Q}_4 \\ \mathbf{Q}_5 \\ \mathbf{Q}_6 \\ \mathbf{Q}_7 \\ \mathbf{Q}_8 \\ \mathbf{Q}_9
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{F}^G_1 \\ \mathbf{F}^G_2 \\ \mathbf{F}^G_3 \\ \mathbf{F}^G_4 \\ \mathbf{F}^G_5 \\ \mathbf{F}^G_6 \\ \mathbf{F}^G_7 \\ \mathbf{F}^G_8 \\ \mathbf{f}^e_9
\end{bmatrix}
$$

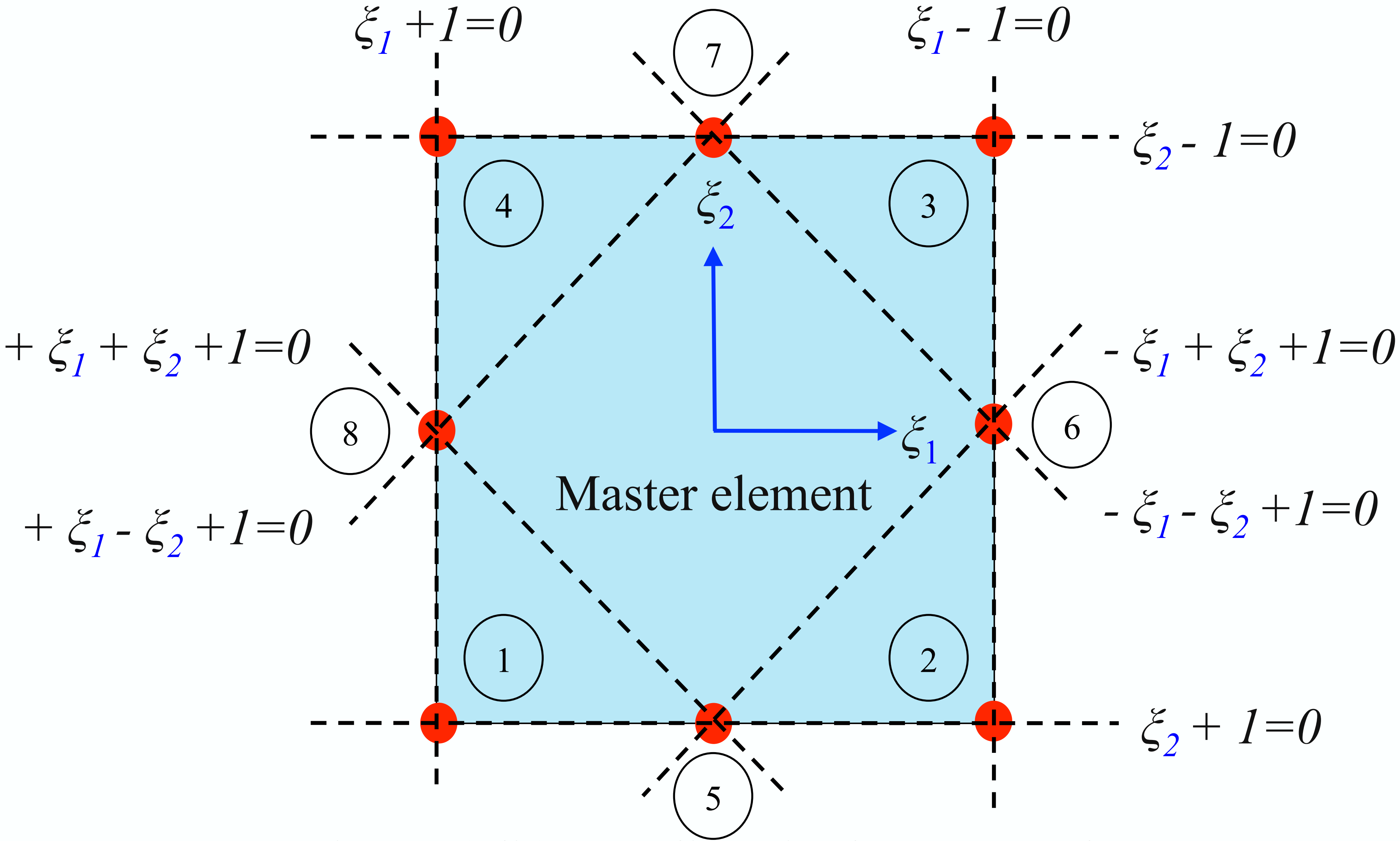
Change element stiffness & force of boundary nodes by:

$$\mathbf{k}^e_{ij} \to \mathbf{k}^e_{ij} - \mathbf{k}^e_{i9}[\mathbf{k}^e_{99}]^{-1}\mathbf{k}^e_{9j}$$

$$\mathbf{f}^e_i \to \mathbf{f}^e_i - \mathbf{k}^e_{i9}[\mathbf{k}^e_{99}]^{-1}\mathbf{f}^e_9$$

Equilibrium equation for node 9 solved immediately

FEM IN 2D: ISOPARAMETRIC 8-NODE QUAD ELEMENTS



Equations of different lines in the master element

FEM IN 2D: ISOPARAMETRIC 8-NODE QUAD ELEMENTS

Shape functions of node I are products of line equations with remaining nodes

$$N_1\left(\xi_1,\xi_2\right) = -\frac{1}{4}(1-\xi_1)(1-\xi_2)(1+\xi_1+\xi_2), \;\; N_5\left(\xi_1,\xi_2\right) = \frac{1}{2}(1-\xi_1^2)(1-\xi_2)$$

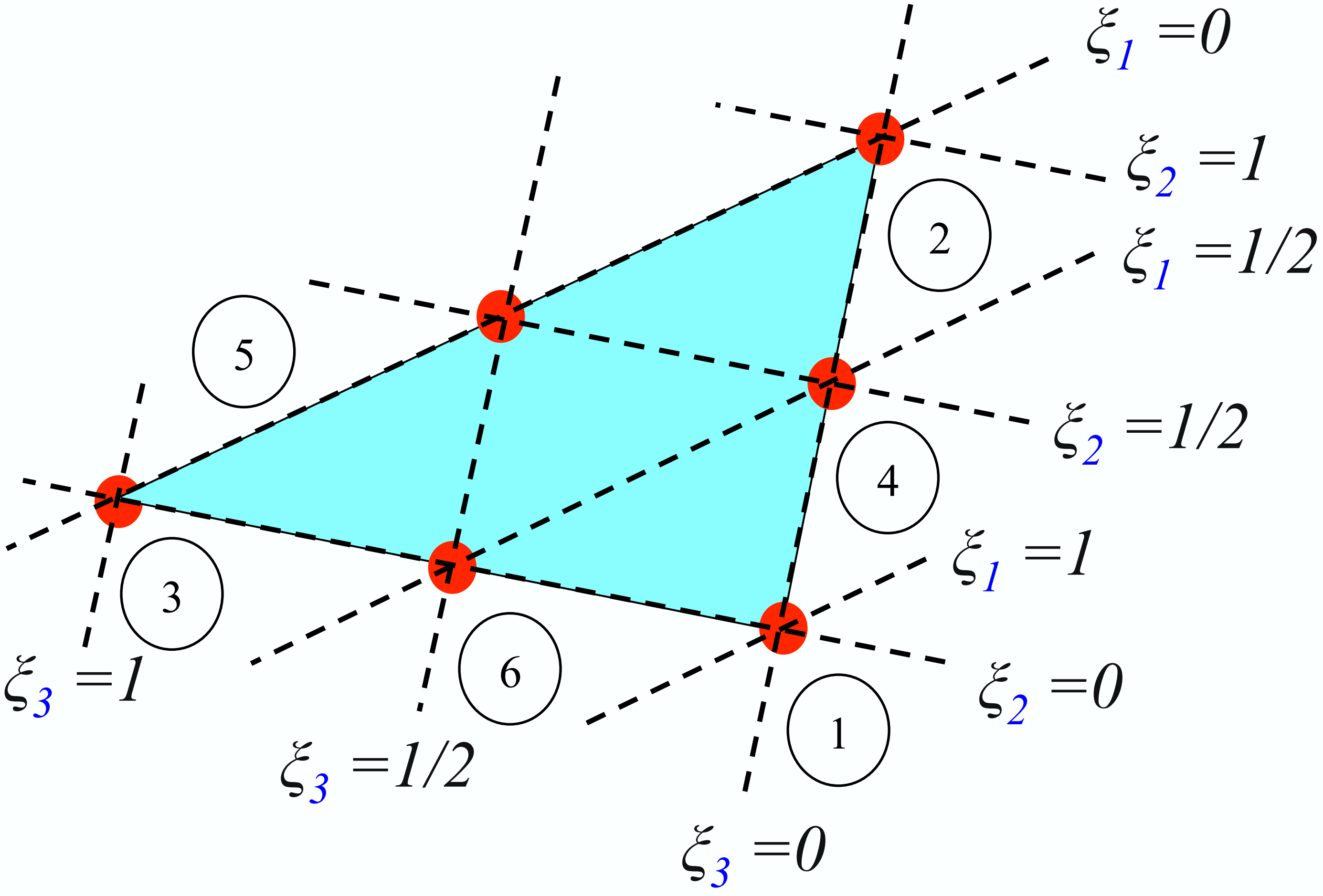
$$N_2\left(\xi_1,\xi_2\right) = -\frac{1}{4}(1+\xi_1)(1-\xi_2)(1-\xi_1+\xi_2), \;\; N_6\left(\xi_1,\xi_2\right) = \frac{1}{2}(1+\xi_1)(1-\xi_2^2)$$

$$N_3\left(\xi_1,\xi_2\right) = -\frac{1}{4}(1+\xi_1)(1+\xi_2)(1-\xi_1-\xi_2), \;\; N_7\left(\xi_1,\xi_2\right) = \frac{1}{2}(1-\xi_1^2)(1+\xi_2)$$

$$N_4\left(\xi_1,\xi_2\right) = -\frac{1}{4}(1-\xi_1)(1+\xi_2)(1+\xi_1-\xi_2), \;\; N_8\left(\xi_1,\xi_2\right) = \frac{1}{2}(1-\xi_1)(1-\xi_2^2)$$

FEM IN 2D: ISOPARAMETRIC 6-NODE TRIANGLES

Shape functions of node I are products of equations avoiding that node



$$N_1 = \xi_1(2\xi_1 - 1)$$

$$N_2 = \xi_2(2\xi_2 - 1)$$

$$N_3 = \xi_3(2\xi_3 - 1)$$

$$N_4 = 4\,\xi_1\xi_2$$

$$N_5 = 4\,\xi_2\xi_3$$

$$N_6 = 4\,\xi_3\xi_1$$

Triangular coordinates satisfy: $\xi_1+\xi_2+\xi_3 = 1$

NUMERICAL INTEGRATION IN 1D AND 2D

NUMERICAL INTEGRATION IN 1D ELEMENTS

$$\text{quadrature}: \boxed{\int_{-1}^{+1} f(\xi)\, d\xi = \sum_{j=1}^{n_I} w_I f(\xi^I) + R \approx \sum_{j=1}^{n_I} w_I f(\xi^I)}$$

Weights **quadrature points** **remainder**

$$\text{trapezoidal}: n_I = 2; \quad \xi^1 = -1,\ \xi^2 = +1$$

accuracy order

$$w_1 = w_2 = 1; \quad R = -\frac{2}{3}\frac{d^2 f}{d\xi^2}(\xi^*); \quad \xi^* \in (-1, 1)$$

$$\text{Simpson}: n_I = 3; \quad \xi^1 = -1,\ \xi^2 = 0,\ \xi^3 = +1$$

$$w_1 = w_3 = \frac{1}{3},\ w_2 = \frac{4}{3}; \quad R = -\frac{1}{90}\frac{d^4 f}{d\xi^4}(\xi^*)$$

GAUSSIAN INTEGRATION (QUADRATURE) IN 1D

Gaussian quadrature of *1, 2 and 3 points*

$$1 \text{ point Gauss} : n_I = 1; \quad \xi^1 = 0$$

$$w_1 = 2; \quad R = \frac{1}{3}\frac{d^2 f}{d\xi^2}(\xi^*); \quad \xi^* \in (-1, 1)$$

$$2 \text{ point Gauss} : n_I = 2; \quad \xi^1 = -\frac{1}{\sqrt{3}},\ \xi^2 = \frac{1}{\sqrt{3}}$$

$$w_1 = w_2 = 1; \quad R = \frac{1}{135}\frac{d^4 f}{d\xi^4}(\xi^*)$$

$$3 \text{ point Gauss} : n_I = 3; \quad \xi^1 = -\frac{\sqrt{3}}{\sqrt{5}},\ \xi_2 = 0,\ \xi^3 = \frac{\sqrt{3}}{\sqrt{5}}$$

$$w_1 = w_3 = \frac{5}{9},\ w_2 = \frac{8}{9}; \quad R = \frac{1}{15750}\frac{d^6 f}{d\xi^6}(\xi^*)$$

GAUSSIAN INTEGRATION (QUADRATURE) IN 1D

General Gaussian quadrature of n_I points is $2n_I$ accurate

$$\text{Gauss points : } \xi^I \text{ are roots of Legendre polynomial } P_{n_I}(\xi^I) = 0$$

$$\text{weight : } w_I = \frac{2}{[(1-(\xi^I)^2]\left[\dfrac{dP_{n_I}}{d\xi}(\xi^I)\right]^2}$$

$$\text{remainder : } R = \frac{2^{(2n_I)}(n_I!)^4}{(2n_I+1)[(2n_I)!]^3}\frac{d^{(2n_I)}f}{d\xi^{(2n_I)}}(\xi^*)$$

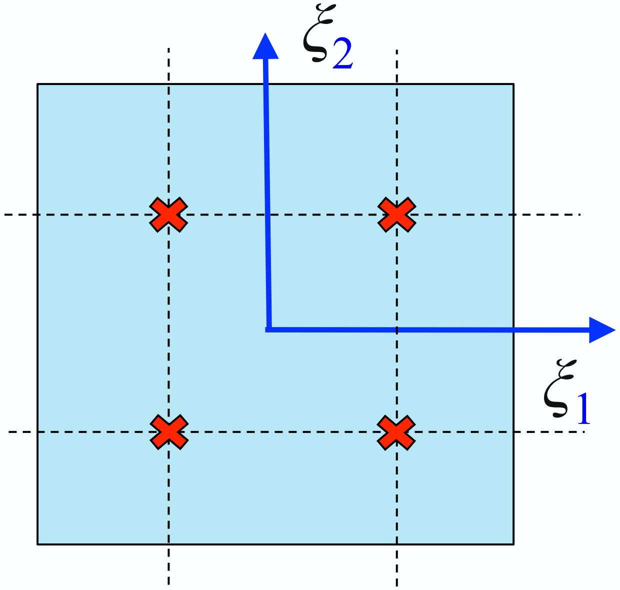
$$n_I \text{ order Legendre : } P_{n_I}(\xi) \equiv \frac{1}{2^{n_I}(n_I)!}\frac{d^{(n_I)}}{d\xi^{(n_I)}}(\xi^2-1)^{n_I}$$

GAUSSIAN INTEGRATION (QUADRATURE) IN 2D

General Gaussian quadrature in 2D uses master element

$$\int_{-1}^{+1}\int_{-1}^{+1} f(\xi_1,\xi_2)\, d\xi_1 d\xi_2 = \sum_{I=1}^{I=n_I}\sum_{J=1}^{J=n_I} w_I w_J f(\xi_1^I,\xi_2^J)$$

$(\xi_1^2,\xi_2^1) = (1/\sqrt{3},-1/\sqrt{3})$ $(\xi_1^2,\xi_2^2) = (1/\sqrt{3},1/\sqrt{3})$



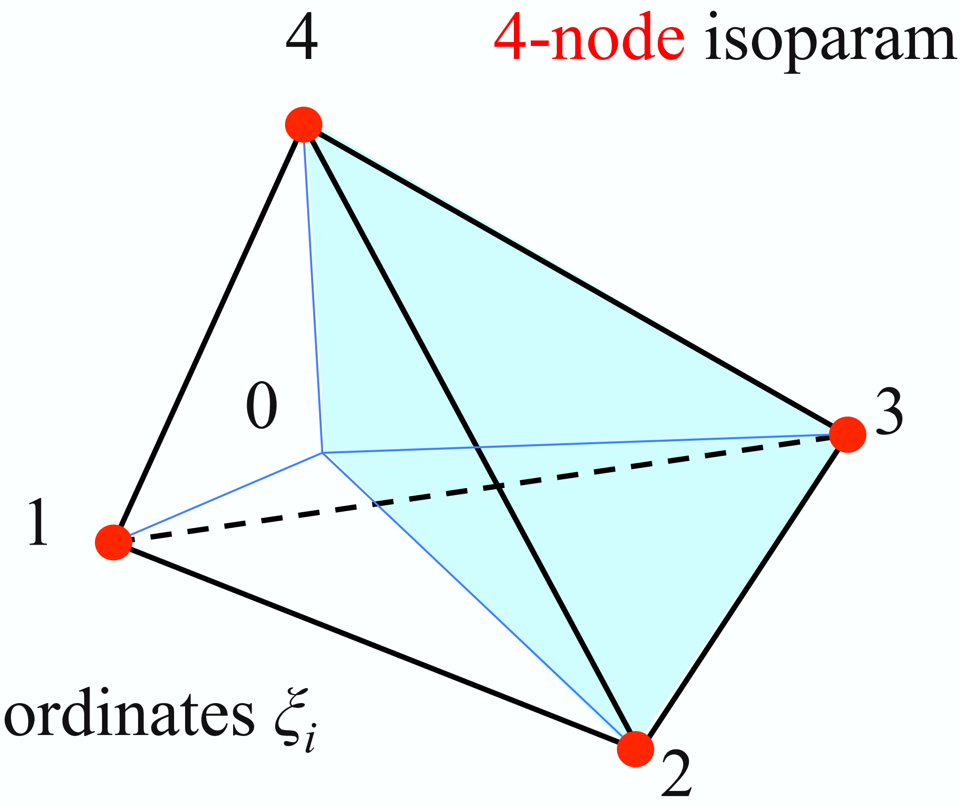
A 2×2 Gauss integration uses grid points with coordinates and weights taken from 1D

$(\xi_1^1,\xi_2^1) = (-1/\sqrt{3},-1/\sqrt{3})$ $(\xi_1^1,\xi_2^2) = (-1/\sqrt{3},1/\sqrt{3})$

ISOPARAMETRIC ELEMENTS IN 3D

FEM IN 3D: ISOPARAMETRIC 4-NODE TETRAHEDRON

4-node isoparametric tetrahedron



$$u_j(\boldsymbol{\xi}) = \sum_{I=1}^{4} N_I(\boldsymbol{\xi}) U_j^I$$

$$x_j(\boldsymbol{\xi}) = \sum_{I=1}^{4} N_I(\boldsymbol{\xi}) x_j^I$$

Coordinates ξ_i

$$N_I(\boldsymbol{\xi}^J) = \delta_{IJ}; \ (I, J = 1, \dots 4)$$

$\xi_1 = V_{0234}/V_{1234}$ – V_{0234} : Volume 0234…

$\xi_2 = V_{0134}/V_{1234}$

$\xi_3 = V_{0124}/V_{1234}$

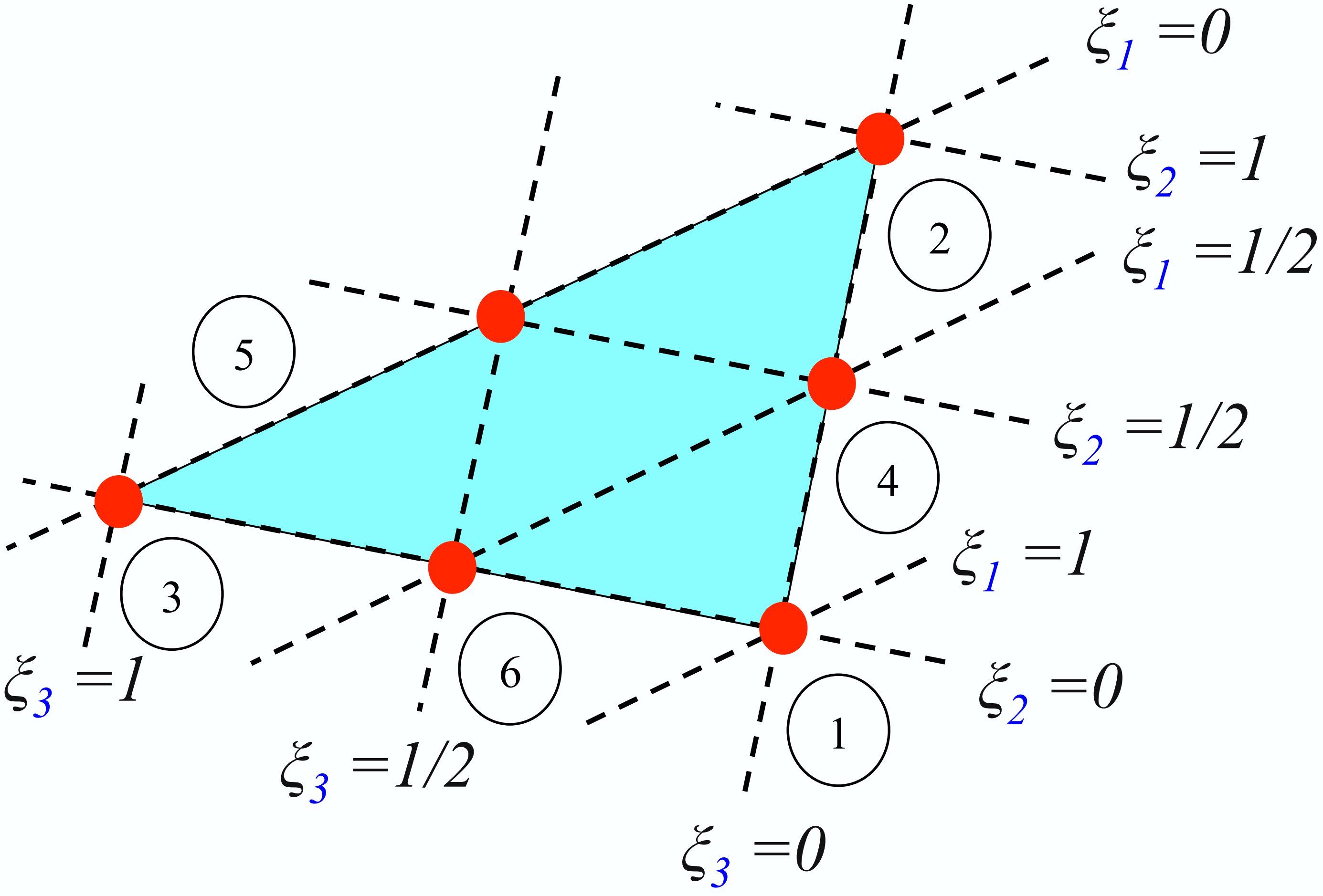
$\xi_4 = V_{0123}/V_{1234}$

$$\boxed{N_I(\boldsymbol{\xi}) = \xi_I; \ (I = 1, \dots 4)}$$

Note: $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 1$, patch test automatically satisfied

RECALL: ISOPARAMETRIC 6-NODE TRIANGLES IN 2D

Shape functions of node I are products of equations avoiding that node



$$N_1 = \xi_1(2\xi_1 - 1)$$

$$N_2 = \xi_2(2\xi_2 - 1)$$

$$N_3 = \xi_3(2\xi_3 - 1)$$

$$N_4 = 4\,\xi_1\xi_2$$

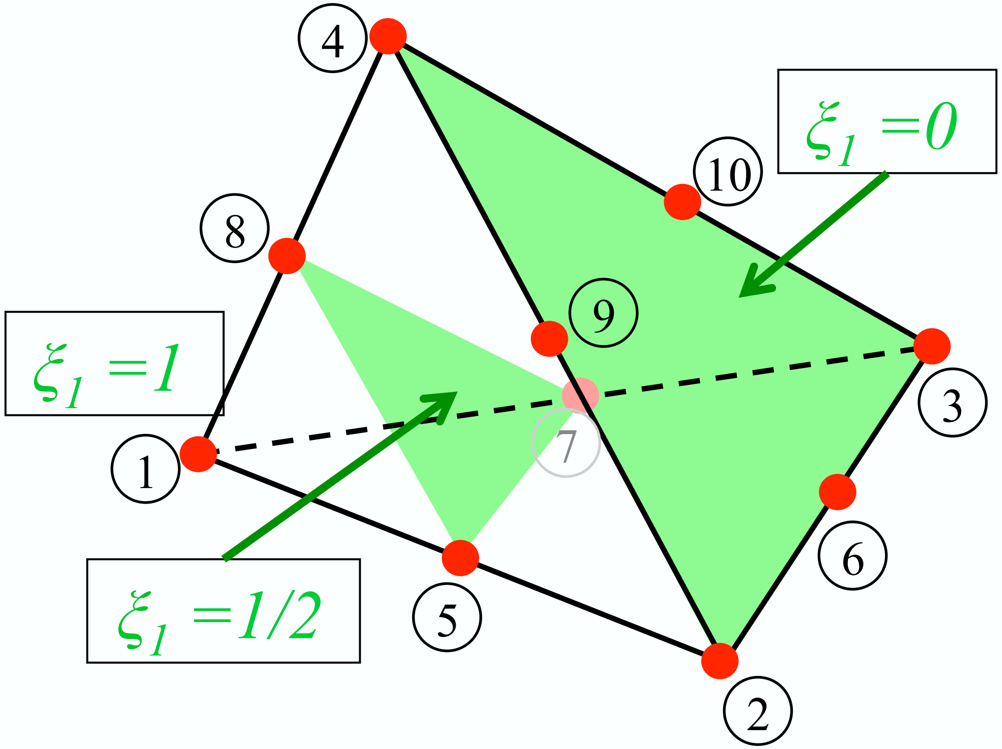
$$N_5 = 4\,\xi_2\xi_3$$

$$N_6 = 4\,\xi_3\xi_1$$

Triangular coordinates satisfy: $\xi_1+\xi_2+\xi_3 = 1$

FEM IN 3D: ISOPARAMETRIC 10-NODE TETRAHEDRON

10-node isoparametric tetrahedron in 3D similarly to 6-node triangle in 2D



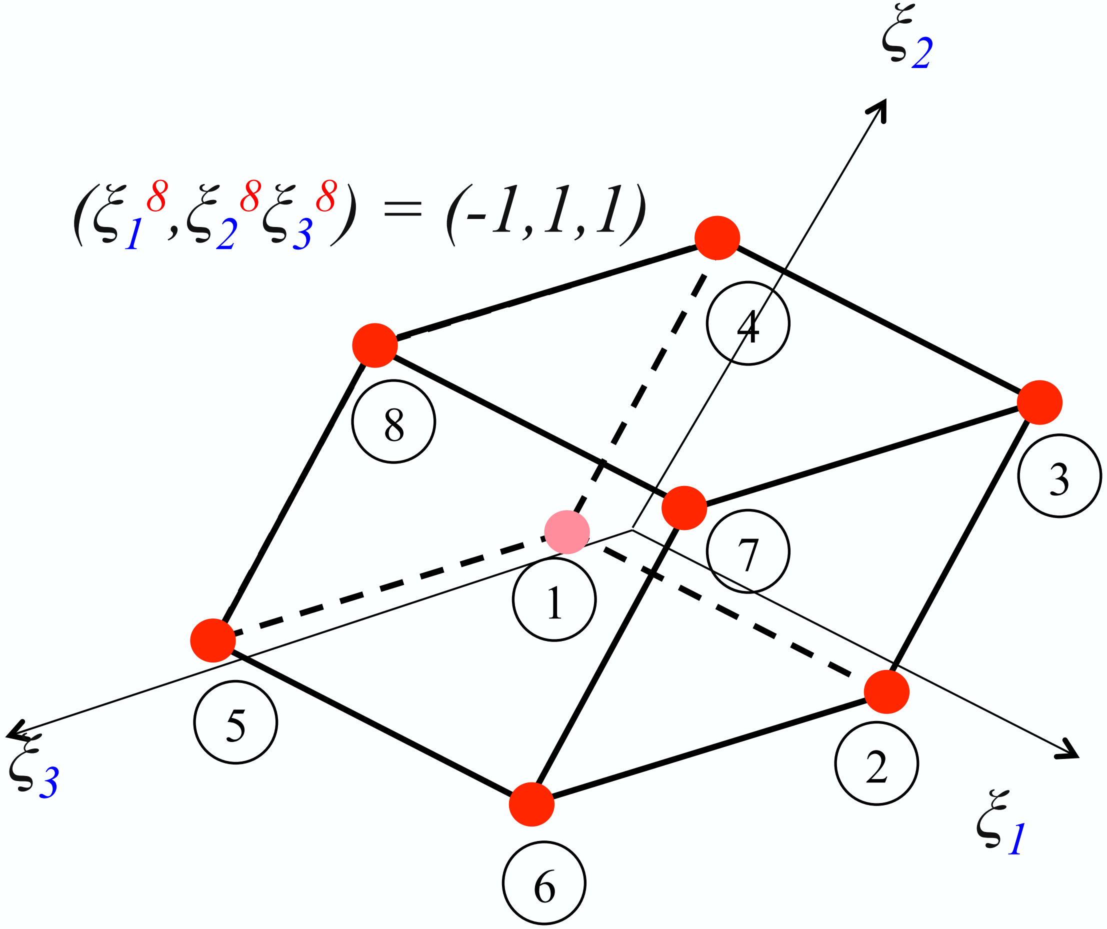
$$N_1 = \xi_1(2\xi_1 - 1)$$
$$N_2 = \xi_2(2\xi_2 - 1)$$
$$N_3 = \xi_3(2\xi_3 - 1)$$
$$N_4 = \xi_4(2\xi_4 - 1)$$
$$N_5 = 4\ \xi_1\xi_2$$
$$N_6 = 4\ \xi_2\xi_3$$
$$N_7 = 4\ \xi_3\xi_1$$
$$N_8 = 4\ \xi_1\xi_4$$
$$N_9 = 4\ \xi_2\xi_4$$
$$N_{10} = 4\ \xi_3\xi_4$$

Shape function of a node is product of equations of planes that do not contain that node; e.g. $N_{10} = 4\ \xi_3\ \xi_4$

Shape functions satisfy: $\sum N_I(\xi) = 1$

FEM IN 3D: ISOPARAMETRIC 8-NODE HEXAHEDRON

8-node isoparametric hexadedron



$(\xi_1^8,\xi_2^8\xi_3^8) = (-1,1,1)$

$(\xi_1^6,\xi_2^6\xi_3^6) = (1,-1,1)$

Shape functions $N_I(\xi)$ $(I = 1,\ldots 8)$

$$\boxed{\begin{aligned} &N_I(\xi_1,\xi_2,\xi_3) = \\ &= \frac{1}{8}(1+\xi_1^I\xi_1)(1+\xi_2^I\xi_2)(1+\xi_3^I\xi_3) \end{aligned}}$$

Shape functions satisfy: $\Sigma N_I(\xi)=1$

FEM IN 3D: NUMERICAL INTEGRATION

isoparametric hexadedron

$$\int_{V(\boldsymbol{\xi})} f(\xi_1,\xi_2,\xi_3)\,d\boldsymbol{\xi} \approx \sum_{I=1}^{n_I}\sum_{J=1}^{n_I}\sum_{K=1}^{n_I} w_I w_J w_K f(\xi_1^I,\xi_2^J,\xi_3^K)$$

1D Gaussian weights and points of [-1,1]

isoparametric tetrahedron

$$\int_{V(\boldsymbol{\xi})} f(\xi_1,\xi_2,\xi_3,\xi_4)\,d\boldsymbol{\xi} \approx \sum_{I=1}^{n_I} w_I f(\xi_1^I,\xi_2^I,\xi_3^I,\xi_4^I)$$

3D Gaussian weights and points calculated in master element

$n_I = 1$ – accuracy $O(h^2)$

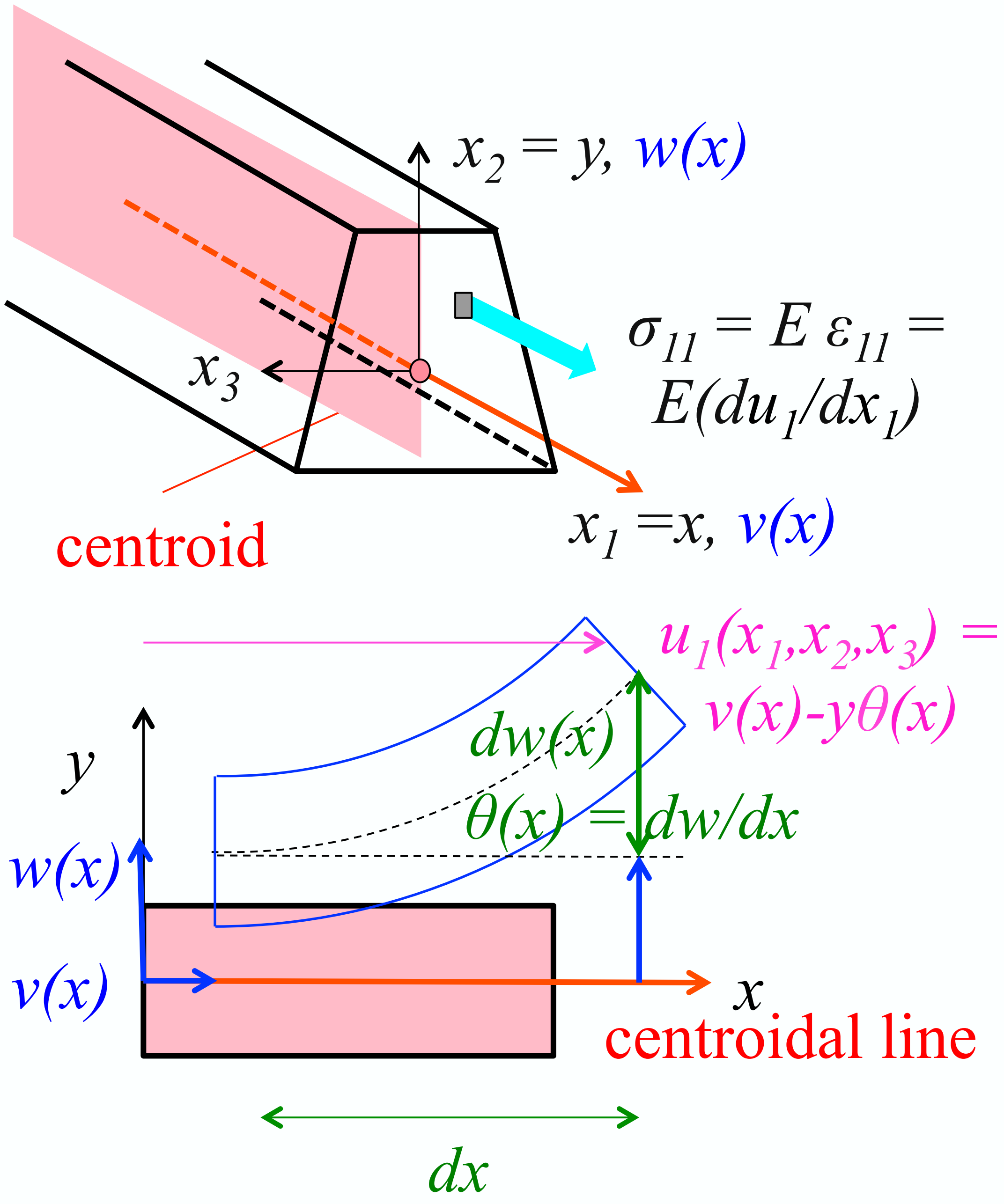
$w_1 = 1,\ \xi^1 = (1/4, 1/4, 1/4, 1/4)$

$n_I = 5$ – accuracy $O(h^4)$

$w_1 = -4/5,\ \xi^1 = (1/4, 1/4, 1/4, 1/4)$

$w_{2,3,4,5} = 9/20,\ \xi^2 = (1/2, 1/6, 1/6, 1/6)$, others by cyclic symmetry

PROBLEMS INVOLVING HIGHER ORDER GRADIENTS
1D (BERNOULLI-EULER-NAVIER) BEAM THEORY



BERNOULLI–EULER–NAVIER

- Uniaxial stress state (only σ_{11})
- Plane sections normal to centroidal line remain plane and normal to the deformed one
- Small (infinitesimal) strain kinematics
- Linear elastic constitutive law (isotropic, can be generalized to transversely isotropic about centroidal line)

DERIVATION OF 1D BEAM THEORY (IN-PLANE BENDING)

$$\mathcal{P}_{int} = \int_V \left[\frac{1}{2}\sigma_{11}\epsilon_{11}\right] dV = \int_0^L \left[\int_A \frac{1}{2}E(\epsilon_{11})^2 \, dA\right] dx$$

$$\text{recall :}\ \epsilon_{11} = \frac{du_1}{dx_1}\ ,\ u_1 = v(x) - y\frac{d}{dx}w(x)\ ; \quad (x \equiv x_1,\ y \equiv x_2)$$

$$\mathcal{P}_{int} = \int_0^L \left[\int_A \frac{1}{2}E\left(\frac{dv}{dx} - y\frac{d^2w}{dx^2}\right)^2 dA\right] dx$$

$$\text{recall :}\ \int_A \overset{\text{section area}}{dA = A}\ ,\quad \int_A \overset{\text{centroid def.}}{y\, dA = 0}\ ,\quad \int_A \overset{\text{section moment of inertia}}{y^2\, dA = I}$$

axial energy

bending energy

$$\mathcal{P}_{int} = \int_0^L \left[\boxed{\frac{1}{2}EA\left(\frac{dv}{dx}\right)^2} + \boxed{\frac{1}{2}EI\left(\frac{d^2w}{dx^2}\right)^2}\right] dx$$

DERIVATION OF 1D BEAM THEORY (IN-PLANE BENDING)

$$\text{Uniaxial stress state}: \; \sigma_{11}(x,y) = E\,\epsilon_{11}(x,y)$$

$$\text{Strain distribution}: \; \epsilon_{11}(x,y) = \epsilon(x) + y\,\kappa(x)$$

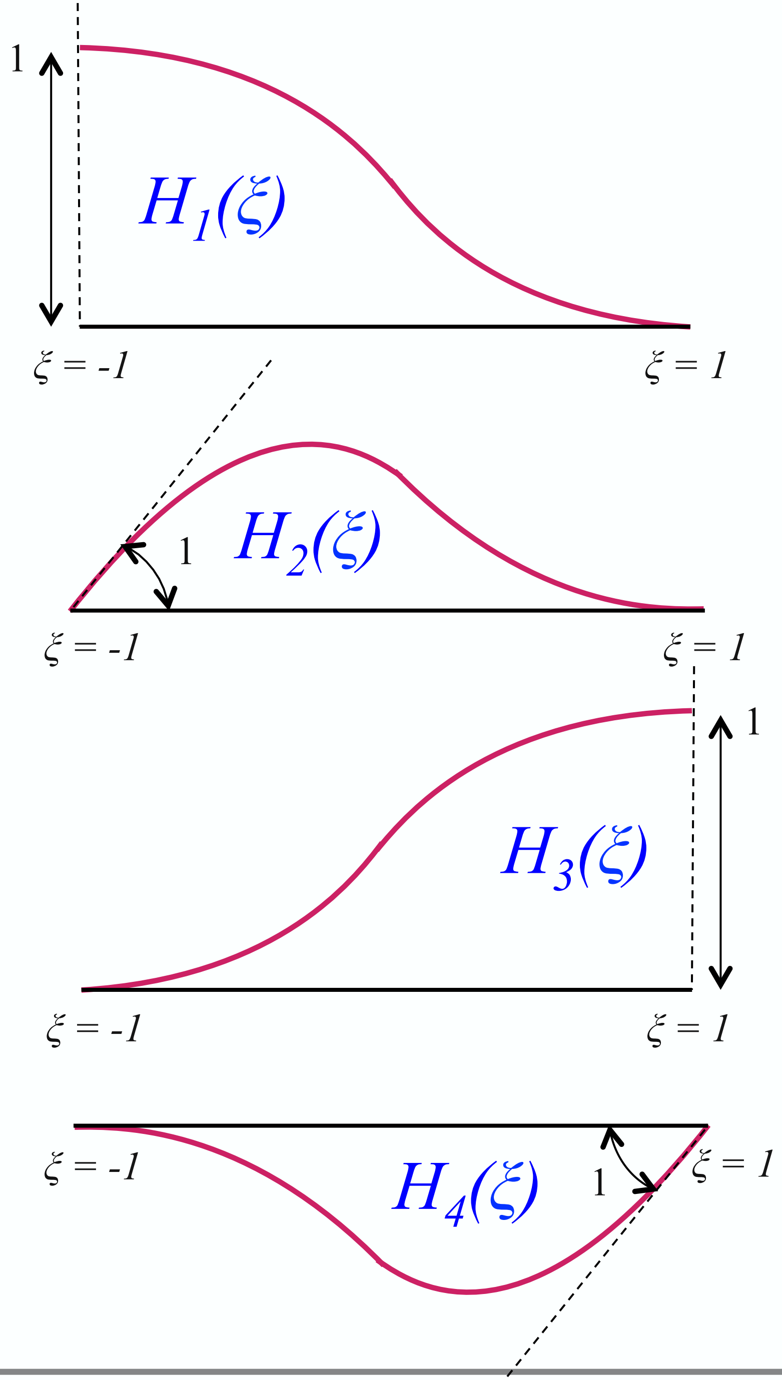
$$\text{Membrane strain}: \epsilon(x) = \frac{dv}{dx}$$

$$\text{Curvature strain}: \; \kappa(x) = -\frac{d^2w}{dx^2}$$

$$\text{Axial resultant}: \; N = \int_A \sigma_{11}\,dy = EA\,\epsilon(x)$$

$$\text{Moment resultant}: \; M = \int_A [\sigma_{11}\,y]\,dy = EI\,\kappa(x)$$

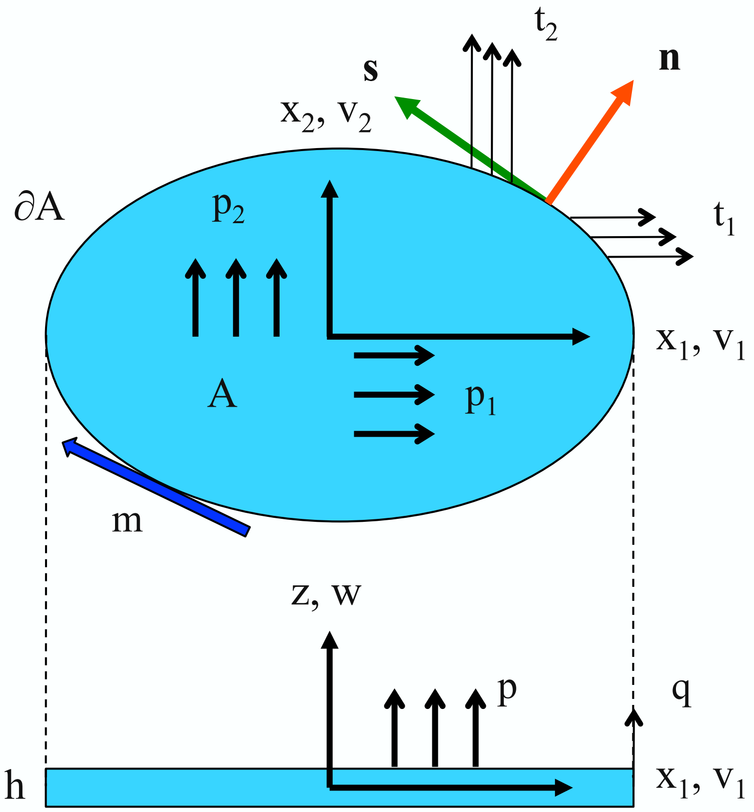
FEM FOR BENDING OF 1D BEAM – HERMITIAN CUBICS



The bending energy – EI$(d^2w/dx^2)^2$ term – dictates C^1 continuity (i.e. continuous *dw/dx*) of the test function *w(x)* in the entire beam. Consequently we must ensure inter-element continuity of both *w(x)* and *dw/dx* at each boundary node. The simplest element functions that do this are Hermitian cubics

$$
\begin{aligned}
H_1(\xi) &= \frac{1}{4}(1-\xi)^2(2+\xi) \\
H_2(\xi) &= \frac{1}{4}(1-\xi)^2(\xi+1) \\
H_3(\xi) &= \frac{1}{4}(1+\xi)^2(2-\xi) \\
H_4(\xi) &= \frac{1}{4}(1+\xi)^2(\xi-1)
\end{aligned}
$$

2D KIRCHHOFF PLATE THEORY



KIRCHHOFF PLATE THEORY:

- Plane stress state (only $\sigma_{\alpha\beta}$ – $\alpha, \beta = 1,2$)
- Normals to the undeformed middle plane remain normal to the deformed middle surface
- Small (infinitesimal) strain kinematics
- Linear elastic constitutive law (isotropic, can be generalized to transverseley isotropic about normal direction)
- Reduces to Bernoulli-Euler-Navier for loading that is independent on x_2 (or x_1)

DERIVATION OF 2D KIRCHHOFF PLATE BENDING THEORY

$$\text{Plane stress state}: \ \sigma_{\alpha\beta}(x_1,x_2,z) = L_{\alpha\beta\gamma\delta}\,\epsilon_{\gamma\delta}(x_1,x_2,z); \ (\text{Greek indexes}: \ 1,2)$$

$$\text{Plane stress moduli}: \ L_{\alpha\beta\gamma\delta} = \frac{E}{1-\nu^2}\left[\frac{1-\nu}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})+\nu\delta_{\alpha\beta}\delta_{\gamma\delta}\right]$$

$$\text{Strain distribution}: \ \epsilon_{\alpha\beta}(x_1,x_2,z) = E_{\alpha\beta}(x_1,x_2)+zK_{\alpha\beta}(x_1,x_2)$$

$$\text{Membrane strains}: \ E_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta}+v_{\beta,\alpha}) \ ; \quad (f_{,\alpha}\equiv\partial f/\partial x_\alpha)$$

$$\text{Curvature strains}: \ K_{\alpha\beta} = -w,_{\alpha\beta}$$

$$\text{Membrane resultants}: \ N_{\alpha\beta} = \int_{-h/2}^{h/2}\sigma_{\alpha\beta}dz = hL_{\alpha\beta\gamma\delta}E_{\gamma\delta}$$

$$\text{Moment resultants}: \ M_{\alpha\beta} = \int_{-h/2}^{h/2}\sigma_{\alpha\beta}zdz = \frac{h^3}{12}L_{\alpha\beta\gamma\delta}K_{\gamma\delta}$$

2D PLATE THEORY REDUCES TO BEAM FOR IN-PLANE BEND.

$$\text{Uniaxial stress state : } \sigma_{11}(x_1, x_2) = E\,\epsilon_{11}(x_1, x_2);$$

$$\text{Strain distribution : } \epsilon_{11}(x_1, x_2) = \epsilon(x_1) + x_2\,\kappa(x_1)$$

$$\text{Membrane strain : } \epsilon(x_1) = \frac{dv}{dx_1}$$

$$\text{Curvature strain : } \kappa(x_1) = -\frac{d^2 w}{d(x_1)^2}$$

$$\text{Axial resultant : } N = \int_A \sigma_{11}\,dx_2 = EA\,\epsilon(x_1)$$

$$\text{Moment resultant : } M = \int_A \sigma_{11} x_2\,dx_2 = EI\,\kappa(x_1)$$

DERIVATION OF 2D KIRCHHOFF PLATE BENDING THEORY

$$\text{Internal energy}: \; \mathcal{P}_{int} \;=\; \int\limits_A [\frac{1}{2}N_{\alpha\beta}E_{\alpha\beta} + \frac{1}{2}M_{\alpha\beta}K_{\alpha\beta})h]dA$$

$$\text{External energy}: \; \mathcal{P}_{ext} \;=\; -\int\limits_A \left[p_\alpha u_\alpha + pw\right] dA - \int\limits_{\partial A} \left[t_\alpha u_\alpha + qw + m(-w,_n)\right] ds$$

$$\text{Potential energy}: \; \mathcal{P} \;=\; \mathcal{P}_{int} + \mathcal{P}_{ext}$$

$$\mathcal{P} \;=\; \frac{1}{2}\int\limits_A [L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}E_{\gamma\delta} + \frac{h^2}{12}K_{\alpha\beta}K_{\gamma\delta})h]dA \;-$$

$$-\int\limits_A \left[p_\alpha u_\alpha + pw\right] dA - \int\limits_{\partial A} \left[t_\alpha u_\alpha + qw + m(-w,_n)\right] ds$$

KIRCHHOFF PLATE BENDING FOR TRANSVERSE LOADING

$$\mathcal{P} = \int_A \left[\frac{1}{2} \left(\frac{h^3}{12} L_{\alpha\beta\gamma\delta} w_{,\alpha\beta}\, w_{,\gamma\delta} \right) - pw \right] dA$$

$$\delta\mathcal{P} = \int_A \left[\left(\frac{h^3}{12} L_{\alpha\beta\gamma\delta} w_{,\alpha\beta\gamma\delta} - pw \right) \delta w \right] dA + \int_{\partial A} M_{nn} \delta w_{,n}\, ds = 0$$

$$\text{E-L} \; : \; \frac{Eh^3}{12(1-\nu^2)} \nabla^4 w = p, \;\; \mathbf{x} \in A$$

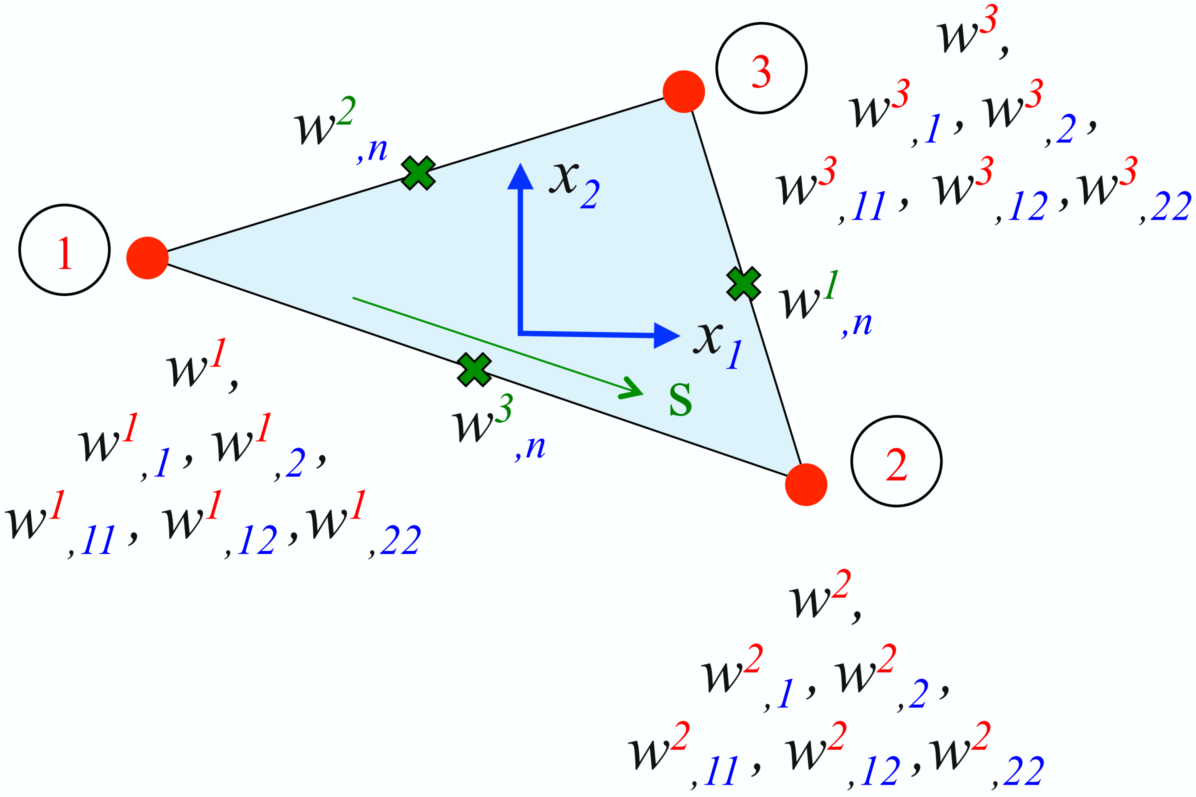
$$\text{E.B.C.} \; : \; w = 0 \; , \;\; \mathbf{x} \in \partial A$$

$$\text{N.B.C.} \; : \; w_{,11}(0, x_2) = w_{,11}(a_1, x_2) = w_{,22}(x_1, 0) = w_{,22}(x_1, a_2) = 0$$

NOTE: Since second order derivatives of the transverse displacement enter the bending energy, need C^1 inter-element continuity!

KIRCHHOFF PLATE BENDING FOR TRANSVERSE LOADING

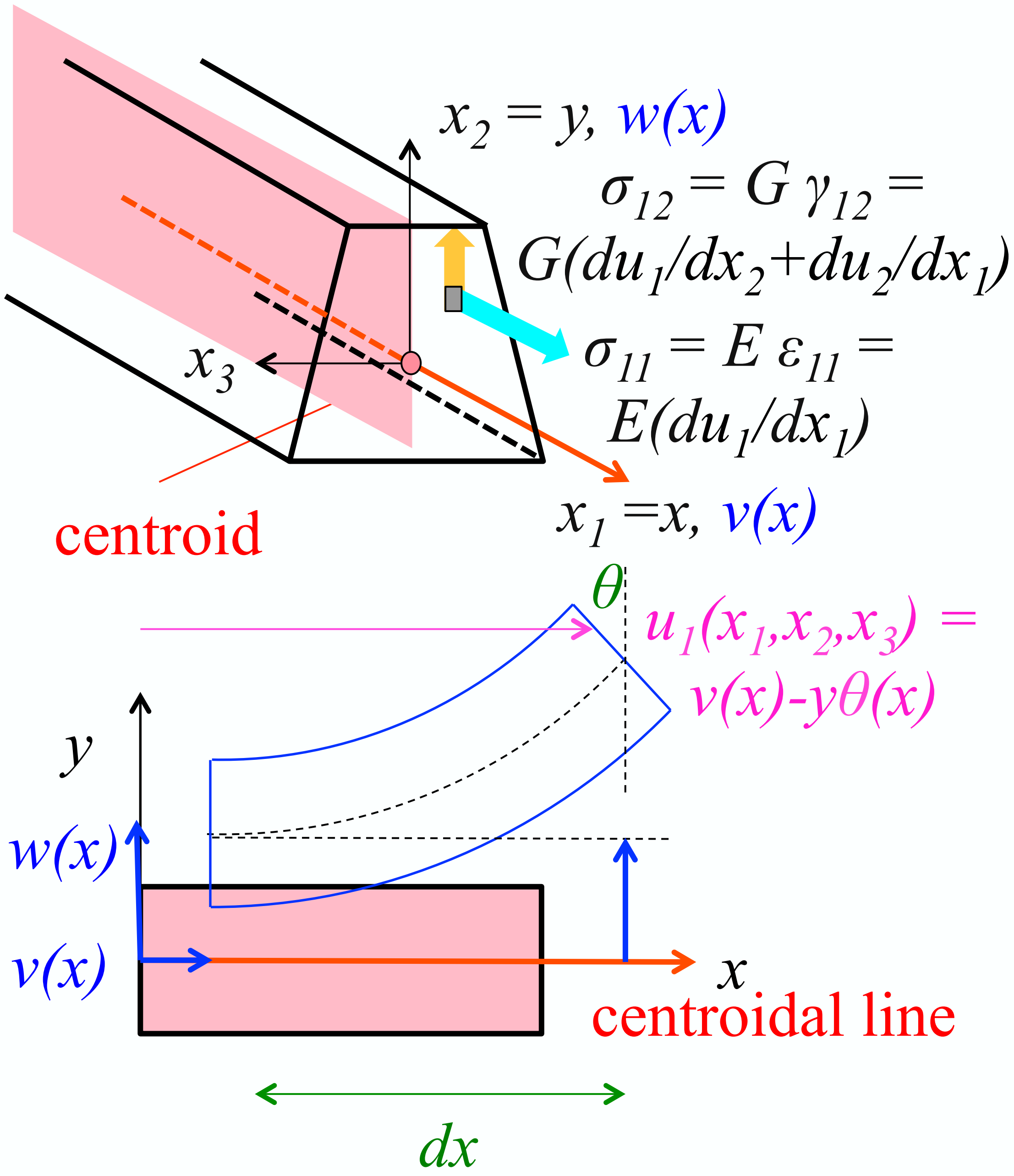
The simplest element that can satisfy C^1 inter-element continuity of $w(x_1,x_2)$ is the Clough Triangle that has 21 degrees of freedom an a full fifth order polynomial shape function



Inter-element continuity of $w(s)$ at each side: 5th order polynomial in s (length coordinate) share the same 6 constants: w, $w_{,s}$, $w_{,ss}$ at each end node

Inter-element continuity of $w_{,n}(s)$ at each side: 4th order polynomial in s share the same 5 constants: $w_{,n}$, $w_{,nn}$ at each end node and $w_{,n}$ at the mid-side node

RELAXING HIGH ORDER INTERELEMENT CONTINUITY: TRANSVERSE SHEAR ENERGY AND ASSOCIATED PENALTY METHODS



TIMOSHENKO BEAM THEORY

- Axial plus shear stresses (σ_{11}, σ_{12})
- Plane sections normal to centroidal line remain plane and rotate by an angle θ
- Small (infinitesimal) strain kinematics
- Linear elastic constitutive law (isotropic, can be generalized to transverseley isotropic about centroidal line)

1D BEAM THEORY WITH TRANSVERSE SHEAR

$$\mathcal{P}_{int} = \int_V \left[\frac{1}{2}\sigma_{11}\epsilon_{11} + \frac{1}{2}\sigma_{12}\gamma_{12} \right] \, dV$$

$$\mathcal{P}_{int} = \int_0^L \left[\int_A \frac{1}{2}E(\epsilon_{11})^2 + \frac{1}{2}G(\gamma_{12})^2 \, dA \right] \, dx$$

$$\text{recall : } \epsilon_{11} = \frac{du_1}{dx_1} = \frac{dv}{dx} - y\frac{d\theta}{dx}$$

$$\text{recall : } \gamma_{12} = \frac{du_1}{dx_2} + \frac{du_2}{dx_1} = -\theta + \frac{dw}{dx}$$

shear correction factor κ
accounts for non-uniform
shrear stress distribution

axial energy

bending energy

shear energy

$$\mathcal{P}_{int} = \frac{1}{2}\int_0^L \left[\boxed{EA\left(\frac{dv}{dx}\right)^2} + \boxed{EI\left(\frac{d\theta}{dx}\right)^2} + \boxed{\kappa GA\left(\frac{dw}{dx} - \theta\right)^2} \right] \, dx$$

$$\boxed{\begin{aligned} w(\xi) &= [N_1(\xi),\ 0,\ N_2(\xi),\ 0]\,\mathbf{q}_e = \mathbf{N}_w \mathbf{q}_e \\ \theta(\xi) &= [0,\ N_1(\xi),\ 0,\ N_2(\xi)]\,\mathbf{q}_e = \mathbf{N}_\theta \mathbf{q}_e \end{aligned}} \longleftarrow$$

NOTE: linear interpolation!

$$\mathbf{q}_e^T = [q_1,\ q_2,\ q_3,\ q_4] = [w_1,\ \theta_1,\ w_2,\ \theta_2]$$

$$\frac{d\theta}{dx}(\xi) = \mathbf{B}_\theta \mathbf{q}_e\ , \qquad \frac{dw}{dx}(\xi) - \theta(\xi) = \mathbf{B}_\gamma \mathbf{q}_e$$

$$\mathcal{P}^e_{int} = \frac{1}{2}\mathbf{q}_e^T \left[\int_{-1}^{+1} \left(EI\mathbf{B}_\theta^T \mathbf{B}_\theta + \kappa GH \mathbf{B}_\gamma^T \mathbf{B}_\gamma \right) \frac{l_e}{2}\, d\xi \right] \mathbf{q}_e$$

$$\mathcal{P}^e_{ext} = \int_{l_e} p\ w\ dx = \mathbf{q}_e^T \left[\int_{-1}^{+1} \mathbf{N}_w^T p\, \frac{l_e}{2}\ d\xi \right]$$

$$\boxed{\left[\mathbf{K}_\theta + \frac{1}{\zeta^2}\mathbf{K}_\gamma\right]\mathbf{Q} = \mathbf{F}} \; ; \quad \zeta \equiv \frac{b}{L} \ll 1 \text{ (aspect ratio)}$$

$$\left[\zeta^2\mathbf{K}_\theta + \mathbf{K}_\gamma\right]\mathbf{Q} = \zeta^2\mathbf{F} \implies \mathbf{Q} = \zeta^2\mathbf{K}_\gamma^{-1}\mathbf{F} + \ldots$$

$$\det[\mathbf{K}_\gamma] \neq 0 \implies \text{locking } (\mathbf{Q} \approx \mathbf{0})$$

$$\det[\mathbf{K}_\gamma] = 0 \implies \text{WORKS}$$

$$\det[\mathbf{K}_\gamma] = 0 \implies \text{underintegration!}$$

NOTE: Each numerical integration point increases the rank of the stiffness matrix $\mathbf{K}_\gamma$ by one (it corresponds to one constraint). We need less constraints than d.o.f. and hence we underintegrate to obtain a singular $\mathbf{K}_\gamma$

MINDLIN THEORY – ADDING TRANSVERSE SHEAR ENERGY

$$\mathcal{P}_{int}^{bend} = \frac{1}{2}\int_A \left[\frac{E}{1-\nu^2}\frac{h^3}{12}K_{\alpha\beta}K_{\alpha\beta} + \boxed{G\chi h\Gamma_\alpha\Gamma_\alpha}\right] dA$$

$$K_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial\theta_\alpha}{\partial x_\beta} + \frac{\partial\theta_\beta}{\partial x_\alpha}\right)$$

$$\Gamma_\alpha = \frac{\partial w}{\partial x_\alpha} - \theta_\alpha$$

Transverse shear energy is added; this is the penalty term that enforces the slope-rotation relation

NOTE energy involves 1st order gradients of d.o.f.

FEM FOR DYNAMICS PROBLEMS

EQUATIONS OF MOTION – HAMILTON'S PRINCIPLE

$$\text{Lagrangian} : \quad \mathcal{L} = \mathcal{K} - \mathcal{P}$$

$$\text{Kinetic} : \quad \mathcal{K} = \int_V \left[\frac{1}{2} \rho \dot{u}_i \dot{u}_i \right] dV$$

$$\text{Potential} : \quad \mathcal{P} = \mathcal{P}_{int} + \mathcal{P}_{ext}$$

$$\text{Internal} : \quad \mathcal{P}_{int} = \int_V W(\epsilon_{ij})\, dV \ ; \quad \sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$$

$$\text{External} : \quad \mathcal{P}_{ext} = -\int_V b_i u_i \, dV - \int_{\partial V_t} t_i u_i \, dS$$

$$\text{Hamilton} : \quad \delta \left[\int_{t_1}^{t_2} \mathcal{L}(\mathbf{u}, \dot{\mathbf{u}}) \, dt \right] = 0 \ ; \quad \mathbf{u}(\mathbf{x}, t_1) = \mathbf{u}(\mathbf{x}, t_2) = \mathbf{0}$$

EQUATIONS OF MOTION – HAMILTON'S PRINCIPLE- CONT.

$$\delta \int_{t_1}^{t_2} \left\{ \int_V \left[\frac{1}{2}\rho \dot{u}_i \dot{u}_i - \frac{1}{2} \boxed{L_{ijkl}\epsilon_{kl}} \epsilon_{ij} + b_i u_i \right] dV + \int_{\partial V_t} t_i u_i \, dS \right\} dt = 0$$

$$\int_{t_1}^{t_2} \left\{ \int_V \left[\rho \dot{u}_i \delta \dot{u}_i - \sigma_{ij} \delta \epsilon_{ij} + b_i \delta u_i \right] dV + \int_{\partial V_t} t_i \delta u_i \, dS \right\} dt = 0$$

$$\int_{t_1}^{t_2} \left\{ \int_V \boxed{\left[-\rho \ddot{u}_i + (\sigma_{ij})_{,j} + b_i \right]} \delta u_i \, dV + \int_{\partial V_t} \boxed{\left[t_i - \sigma_{ij} n_j \right]} \delta u_i \, dS \right\} dt = 0$$

Euler-Lagrange equation of motion – pointwise

Natural boundary condition – pointwise

EQUATIONS OF MOTION – HAMILTON'S PRINCIPLE - FEM

$$\text{Discretization}: \quad \mathbf{u}(\mathbf{x},t)=\mathbf{N}(\mathbf{x})\mathbf{Q}(t)\,, \quad \dot{\mathbf{u}}(\mathbf{x},t)=\mathbf{N}(\mathbf{x})\dot{\mathbf{Q}}(t)$$

FEM degrees of freedom

FEM mass matrix

$$\text{Kinetic}: \quad \mathcal{K}=\frac{1}{2}\dot{\mathbf{Q}}^T\boxed{\mathbf{M}}\dot{\mathbf{Q}}$$

FEM stiffness matrix

$$\text{Potential}: \quad \mathcal{P}=\mathcal{P}_{int}+\mathcal{P}_{ext}=\frac{1}{2}\mathbf{Q}^T\boxed{\mathbf{K}}\mathbf{Q}-\mathbf{Q}^T\boxed{\mathbf{F}}$$

FEM force vector

$$\text{Lagrangian}: \quad \mathcal{L}=\mathcal{K}-\mathcal{P}=\frac{1}{2}\dot{\mathbf{Q}}^T\mathbf{M}\dot{\mathbf{Q}}-\frac{1}{2}\mathbf{Q}^T\mathbf{K}\mathbf{Q}+\mathbf{Q}^T\mathbf{F}$$

$$\text{Hamilton}: \quad \delta\left[\int_{t_1}^{t_2}\mathcal{L}(\mathbf{Q},\dot{\mathbf{Q}})\,dt\right]=0\,; \quad \mathbf{Q}(t_1)=\mathbf{Q}(t_2)=\mathbf{0}$$

FEM equations of motion

$$\text{Hamilton}: \quad \int_{t_1}^{t_2}\left[\delta\mathbf{Q}^T\boxed{(-\mathbf{M}\ddot{\mathbf{Q}}-\mathbf{K}\mathbf{Q}+\mathbf{F})}\right]dt=0$$

FEM CALCULATIONS OF EIGENVALUES & MODES

EIGENVALUE PROBLEM USING RALEIGH QUOTIENTS - FEM

$$\text{Vibration equations} \ : \quad \mathbf{M}\ddot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} = \mathbf{0}$$

$$\text{Solution in form} \ : \quad \mathbf{Q}(t) = \exp[i\boxed{\omega}t]\boxed{\mathbf{U}}$$

eigenmode

eigenfrequency

$$\text{Eigenvalue problem} \ : \quad \left[\mathbf{K} - (\omega)^2\mathbf{M}\right]\mathbf{U} = \mathbf{0}$$

$$\text{Rayleigh algorithm} \ : \quad (\omega)^2_{max} = \lim_{j\to\infty} \frac{\mathbf{U}_j^T\mathbf{K}\mathbf{U}_j}{\mathbf{U}_j^T\mathbf{M}\mathbf{U}_j}$$

$$\text{Rayleigh algorithm} \ : \quad \boxed{\mathbf{M}}\mathbf{U}_{j+1} = \boxed{\mathbf{K}}\mathbf{U}_j$$

$$\text{Always converges} \ : \quad \mathbf{U}_0 \quad \text{arbitrary}$$

Positive definite mass matrix,
Easy to invert (lumping…)

Positive definite stiffness matrix

INCREMENTAL NEWTON-RAPHSON FOR NONLINEAR PROBLEMS IN FEM

NEWTON-RAPHSON FOR NONLINEAR EQUATIONS

- The Newton-Raphson algorithm is used to solve a nonlinear set of equations $\mathbf{f}(\mathbf{u}) = \mathbf{0}$, where $\mathbf{u}$ is the n-dimensional degree of freedom vector. Since our FEM system comes from a minimization principle, vetor $\mathbf{f}$ is the derivative of the potential energy $\mathcal{P}(\mathbf{u})$, i.e. $\mathbf{f} = \partial\mathcal{P}/\partial\mathbf{u}$

- Method requires the construction of the tangent stiffness matrix $\mathbf{K}(\mathbf{u})$ (where $\mathbf{K}(\mathbf{u}) = \partial\mathbf{f}/\partial\mathbf{u} = \partial^2\mathcal{P}/\partial\mathbf{u}\partial\mathbf{u}$) that has to be re-assembled at every step of the solution process

- Although this algorithm has a rapid convergence, it requires a good initial guess. If the initial guess is outside the domain of convergence, the algorithm fails. So a method to provide a reliable initial guess is needed

INCREMENTAL NEWTON-RAPHSON METHOD IN FEM

$$\mathcal{P}(u,\lambda): \quad \text{continuum energy at displ. } u(\mathbf{x}) \in U,\ \text{load } \lambda \geq 0$$

$$\mathcal{P}(\mathbf{u},\lambda): \quad \text{discretized energy at displ. } \mathbf{u} \in \mathbb{R}^n,\ \text{load } \lambda \geq 0$$

$$\mathbf{u} = \{u_i\}_{i=1}^{n}: \quad u(\mathbf{x}) = \sum_{i=1}^{n} u_i \varphi_i(\mathbf{x}),\ u_i:\ \text{d.o.f},\ \text{(FEM method)}$$

$$\mathcal{P}_{,\mathbf{u}}(\mathbf{u},\lambda) = \mathbf{0}: \quad \partial\mathcal{P}/\partial u_i = 0,\ i = 1\ldots n:\ \text{equilibrium equations}$$

$$\text{Start at}: \quad \lambda = 0, \quad \mathbf{u} = \mathbf{0}$$

INCREMENTAL NEWTON-RAPHSON METHOD IN FEM

$$\text{Newton} - \text{Raphson}: \quad \mathbf{0} = \mathcal{P}_{,\mathbf{u}}(\mathbf{u} + \Delta\mathbf{u}, \lambda) \approx \mathcal{P}_{,\mathbf{u}}(\mathbf{u}, \lambda) + \mathcal{P}_{,\mathbf{uu}}(\mathbf{u}, \lambda)\Delta\mathbf{u}$$

$$\Delta\mathbf{u} \equiv \mathbf{u}_{(i)}^{(1)} - \mathbf{u}_{(i)}^{(0)} = -\boxed{\left[\mathcal{P}_{,\mathbf{uu}}\left(\mathbf{u}_{(i)}^{(0)}, \lambda_{(i)}\right)\right]}^{-1} \boxed{\mathcal{P}_{,\mathbf{u}}\left(\mathbf{u}_{(i)}^{(0)}, \lambda_{(i+1)}\right)},$$

Stiffness matrix (needs inversion at each step)

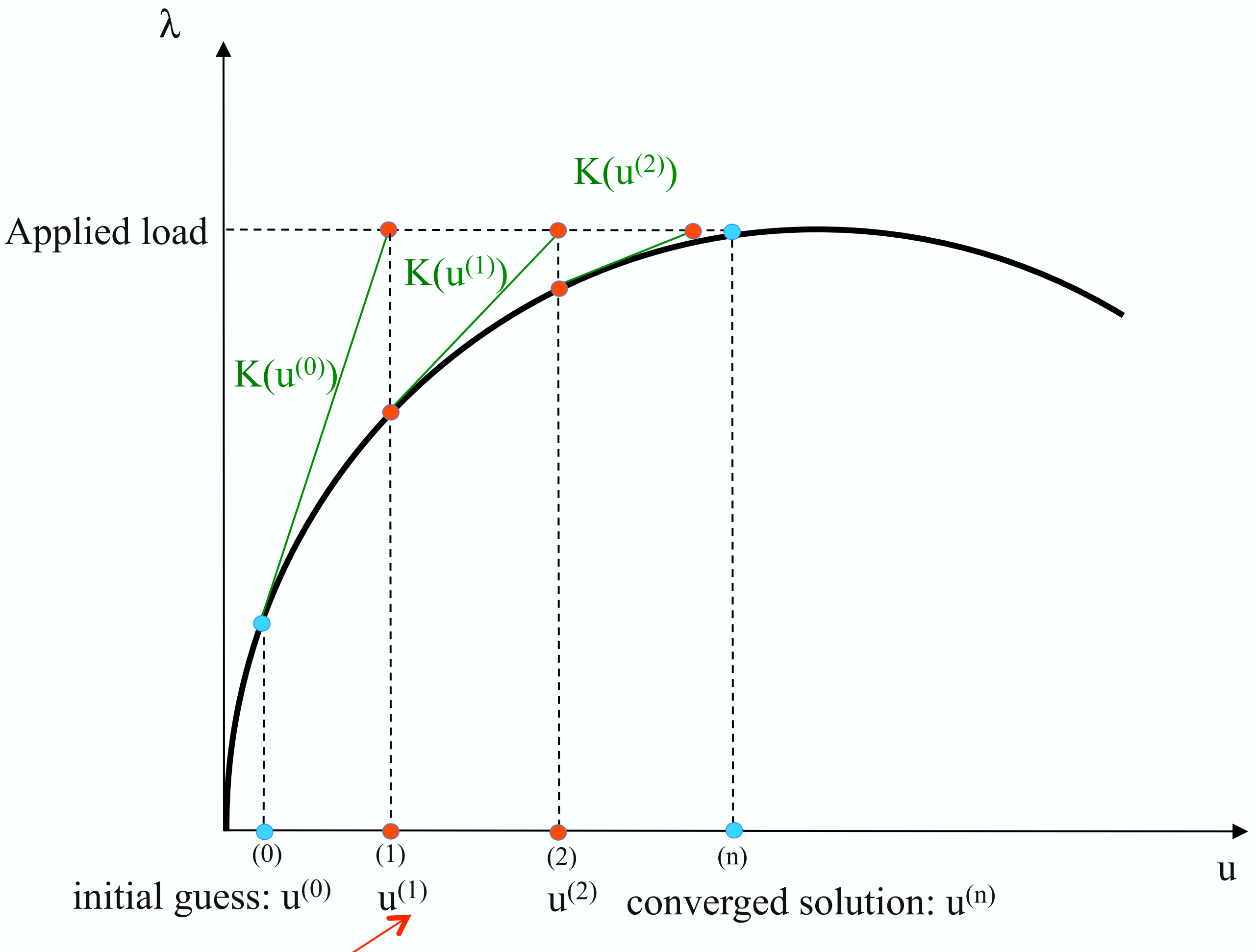
$$\text{start at} \ : \mathbf{u}_{(i)}^{(0)} = \mathbf{u}_{(i)} \equiv \mathbf{u}(\lambda_{(i)}), \ \text{where} \ \mathbf{u}_{(i)}^{(j)} : \ \text{increment} \ (i) \ \text{iteration} \ (j)$$

Load vector

$$\Delta\mathbf{u} \equiv \mathbf{u}_{(i)}^{(j+1)} - \mathbf{u}_{(i)}^{(j)} = -\boxed{\left[\mathcal{P}_{,\mathbf{uu}}\left(\mathbf{u}_{(i)}^{(j)}, \lambda_{(i+1)}\right)\right]}^{-1} \boxed{\mathcal{P}_{,\mathbf{u}}\left(\mathbf{u}_{(i)}^{(j)}, \lambda_{(i+1)}\right)},$$

$$\text{end at} \ : \mathbf{u}_{(i)}^{(j+1)} = \mathbf{u}_{(i+1)} \equiv \mathbf{u}(\lambda_{(i+1)}), \ \text{when} : \ \left\| \mathcal{P}_{,\mathbf{u}}\left(\mathbf{u}_{(i)}^{(j)}, \lambda_{(i+1)}\right) \right\| < \varepsilon$$

NEWTON-RAPHSON FOR NONLINEAR EQUATIONS



Superscript: iteration number

Equilibrium solution of continuum problem is amenable to solving a finite d.o.f. nonlinear problem through FEM discretization

For the Newton-Raphson method to work, a good initial displacement guess is needed to guarantee convergence (this is a fixed-point method with quadratic convergence)

INCREMENTAL METHOD FOR NONLINEAR EQUATIONS

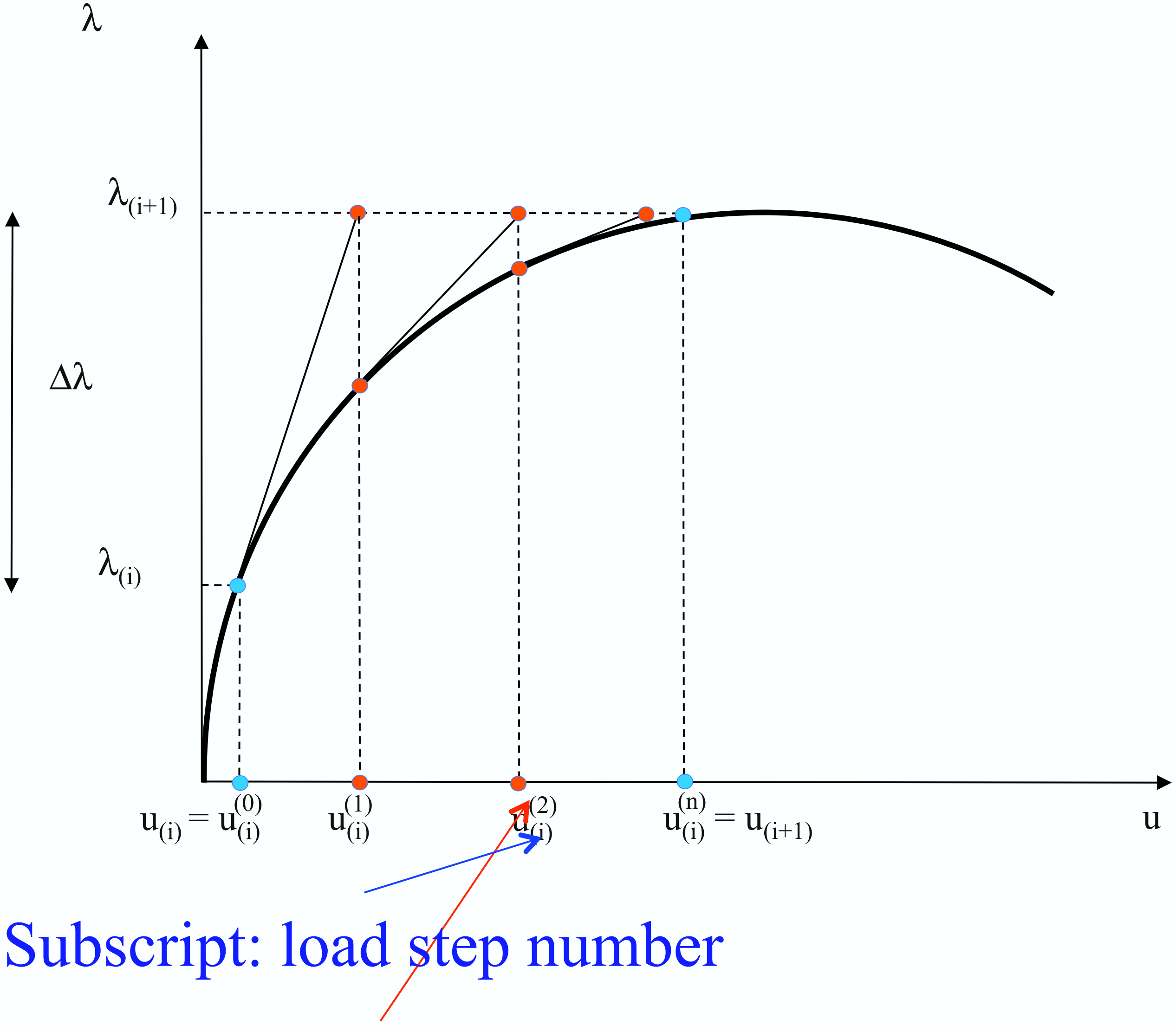
- Although this algorithm has a rapid convergence, it requires a good initial guess. If the initial guess is outside the domain of convergence, the algorithm fails. So a method to provide a reliable initial guess is needed

- Physics come to the rescue. Loading can often be parametrized by a monotonically increasing positive scalar $\lambda \geq 0$ (termed load parameter) and the potential energy is $\mathcal{P}(\mathbf{u}, \lambda)$. Solution is $\mathbf{u}(\lambda)$

- For $\lambda = 0$ (unloaded configuration) $\mathbf{u}(0) = \mathbf{0}$, which is our starting point

- By increasing the load in small increments $\Delta\lambda = \lambda_{i+1} - \lambda_i$ one can use the converged solution of the previous load step $\mathbf{u}(\lambda_i)$ as an accurate initial guess to calculate the solution for the current load step $\mathbf{u}(\lambda_{i+1})$. This is the incremental part of the algorithm that guarantees convergence if $\Delta\lambda$ is adequately small

INCREMENTAL METHOD FOR NONLINEAR EQUATIONS



Subscript: load step number

Superscript: iteration number

Incremental method starting from zero load/displacement is needed to guarantee convergence of Newton-Raphson (works for small enough load steps)

Within each increment, we use the converged solution of the previous load step as an accurate initial guess to calculate the solution for the current load step

FINITE STRAIN ELASTICITY (2D)

FINITE STRAIN ELASTICITY – 2D

Finite strain elasticity: $W(\mathbf{F}) = W(\mathbf{C})$, where $\mathbf{C} = \mathbf{F}^{T}\cdot\mathbf{F}$ (right Cauchy-Green)

Isotropic case: $W(I_1, I_2)$ where are invariants of **C** ($I_1 = \mathrm{tr}\,\mathbf{C}$, $I_2 = \det\mathbf{C}$)

Element force vector and stiffness matrices easily constructed by calculating the first and second derivatives of W with respect to **F**

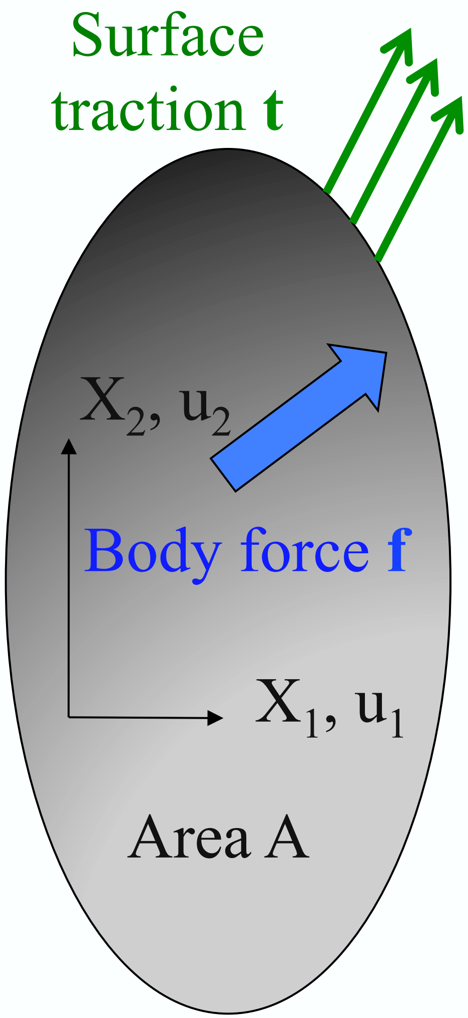
ATTENTION: Nonlinear problems have **non-unique** solutions (**buckling**)

ATTENTION: Constitutive equations have to satisfy certain properties (**rank-one convexity**) otherwise discontinuous solutions are possible and FEM **results depend on the mesh** you use!

You cannot use nonlinear FEM problems without understanding your mechanics, otherwise you get nonsensical results

FINITE STRAIN ELASTICITY – 2D

REFERENCE CONFIGURATION



boundary ∂A

$$\mathcal{P}(u,\lambda) = \mathcal{P}_{int} + \mathcal{P}_{ext},$$

Displacement : $u = \mathbf{u}(\mathbf{X}) = (u_1(\mathbf{X}), u_2(\mathbf{X}))$

At reference configuration point : $\mathbf{X} = (X_1, X_2,)$

$$\mathcal{P}_{int} = \int_A W(\mathbf{C}(\mathbf{X}), \mathbf{X})\, dA\ , \quad \mathbf{C} \equiv \mathbf{F}^T \cdot \mathbf{F}$$

$$\mathcal{P}_{ext} = -\left[\int_A f_i(\lambda) u_i dA + \int_{\partial A} t_i(\lambda) u_i ds\right]$$

Elastic energy density/ ref. volume : $W(\mathbf{C}(\mathbf{X}), \mathbf{X})$

Isotropic case : $W(I_1, I_2, \mathbf{X})\ ; \quad I_1 = \text{tr}\ \mathbf{C},\ I_2 = \mathbf{C}$

FINITE STRAIN ELASTICITY – 2D

$$u_i = \mathbf{N}_i \mathbf{q}_e \ , \quad u_{i,j} \equiv \partial u_i / \partial X_j = \mathbf{B}_{ij} \delta \mathbf{q}_e$$

FEM force vector

$$\mathcal{P}^e_{,u} \delta u = \boxed{\left\{ \int_{A_e} \left[\frac{\partial W}{\partial F_{ij}} \mathbf{B}_{ij} - f_i \mathbf{N}_i \right] dA - \int_{\partial A_e} \left[t_i \mathbf{N}_i \right] ds \right\}} \mathbf{q}_e$$

$$(\mathcal{P}^e_{,uu} \Delta u) \delta u = (\Delta \mathbf{q}_e)^T \boxed{\left\{ \int_{A_e} \left[\mathbf{B}_{ij} \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}} \mathbf{B}_{kl} \right] dA \right\}} \delta \mathbf{q}_e$$

FEM tangent stiffness matrix

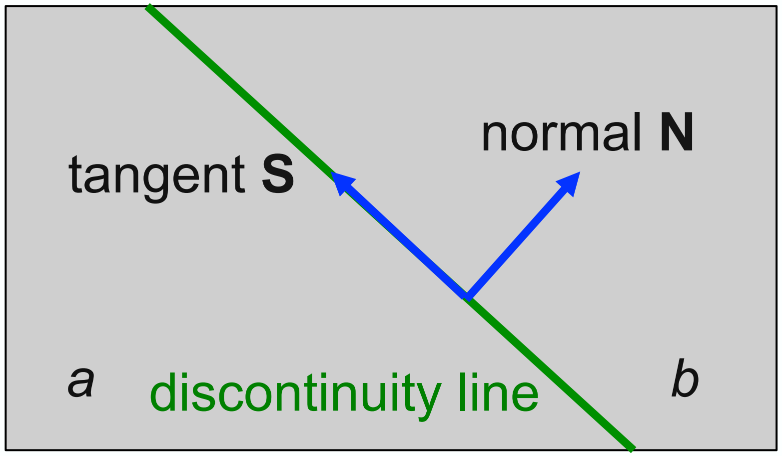
$$L_{ijkl}(\mathbf{F}) \equiv \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}} (\mathbf{F}) \ ; \quad \text{tangent moduli}$$

NOTE: *W*(**F**) not convex in **F** – $A_{ij} L_{ijkl}$ (**F**) A_{kl} sign depends on **A** (arbitrary)

FINITE STRAIN ELASTICITY – 2D

Finite strain elasticity: *W(***F***)* cannot be convex in **F** (otherwise all nonlinear elasticity problems would have had a unique solution)!

However *W(***F***)* has to be rank-one convex (elliptic system of incremental equations) in order to avoid the appearance of discontinuous solutions)



$$\text{definition}: [\![f]\!] \equiv f_b - f_a$$

$$\text{equilibrium}: \mathbf{N}\cdot[\![\mathbf{\Pi}]\!] = 0$$

$$\text{kinematics}: [\![\mathbf{F}]\!]\cdot\mathbf{S} = 0 \implies [\![\mathbf{F}]\!] = \mathbf{A}\otimes\mathbf{N}$$

$$\text{constitutive}: \dot{\mathbf{\Pi}} = \mathbf{L} : \dot{\mathbf{F}} \quad \left(\text{recall}: \mathbf{\Pi} = \frac{\partial W}{\partial \mathbf{F}}\right)$$

$$\text{moduli}: \mathbf{L} = \frac{\partial^2 W}{\partial\mathbf{F}\partial\mathbf{F}}$$

$$\text{loss of ellipticity}: (\mathbf{A}\otimes\mathbf{N}) : \mathbf{L} : (\mathbf{A}\otimes\mathbf{N}) = 0$$

POWERFULL CODES PRODUCE NICE PICTURES, BUT

1. Need to know the code behind them!
2. You need to understand your model! Very frequently the code gives correct results but you are unable to interpret them…
3. In linear problems with constraints or higher order displacement gradients you must avoid locking phenomena (underintegrate)
4. In nonlinear problems solutions not unique! attention to buckling and localization phenomena (mesh-dependence)!
5. Goal of the class to help you understand mechanics and how we translate the mathematical model to a functioning algorithm

TWO IDEAS I WANT YOU TO REMEMBER:

1. **Need to know the code behind FEM packages you use!**

2. **Understand your mechanics very well!**

THANKS FOR TAKING THIS CLASS, ALWAYS AT YOUR

DISPOSAL IF YOU ARE INETERESTED IN THIS MATERIAL!