TOPICS COVERED IN THIS LECTURE

1. FEM FOR DYNAMICS PROBLEMS
2. CALCULATIONS OF EIGENVALUES & EIGENMODES
3. EIGENVALUES AND EIGENMODES OF BEAMS
4. EIGENVALUES AND EIGENMODES OF PLATES

FEM FOR DYNAMICS PROBLEMS

Lagrangian : $\mathcal{L} = \mathcal{K} - \mathcal{P}$

Kinetic : $\mathcal{K} = \int_V \left[\frac{1}{2}\rho \dot{u}_i \dot{u}_i\right] dV$

Potential : $\mathcal{P} = \mathcal{P}_{int} + \mathcal{P}_{ext}$

Internal : $\mathcal{P}_{int} = \int_V W(\epsilon_{ij})\, dV \; ; \quad \sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$

External : $\mathcal{P}_{ext} = -\int_V b_i u_i \, dV - \int_{\partial V_t} t_i u_i \, dS$

Hamilton : $\delta \left[\int_{t_1}^{t_2} \mathcal{L}(\mathbf{u}, \dot{\mathbf{u}})\, dt\right] = 0 \; ; \quad \mathbf{u}(\mathbf{x}, t_1) = \mathbf{u}(\mathbf{x}, t_2) = \mathbf{0}$

$$\delta \int_{t_1}^{t_2} \left\{ \int_V \left[\frac{1}{2}\rho \dot{u}_i \dot{u}_i - \frac{1}{2} L_{ijkl}\epsilon_{kl}\epsilon_{ij} + b_i u_i \right] dV + \int_{\partial V_t} t_i u_i \, dS \right\} dt = 0$$

$$\int_{t_1}^{t_2} \left\{ \int_V \left[\rho \dot{u}_i \delta \dot{u}_i - \sigma_{ij}\delta\epsilon_{ij} + b_i \delta u_i \right] dV + \int_{\partial V_t} t_i \delta u_i \, dS \right\} dt = 0$$

$$\int_{t_1}^{t_2} \left\{ \int_V \left[-\rho \ddot{u}_i + (\sigma_{ij})_{,j} + b_i \right] \delta u_i \, dV + \int_{\partial V_t} \left[t_i - \sigma_{ij} n_j \right] \delta u_i \, dS \right\} dt = 0$$

Euler-Lagrange equation of motion – pointwise

Natural boundary condition – pointwise

$$\text{Discretization}: \quad \mathbf{u}(\mathbf{x},t) = \mathbf{N}(\mathbf{x})\mathbf{Q}(t)\ , \quad \dot{\mathbf{u}}(\mathbf{x},t) = \mathbf{N}(\mathbf{x})\dot{\mathbf{Q}}(t)$$

FEM degrees of freedom

FEM mass matrix

$$\text{Kinetic}: \quad \mathcal{K} = \frac{1}{2}\dot{\mathbf{Q}}^T\mathbf{M}\dot{\mathbf{Q}}$$

FEM stiffness matrix

$$\text{Potential}: \quad \mathcal{P} = \mathcal{P}_{int} + \mathcal{P}_{ext} = \frac{1}{2}\mathbf{Q}^T\mathbf{K}\mathbf{Q} - \mathbf{Q}^T\mathbf{F}$$

FEM force vector

$$\text{Lagrangian}: \quad \mathcal{L} = \mathcal{K} - \mathcal{P} = \frac{1}{2}\dot{\mathbf{Q}}^T\mathbf{M}\dot{\mathbf{Q}} - \frac{1}{2}\mathbf{Q}^T\mathbf{K}\mathbf{Q} + \mathbf{Q}^T\mathbf{F}$$

$$\text{Hamilton}: \quad \delta\left[\int_{t_1}^{t_2} \mathcal{L}(\mathbf{Q},\dot{\mathbf{Q}})\ dt\right] = 0\ ; \quad \mathbf{Q}(t_1) = \mathbf{Q}(t_2) = \mathbf{0}$$

FEM equations of motion

$$\text{Hamilton}: \quad \int_{t_1}^{t_2}\left[\delta\mathbf{Q}^T(-\mathbf{M}\ddot{\mathbf{Q}} - \mathbf{K}\mathbf{Q} + \mathbf{F})\right] dt = 0$$

CALCULATIONS OF EIGENVALUES & EIGENMODES

Free vibrations : $\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} = \mathbf{0}$

eigenfrequency

Solution : $\mathbf{Q}(t) = \exp[i\omega t]\mathbf{U}$ eigenmode

Eigenproblem : $\left[\mathbf{K} - (\omega)^2\mathbf{M}\right]\mathbf{U} = \mathbf{0}$

Rayleigh ratio : $(\omega)^2_{max} = \lim_{j\to\infty} \dfrac{\mathbf{U}_j^T\mathbf{K}\mathbf{U}_j}{\mathbf{U}_j^T\mathbf{M}\mathbf{U}_j} = (\omega_n)^2$

$\mathbf{U}_0$ arbitrary : $\mathbf{M}\mathbf{U}_{j+1} = \mathbf{K}\mathbf{U}_j \; ; \; \mathbf{U}_n = \lim_{j\to\infty} \mathbf{U}_j$

Positive definite mass matrix, Easy to invert (lumping…)

Positive definite stiffness matrix

$\mathbf{U}_0 \perp \mathbf{U}_n$: $(\omega_{n-1})^2 = \lim_{j\to\infty} \dfrac{\mathbf{U}_j^T\mathbf{K}\mathbf{U}_j}{\mathbf{U}_j^T\mathbf{M}\mathbf{U}_j} \; ; \; \mathbf{U}_{n-1} = \lim_{j\to\infty} \mathbf{U}_j$

subspace method

EIGENVALUES & EIGENMODES OF BEAMS

$$\text{Hamilton} \quad : \quad \delta\left[\int_{t_1}^{t_2} \mathcal{L}(\mathbf{u}, \dot{\mathbf{u}})\; dt\right] = 0\;; \qquad \mathbf{u}(\mathbf{x}, t_1) = \mathbf{u}(\mathbf{x}, t_2) = \mathbf{0}$$

$$\text{Kinetic} : \quad \mathcal{K} = \frac{1}{2}\int_0^L \left\{\rho A \left[\left(\frac{\partial u_1}{\partial t}\right)^2 + \left(\frac{\partial u_2}{\partial t}\right)^2\right]\right\} dx_1$$

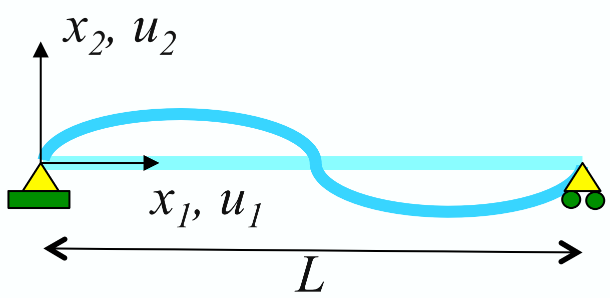
$$\text{Potential} \quad : \quad \mathcal{P} = \frac{1}{2}\int_0^L \left\{EA\left(\frac{\partial u_1}{\partial x_1}\right)^2 + EI\left(\frac{\partial^2 u_2}{\partial x_1^2}\right)^2\right\} dx_1$$

$$\text{axial vibration} : \quad \boxed{EA\frac{\partial^2 u_1}{\partial x_1^2} - \rho A\frac{\partial^2 u_1}{\partial t^2} = 0}\,, \text{(and b.c.} - \ \delta u_1 \text{ terms)}$$

$$\text{bending vibration} : \quad \boxed{EI\frac{\partial^4 u_2}{\partial x_1^4} + \rho A\frac{\partial^2 u_2}{\partial t^2} = 0}\,, \text{(and b.c.} - \ \delta u_2 \text{ terms)}$$

bending vibration : $$EI\frac{\partial^4 u_2}{\partial x_1^4} + \rho A \frac{\partial^2 u_2}{\partial t^2} = 0\ ,$$

simply supported beam



essential b.c. : $$u_2(0,t) = u_2(L,t) = 0\ ,$$

natural b.c. : $$EI\frac{\partial^2 u_2}{\partial x_1^2}(0,t) = EI\frac{\partial^2 u_2}{\partial x_1^2}(L,t) = 0\ .$$

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variable separation : $$u_2(x_1,t) = X(x_1)T(t)$$

$$T(t) = \sin(\omega t - \alpha)\ ;$$

$$\mu^4 = \left(\frac{\rho A}{EI}\right)\omega^2\ ;$$

$$X(x_1) = A\cos(\mu x_1) + B\sin(\mu x_1) + C\cosh(\mu x_1) + D\sinh(\mu x_1)$$

$$X(x_1) = A\cos(\mu x_1) + B\sin(\mu x_1) + \; C\cosh(\mu x_1) + D\sinh(\mu x_1)$$

$$u_2(0,t) = EI\frac{\partial^2 u_2}{\partial x_1^2}(0,t) = 0 \implies A = C = 0$$

$$u_2(L,t) = EI\frac{\partial^2 u_2}{\partial x_1^2}(L,t) = 0 \implies B \neq 0,\; D \neq 0\;,\; \text{where :}$$

$$\det\begin{bmatrix} \sin(\mu L) & \sinh(\mu L) \\ \mu^2\sin(\mu L) & \mu^2\sinh(\mu L) \end{bmatrix} = 0 \implies 2\mu^2\sin(\mu L)\sinh(\mu L) = 0$$

$$\sin(\mu L) = 0 \implies \mu_n L = n\pi\;, n = 1, 2, 3, \cdots$$

$$\text{eigenmode : } X(x_1) = B\sin(\mu_n t)\;; \qquad \text{c. frequency : } \omega_n = \left(\frac{n\pi}{L}\right)^2\left(\frac{EI}{\rho A}\right)^{1/2}$$

FREE VIBRATIONS OF TIMOSHENKO BEAM

$$\text{Hamilton} \; : \quad \delta \left[\int_{t_1}^{t_2} \mathcal{L}(\mathbf{u}, \dot{\mathbf{u}}) \, dt \right] = 0 \; ; \quad \mathbf{u}(\mathbf{x}, t_1) = \mathbf{u}(\mathbf{x}, t_2) = \mathbf{0}$$

$$\text{Kinetic} : \quad \mathcal{K} = \frac{1}{2} \int_0^L \left\{ \rho A \left[\left(\frac{\partial u_1}{\partial t} \right)^2 + \left(\frac{\partial u_2}{\partial t} \right)^2 \right] + \rho I \left[\left(\frac{\partial \theta}{\partial t} \right)^2 \right] \right\} dx_1$$

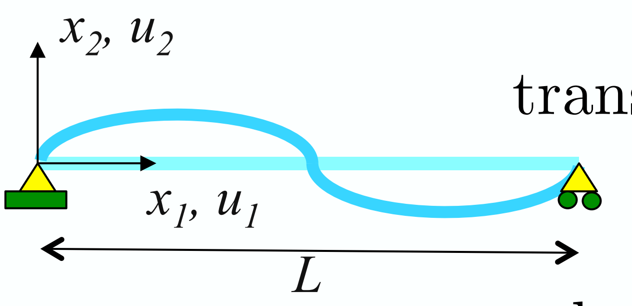
$$\text{Potential} \; : \quad \mathcal{P} = \frac{1}{2} \int_0^L \left\{ EA \left(\frac{\partial u_1}{\partial x_1} \right)^2 + \kappa GA \left(\frac{\partial u_2}{\partial x_1} - \theta \right)^2 + EI \left(\frac{\partial \theta}{\partial x_1} \right)^2 \right\} dx_1$$

$$\text{axial} : \quad \boxed{EA \frac{\partial^2 u_1}{\partial x_1^2} - \rho A \frac{\partial^2 u_1}{\partial t^2} = 0}, \text{(and b.c.} - \; \delta u_1 \text{ terms)}$$

$$\text{transverse} : \quad \boxed{\kappa GA \left(\frac{\partial^2 u_2}{\partial x_1^2} - \frac{\partial \theta}{\partial x_1} \right) - \rho A \frac{\partial^2 u_2}{\partial t^2} = 0}, \text{(and b.c.} - \; \delta u_2 \text{ terms)}$$

$$\text{bending} : \quad \boxed{EI \frac{\partial^2 \theta}{\partial x_1^2} + \kappa GA \left(\frac{\partial u_2}{\partial x_1} - \theta \right) - \rho I \frac{\partial^2 \theta}{\partial t^2} = 0}, \text{(and b.c.} - \; \delta \theta \text{ terms)}$$

simply supported beam



$$\text{transverse}: \quad \kappa G A \left(\frac{\partial^2 u_2}{\partial x_1^2} - \frac{\partial \theta}{\partial x_1} \right) - \rho A \frac{\partial^2 u_2}{\partial t^2} = 0$$

$$\text{bending}: \quad EI \frac{\partial^2 \theta}{\partial x_1^2} - \kappa G A \left(\frac{\partial u_2}{\partial x_1} - \theta \right) + \rho I \frac{\partial^2 \theta}{\partial t^2} = 0$$

$$\text{essential b.c.} \quad : \quad u_2(0,t) = u_2(L,t) = 0 \; , \; (\text{for tranverse})$$

$$\text{natural b.c.} \quad : \quad EI \frac{\partial \theta}{\partial x_1}(0,t) = EI \frac{\partial \theta}{\partial x_1}(L,t) = 0 \; , \; (\text{for bending})$$

$$u_2(x_1,t) = X(x_1) \sin(\omega t - \alpha) : \quad \boxed{X(x_1) = A \sin(\lambda x_1) + B \sinh(\lambda x_1)}$$

$$\theta(x_1,t) = \Theta(x_1) \sin(\omega t - \alpha) : \quad \boxed{\Theta(x_1) = C \sin(\lambda x_1) + D \sinh(\lambda x_1)}$$

Relation between wavelength and c. frequency more complicated than Euler beam!

$$d \equiv \rho\omega^2 \left(\frac{1}{E} + \frac{1}{\kappa G} \right) , \quad e \equiv \frac{\rho\omega^2}{E} \left(\frac{\rho\omega^2}{\kappa G} - \frac{A}{I} \right)$$

$$(\lambda_-)^2 \equiv \left| \frac{d - \sqrt{d^2 - 4e}}{2} \right| , \quad (\lambda_+)^2 \equiv \left| \frac{d + \sqrt{d^2 - 4e}}{2} \right|$$

$$\longrightarrow \begin{cases} \det \begin{bmatrix} \sinh \lambda_- L & \sin \lambda_+ L \\ \lambda_-^2 \sinh \lambda_- L & -\lambda_+^2 \sin \lambda_+ L \end{bmatrix} = 0 & \text{for} \quad \omega < \sqrt{\dfrac{\kappa G A}{\rho I}} \\ \det \begin{bmatrix} \sin \lambda_- L & \sin \lambda_+ L \\ -\lambda_-^2 \sin \lambda_- L & -\lambda_+^2 \sin \lambda_+ L \end{bmatrix} = 0 & \text{for} \quad \omega > \sqrt{\dfrac{\kappa G A}{\rho I}} \end{cases}$$

Bernoulli-Euler-Navier & Timoshenko beam share axial vibration equations!

$$\text{axial vibration}: \quad EA\frac{\partial^2 u_1}{\partial x_1^2} - \rho A\frac{\partial^2 u_1}{\partial t^2} = 0 \ ,$$

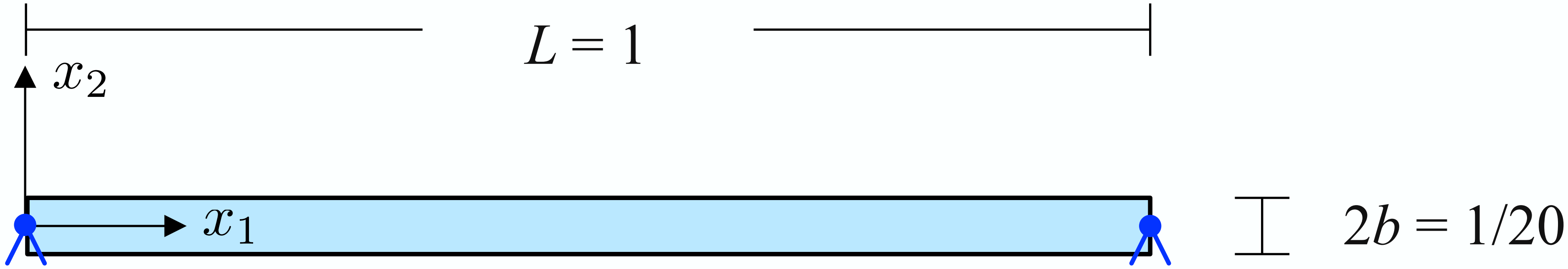
$$\text{essential b.c.}: \quad u_1(0,t) = 0 \ , \quad \text{(fixed end)}$$

$$\text{natural b.c.}: \quad EA\frac{\partial u_1}{\partial x_1}(L,t) = 0 \ , \quad \text{(free end)}$$

$$\text{variable separation}: \quad u_2(x_1,t) = X(x_1)\sin(\omega t - \alpha) \ .$$

$$\text{eigenmode}: \quad \boxed{X(x_1) = A\sin\left[\left(n - \frac{1}{2}\right)\frac{\pi x_1}{L}\right] \ ; \quad n = 1,2,3\cdots}$$

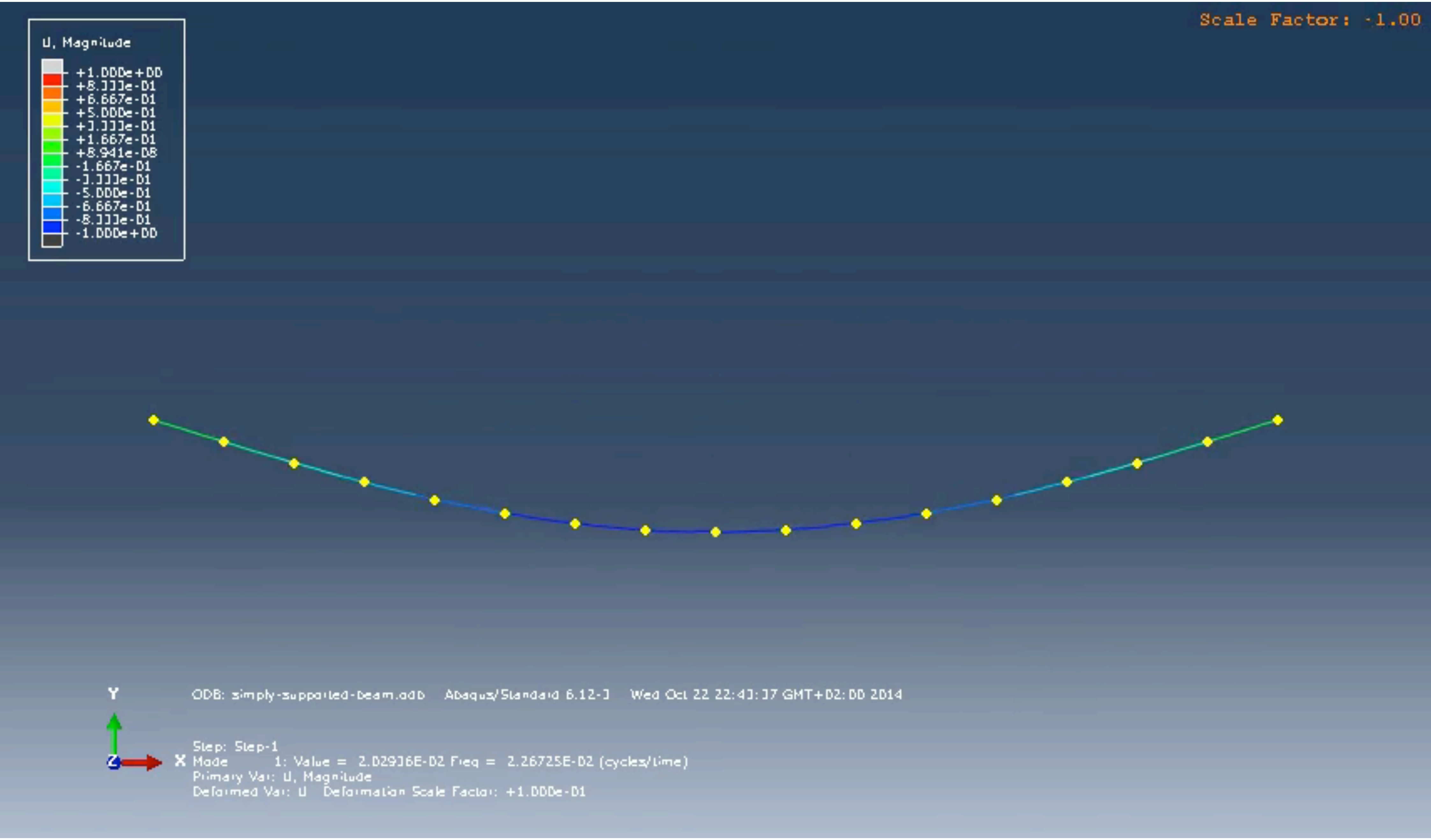
$$\text{c. frequency}: \quad \boxed{\omega_n = \left(n - \frac{1}{2}\right)\frac{\pi}{L}\sqrt{\frac{E}{\rho}} \ ; \quad n = 1,2,3\cdots}$$

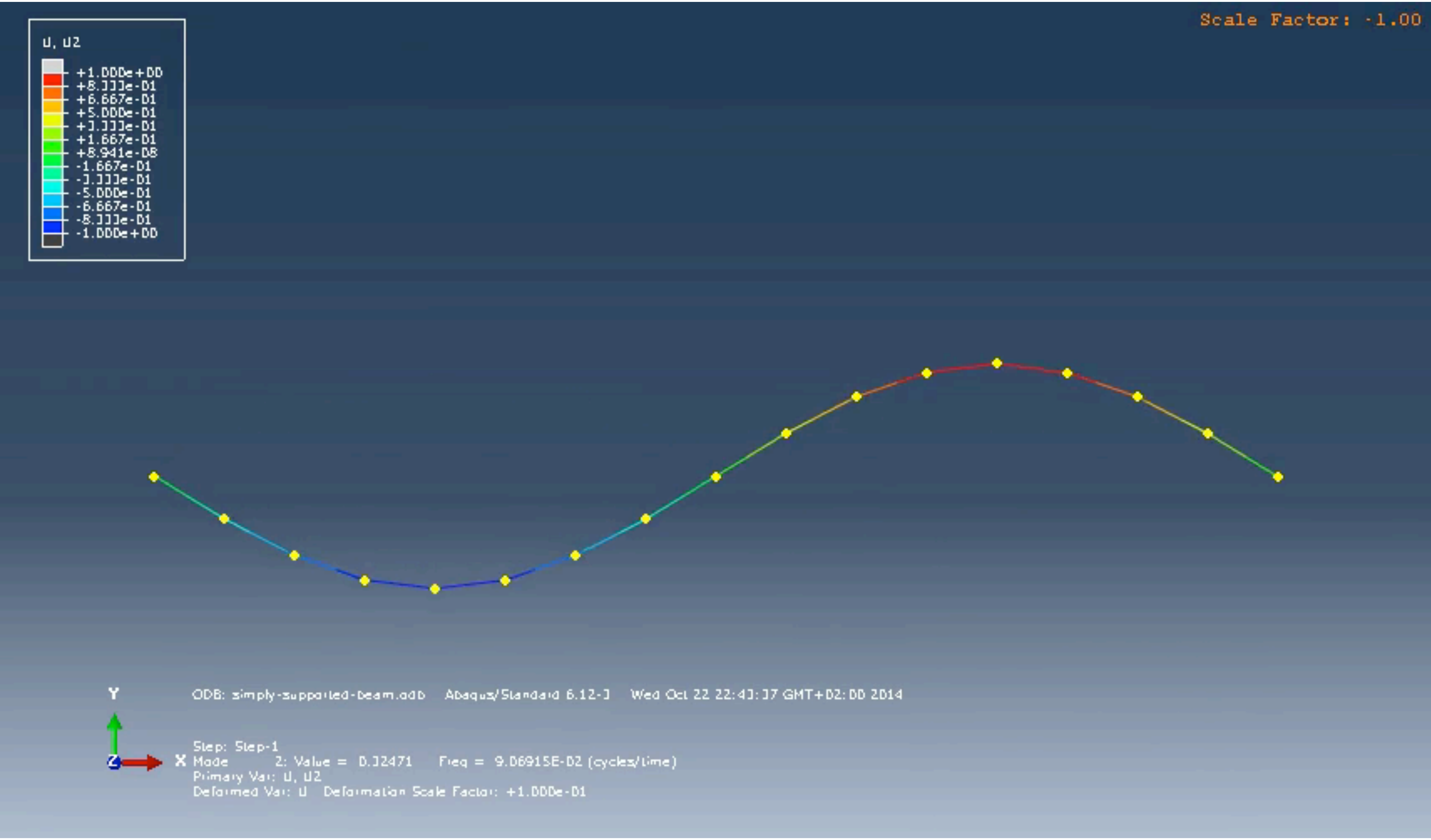


$\rho = 1,\ E = 1,\ \nu = 0.3$

$A = 2\ ba,\ I = 2/3\ b^3 a$

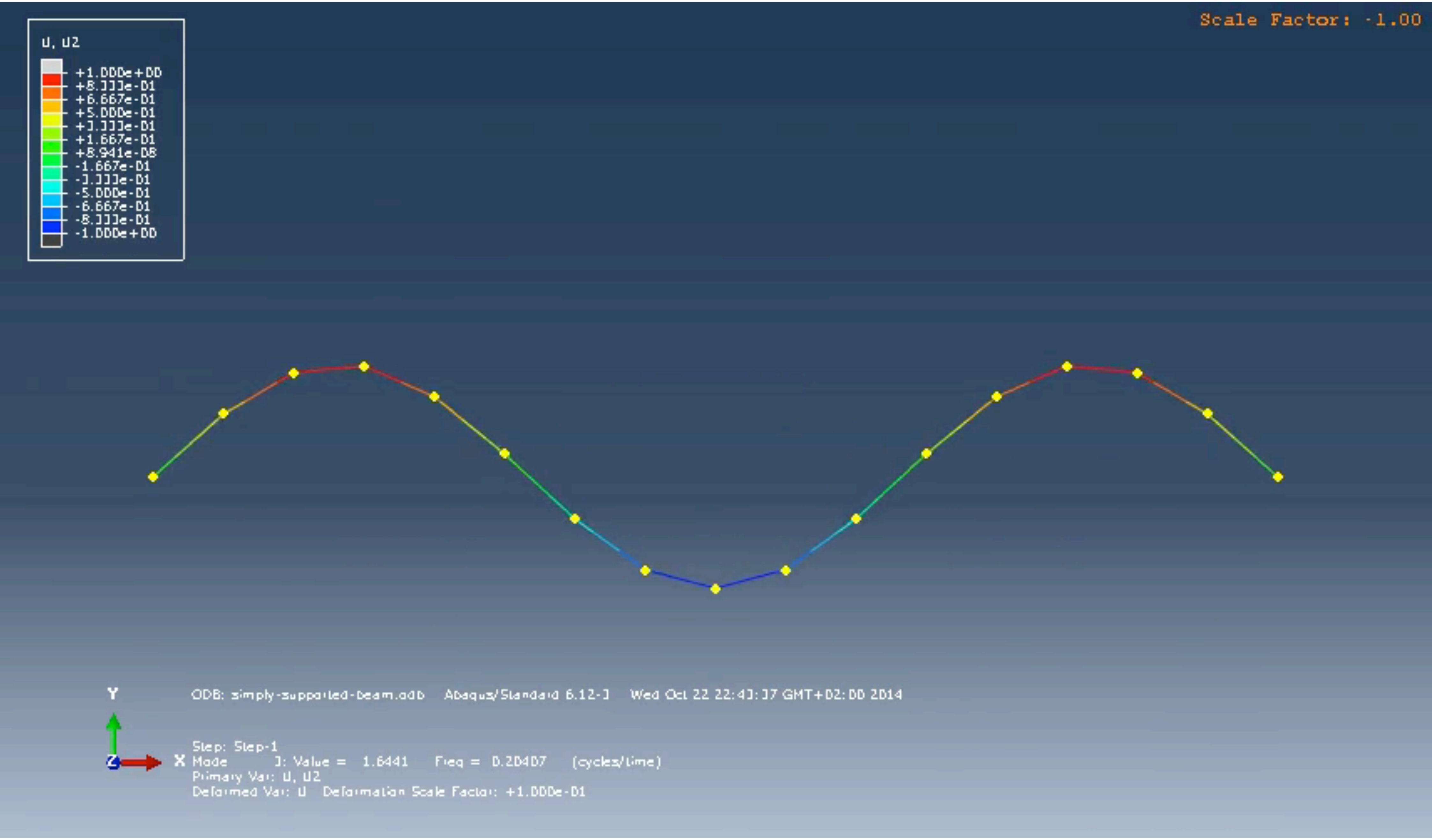
1st BENDING EIGENMODE OF S. S. EULER BEAM



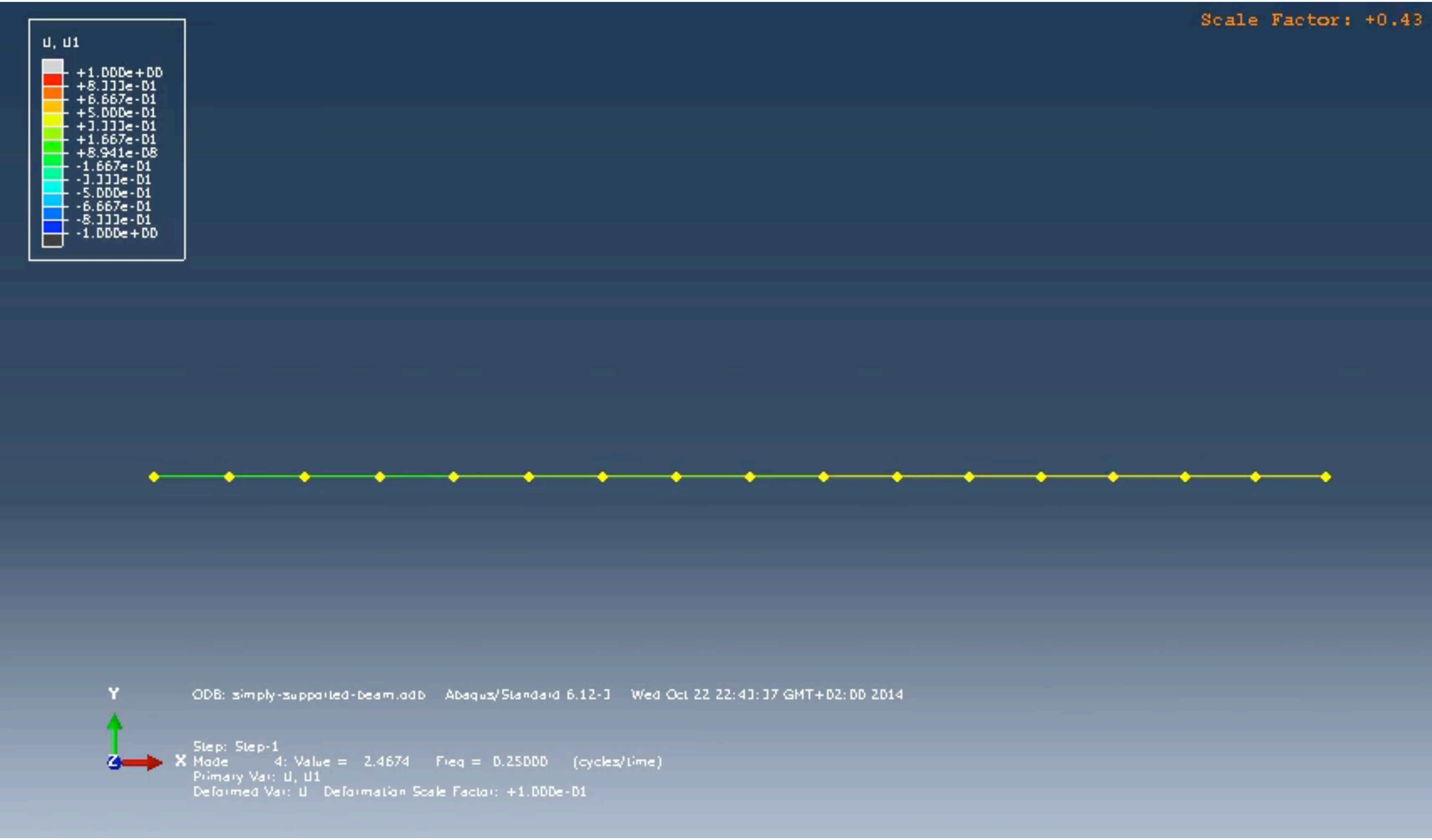


U, U2
+1.000e+00
+8.333e-01
+6.667e-01
+5.000e-01
+3.333e-01
+1.667e-01
+8.941e-08
-1.667e-01
-3.333e-01
-5.000e-01
-6.667e-01
-8.333e-01
-1.000e+00
Scale Factor: -1.00
Y
X
Z
ODB: simply-supported-beam.odb Abaqus/Standard 6.12-3 Wed Oct 22 22:43:37 GMT+02:00 2014
Step: Step-1
Mode 2: Value = 0.32471 Freq = 9.06915E-02 (cycles/time)
Primary Var: U, U2
Deformed Var: U Deformation Scale Factor: +1.000e-01

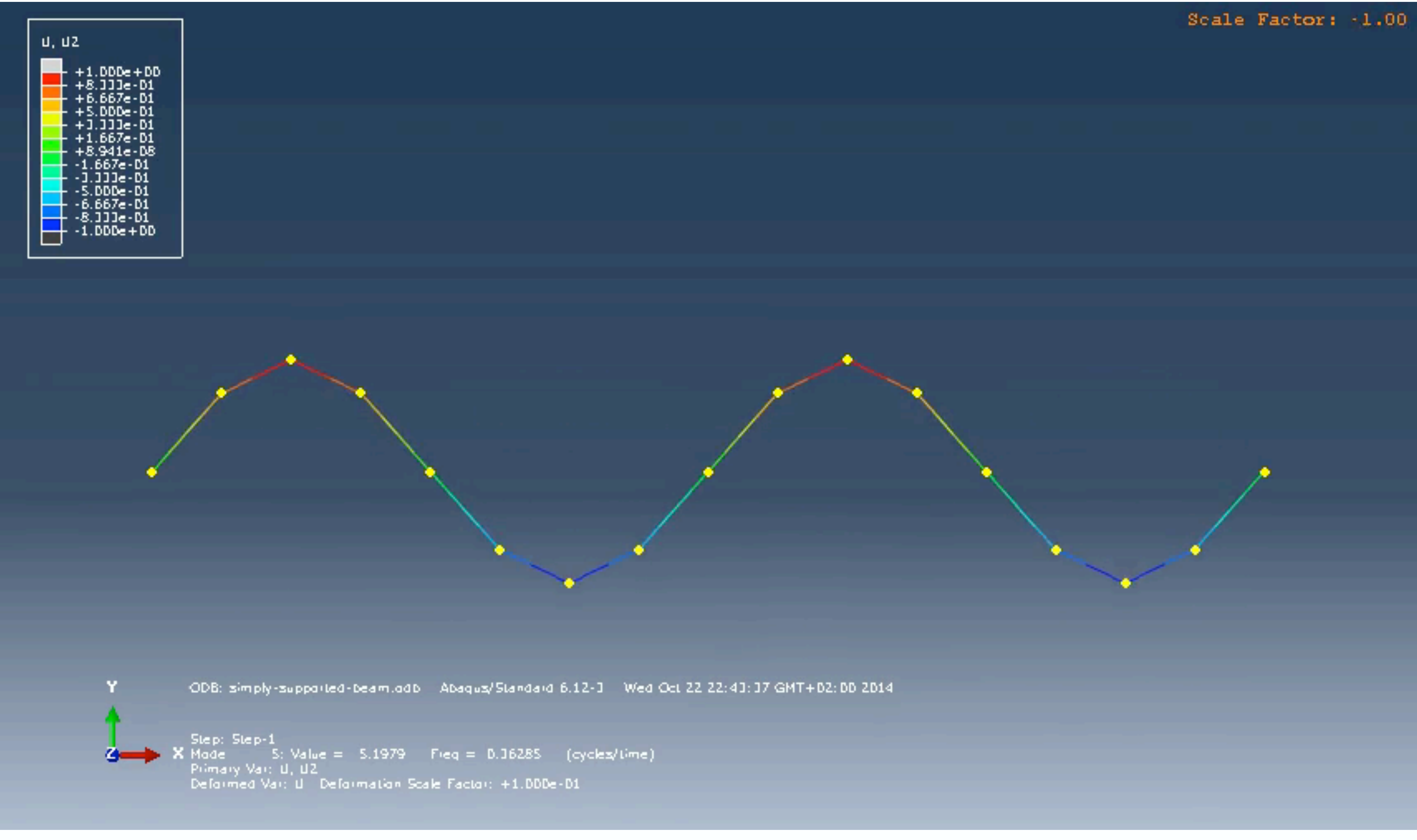
3rd BENDING EIGENMODE OF S. S. EULER BEAM



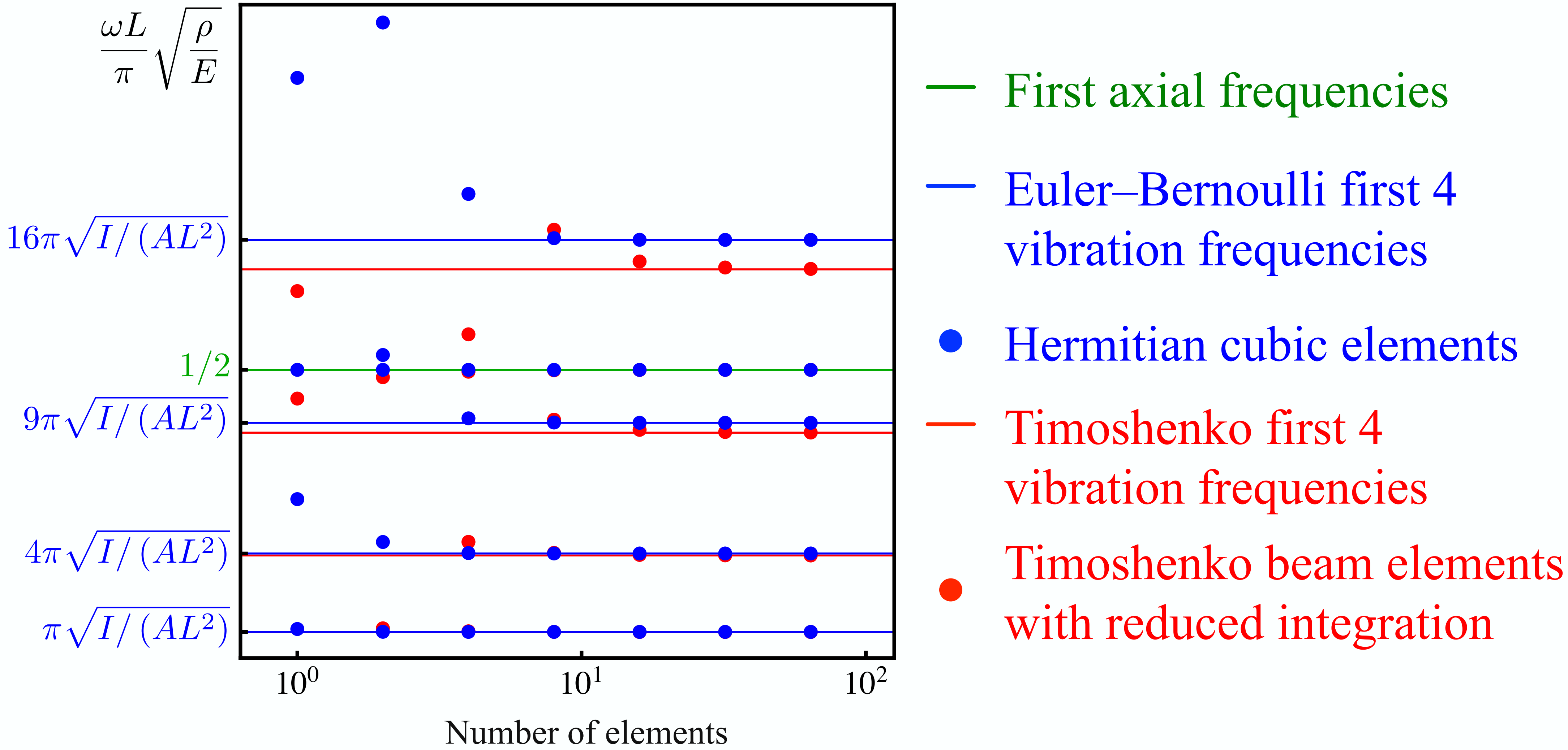
1st AXIAL EIGENMODE OF S. S. EULER BEAM



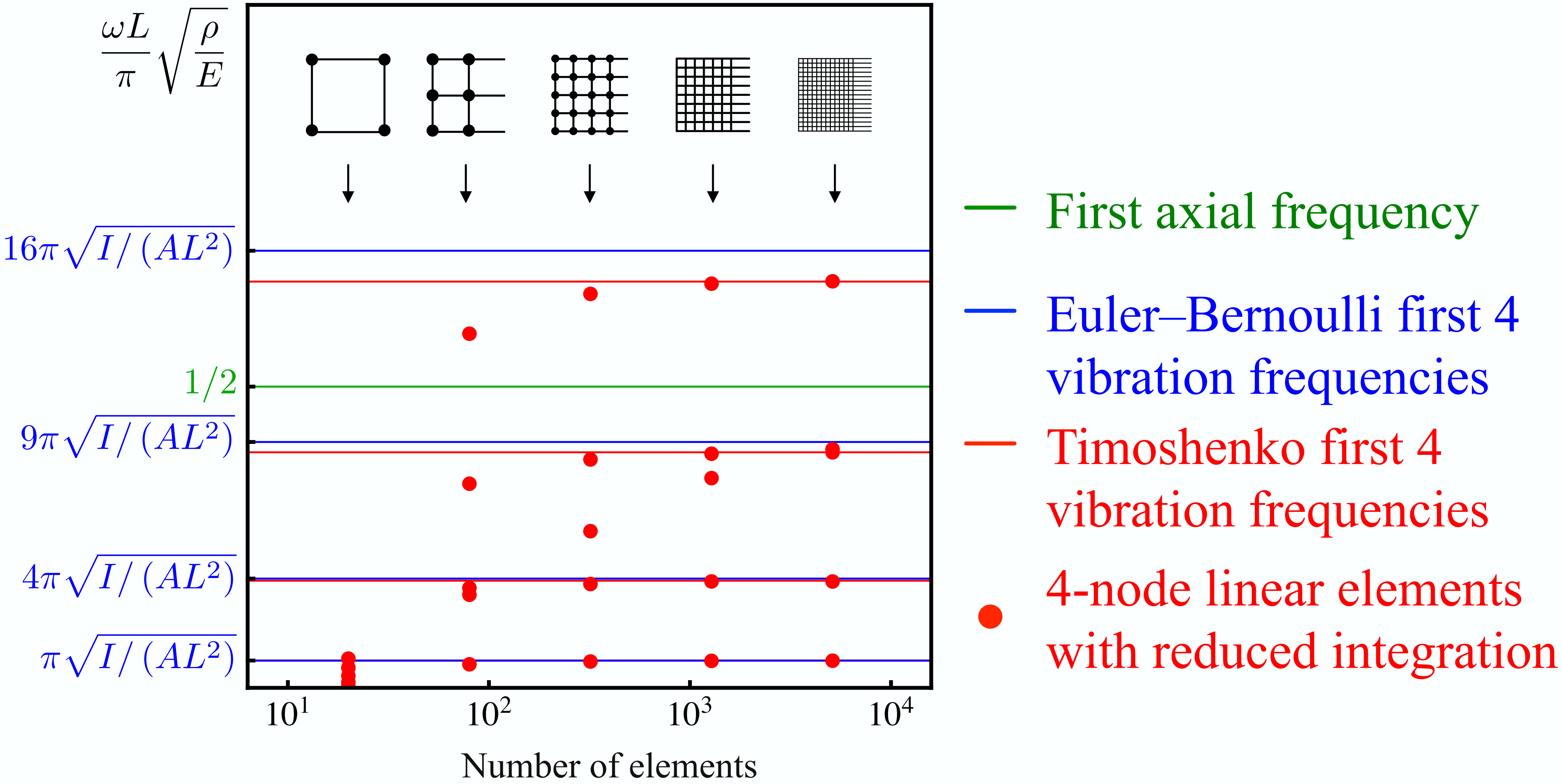
4^{th} BENDING EIGENMODE OF S. S. EULER BEAM

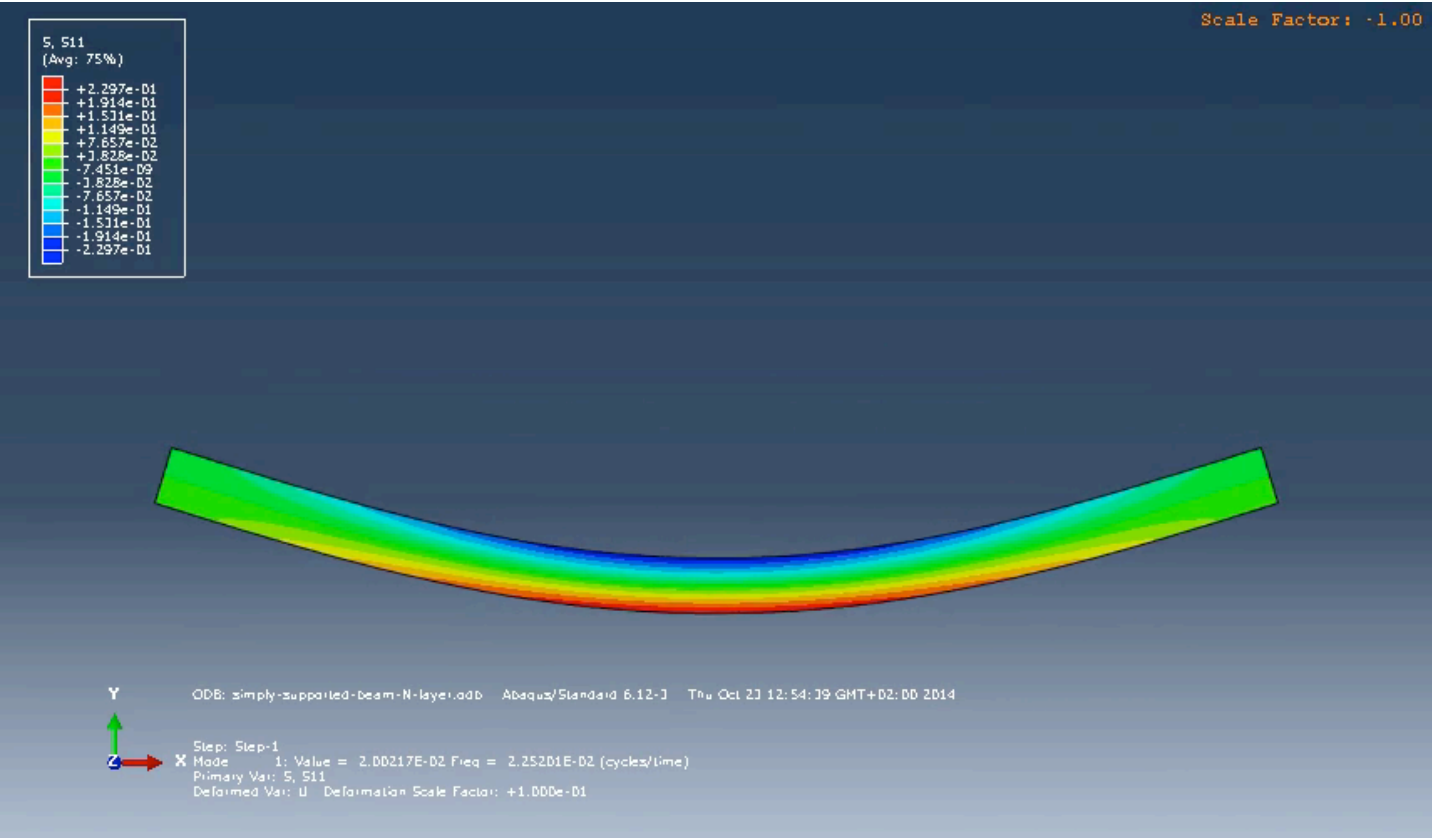


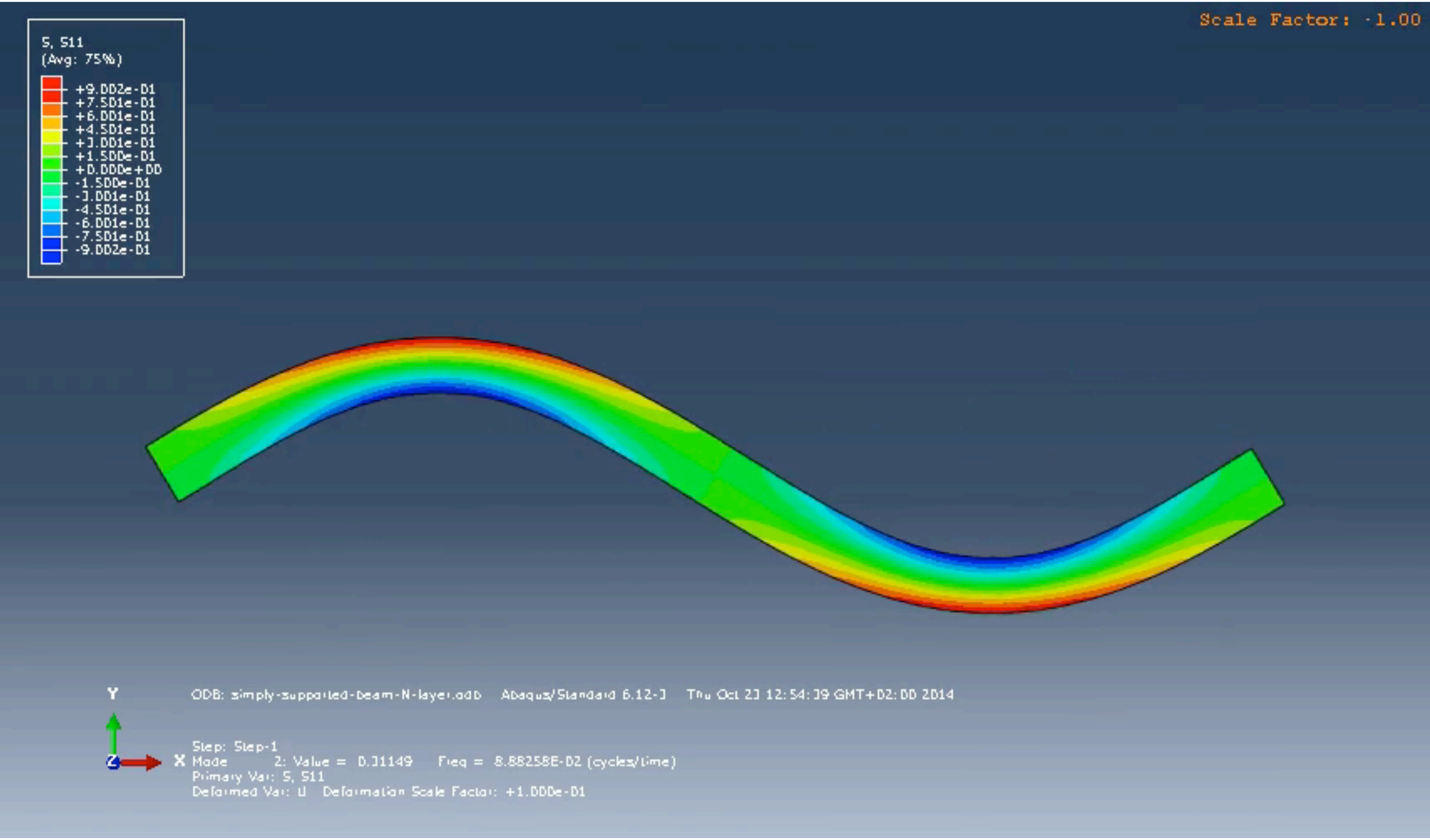
Eigenvalues of 1D thin beam – subspace method



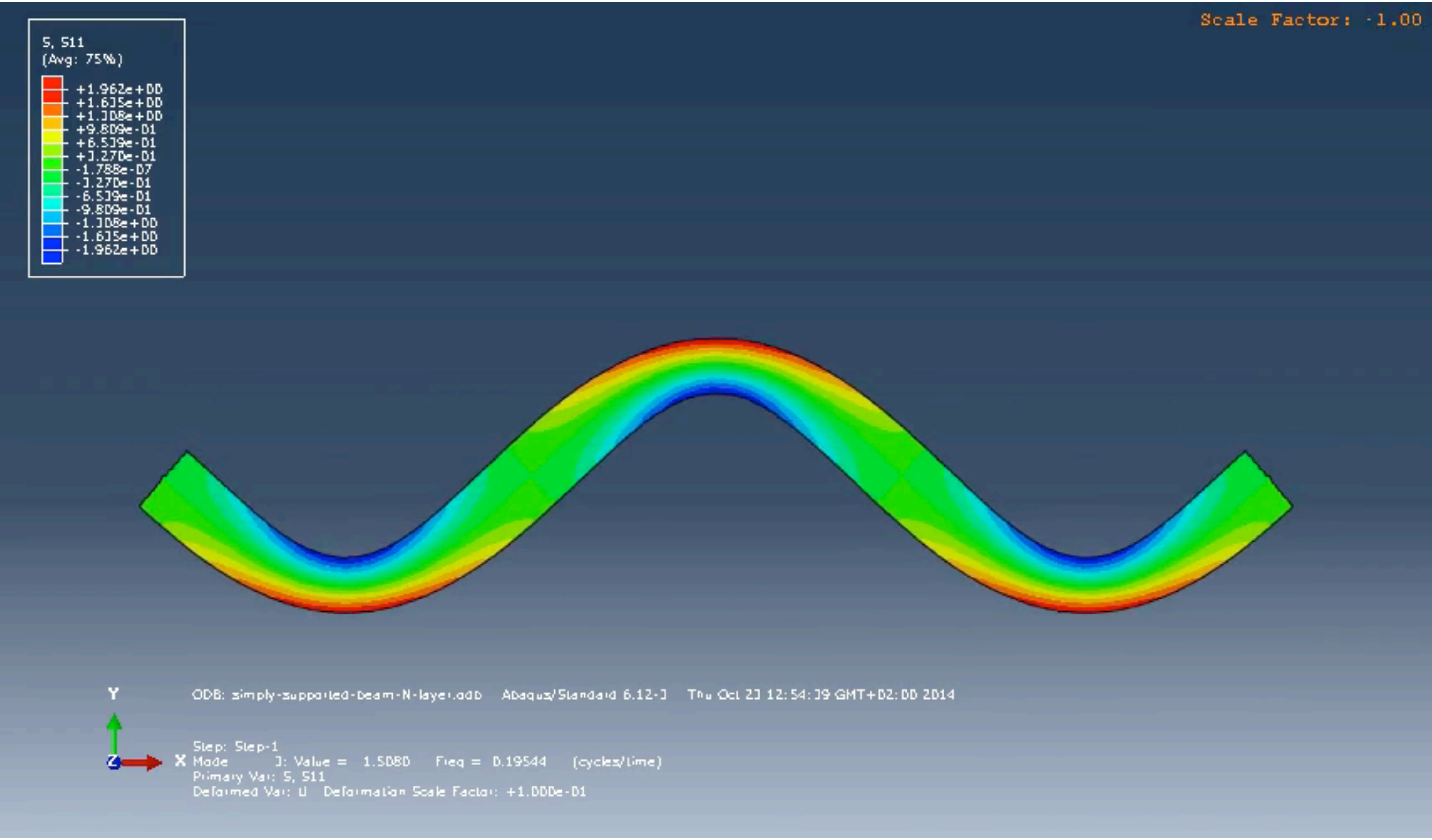
Eigenvalues of 2D FEM thin beam – subspace method




Scale Factor: -1.00
S, S11
(Avg: 75%)
+2.297e-01
+1.914e-01
+1.531e-01
+1.149e-01
+7.657e-02
+3.828e-02
-7.451e-09
-3.828e-02
-7.657e-02
-1.149e-01
-1.531e-01
-1.914e-01
-2.297e-01
Y
X
Z
ODB: simply-supported-beam-N-layer.odb Abaqus/Standard 6.12-3 Thu Oct 23 12:54:39 GMT+02:00 2014
Step: Step-1
Mode 1: Value = 2.00217E-02 Freq = 2.25201E-02 (cycles/time)
Primary Var: S, S11
Deformed Var: U Deformation Scale Factor: +1.000e-01

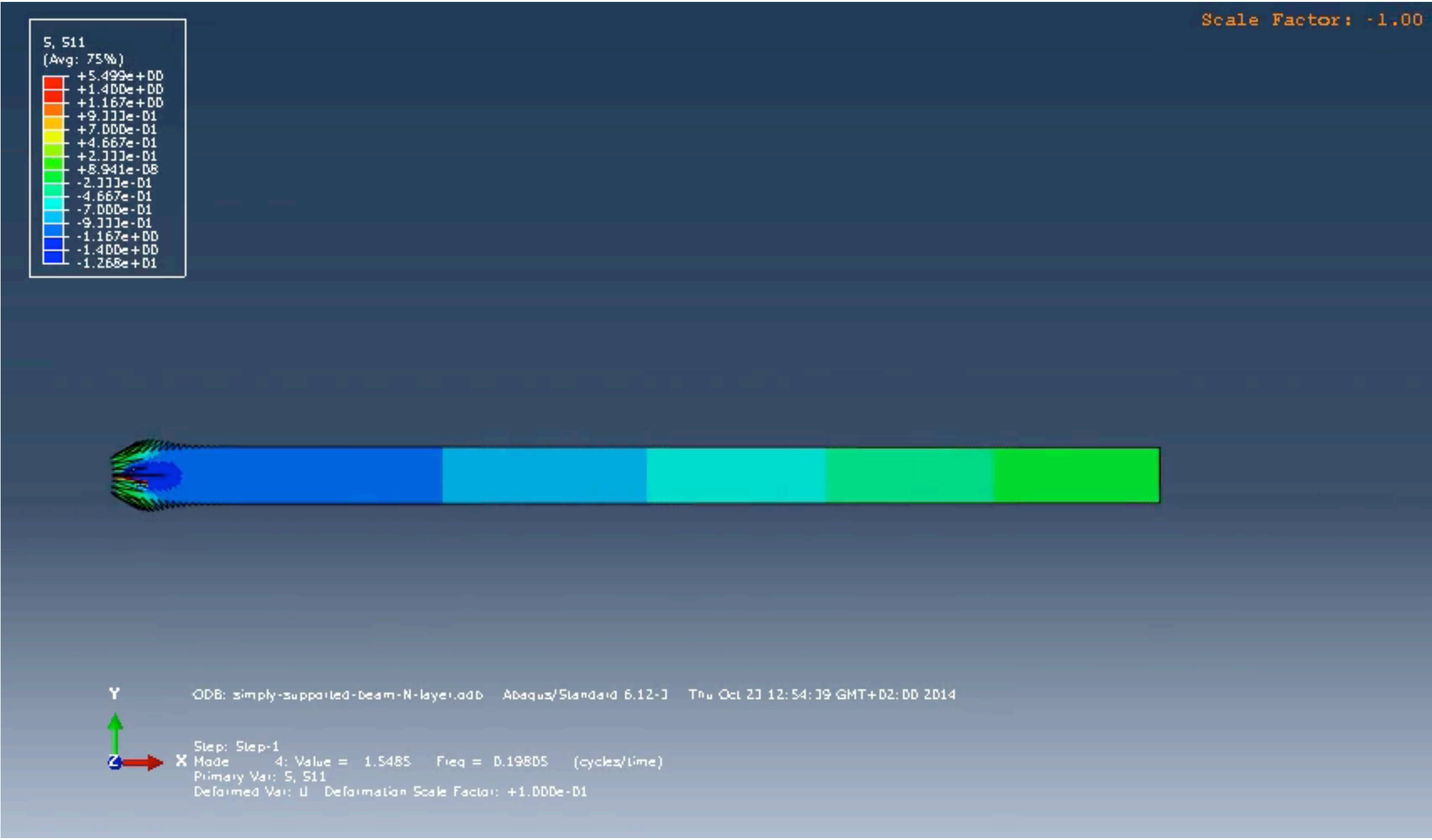


S, S11
(Avg: 75%)
+9.002e-01
+7.501e-01
+6.001e-01
+4.501e-01
+3.001e-01
+1.500e-01
+0.000e+00
-1.500e-01
-3.001e-01
-4.501e-01
-6.001e-01
-7.501e-01
-9.002e-01
Scale Factor: -1.00
Y
X
Z
ODB: simply-supported-beam-N-layer.odb Abaqus/Standard 6.12-3 Thu Oct 23 12:54:39 GMT+02:00 2014
Step: Step-1
Mode 2: Value = 0.31149 Freq = 8.88258E-02 (cycles/time)
Primary Var: S, S11
Deformed Var: U Deformation Scale Factor: +1.000e-01

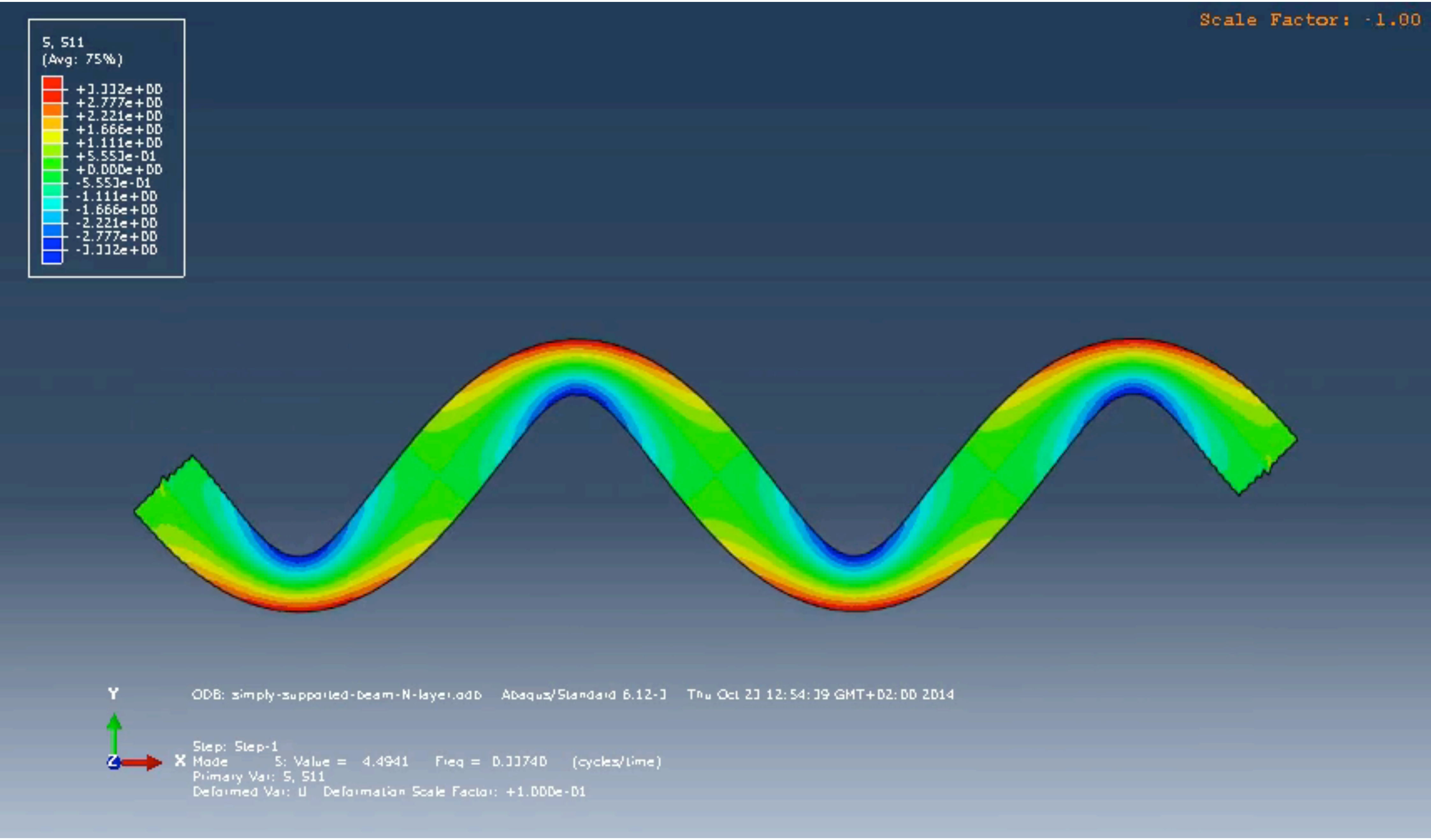


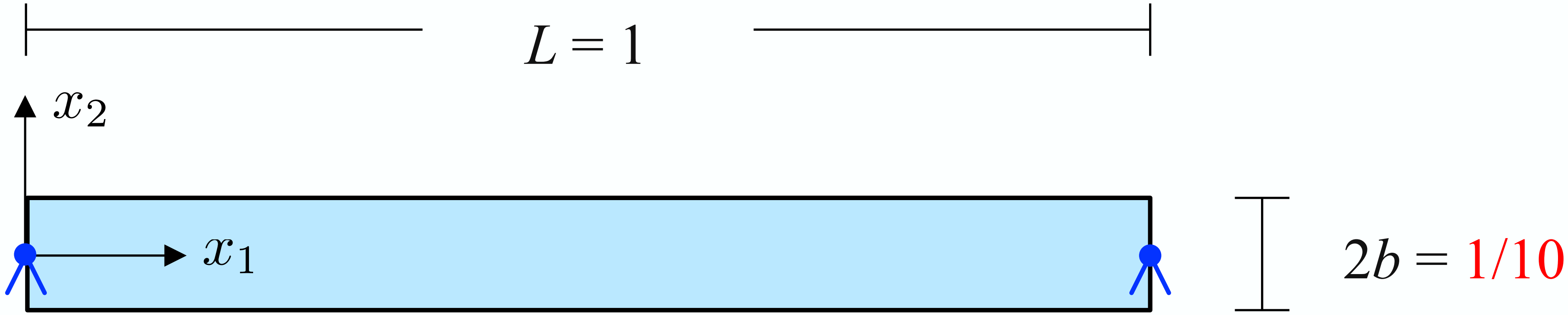
Scale Factor: -1.00
S, S11
(Avg: 75%)
+1.962e+00
+1.635e+00
+1.308e+00
+9.809e-01
+6.539e-01
+3.270e-01
-1.788e-07
-3.270e-01
-6.539e-01
-9.809e-01
-1.308e+00
-1.635e+00
-1.962e+00
Y
X
Z
ODB: simply-supported-beam-N-layer.odb Abaqus/Standard 6.12-3 Thu Oct 23 12:54:39 GMT+02:00 2014
Step: Step-1
Mode 3: Value = 1.5080 Freq = 0.19544 (cycles/time)
Primary Var: S, S11
Deformed Var: U Deformation Scale Factor: +1.000e-01

1st (AXIAL) EIGENMODE OF S. S. SLENDER BEAM (FEM)



4th (BENDING) EIGENMODE OF S. S. SLENDER BEAM (FEM)

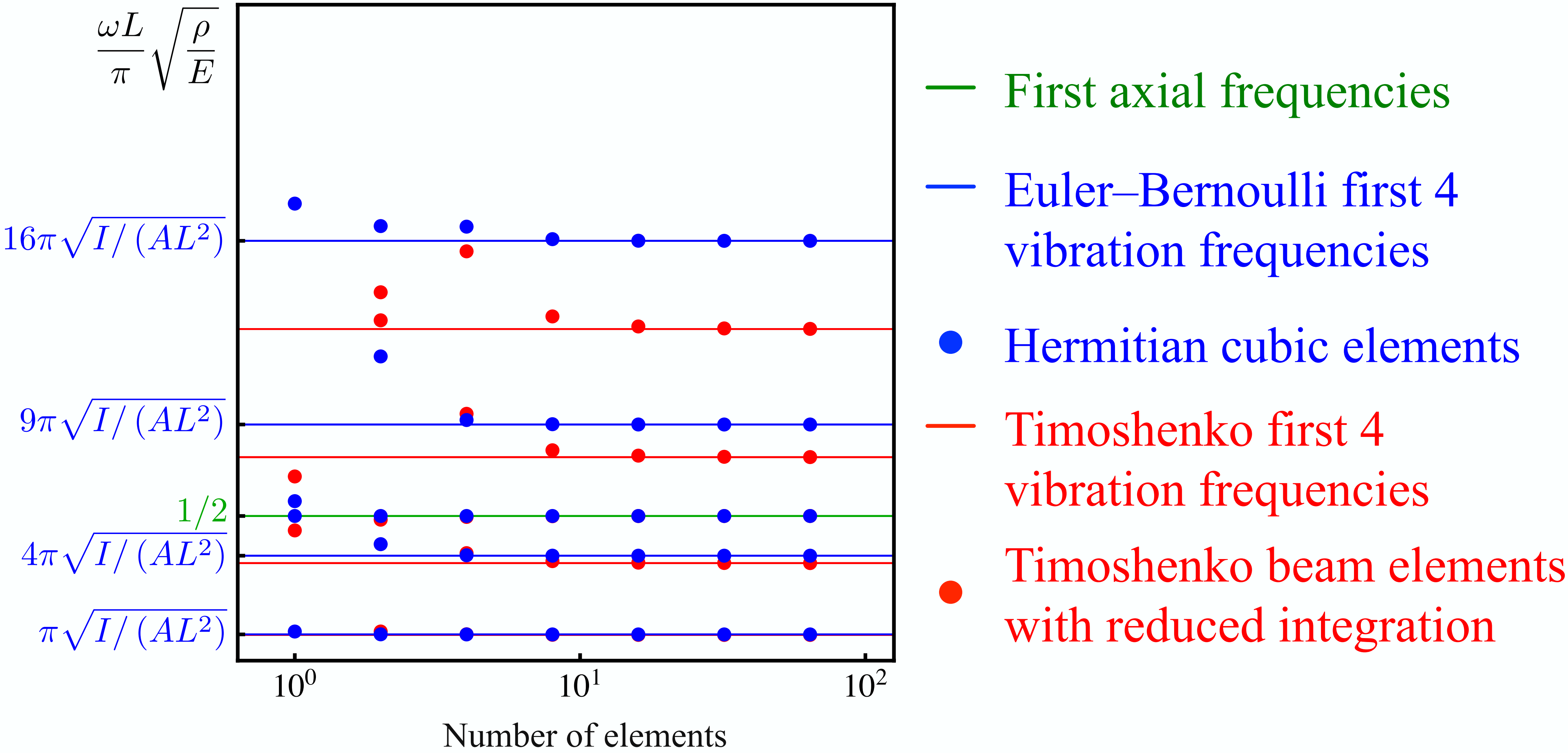




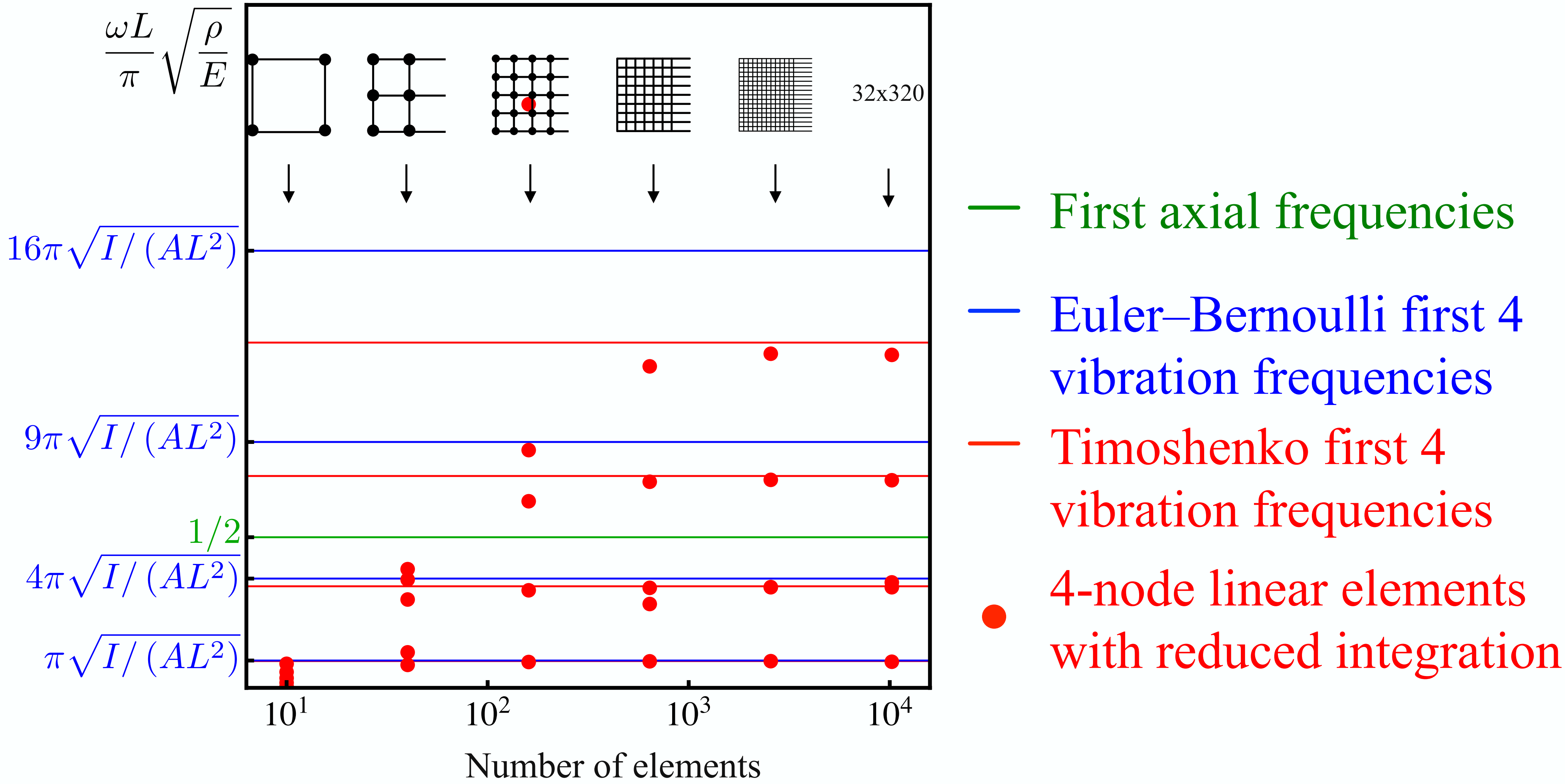
$\rho = 1, E = 1, \nu = 0.3$

$A = 2\, b\, a, I = 2/3\, b^3 a$

Eigenvalues of 1D thick beam – subspace method



Eigenvalues of 2D FEM thick beam – subspace method



EIGENVALUES & EIGENMODES OF PLATES

Kirchhoff-Love Plate Theory – Plate dimensions: L x L x 2b

$$D\left(\frac{\partial^4 u_3}{\partial x_1^4} + 2\frac{\partial^4 u_3}{\partial x_1^2 x_2^2} + \frac{\partial^4 u_3}{\partial x_2^4}\right) + 2\rho b\frac{\partial^2 u_3}{\partial t^2} = 0 \; ; \qquad D \equiv \frac{2Eb^3}{3(1-\nu^2)}$$

$$u_3(x_1, 0) = u_3(x_1, L) = u_3(0, x_2) = u_3(L, x_2) = 0 \; ; \qquad \text{essential b.c.}$$

$$D\frac{\partial^2 u_3}{\partial x_1^2}(0, x_2) = D\frac{\partial^2 u_3}{\partial x_1^2}(L, x_2) = 0 \; ; \qquad \text{natural b.c. at} : x_1 = 0, L$$

$$D\frac{\partial^2 u_3}{\partial x_2^2}(x_1, 0) = D\frac{\partial^2 u_3}{\partial x_2^2}(x_1, L) = 0 \; ; \qquad \text{natural b.c. at} : x_2 = 0, L$$

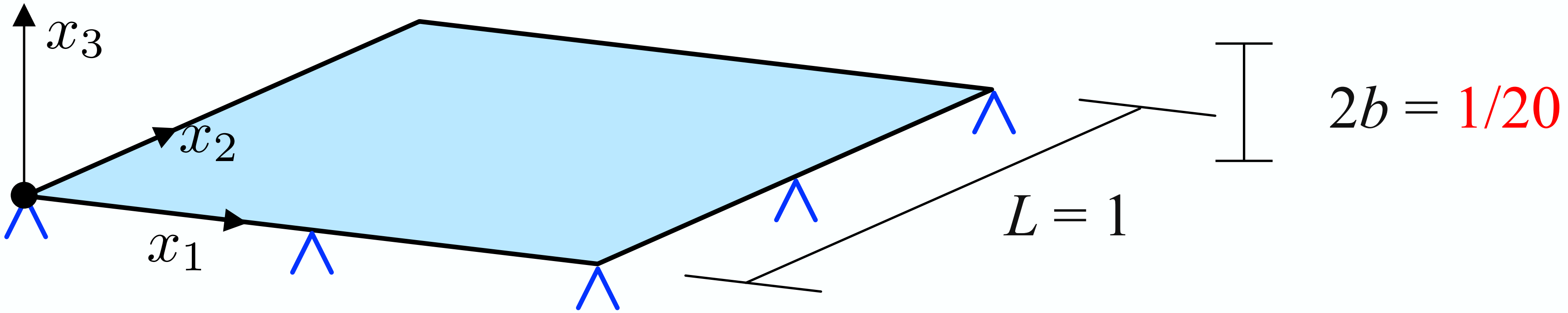
$$u_3(x_1, x_2, t) = X(x_1)Y(x_2)T(t) \; ; \qquad \text{variable separation}$$

Eigenmodes and cyclic frequencies of simply supported square Kirchhoff plate under transverse free vibrations

Eigenmode: $$u_3 = \sin\left(\frac{m\pi x_1}{L}\right)\sin\left(\frac{n\pi x_2}{L}\right)\sin\left(\omega_{mn} t\right)$$

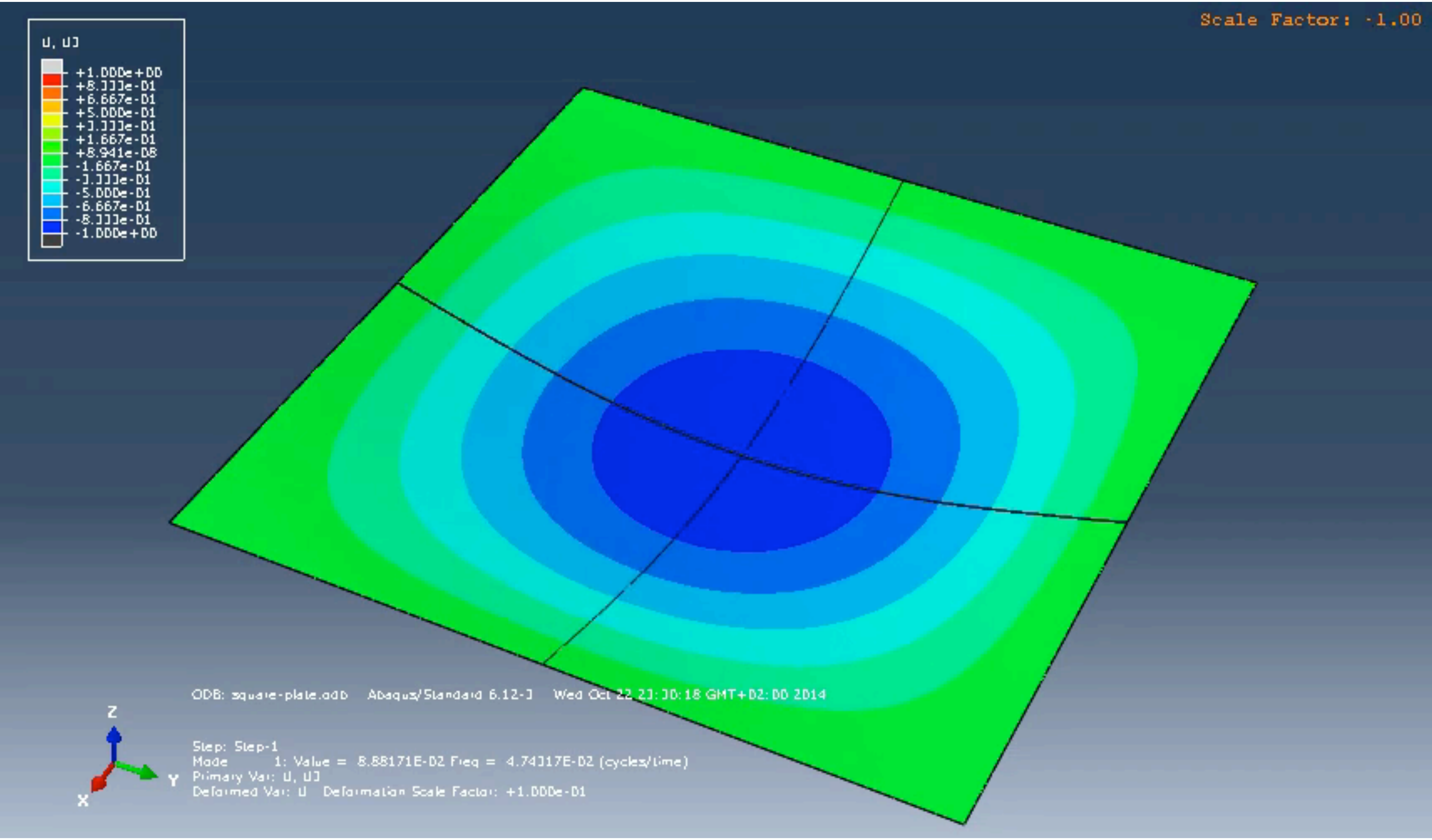
Cyclic frequencies: $$\omega_{mn} = \frac{\pi^2}{L^2}\sqrt{\frac{D}{2\rho b}}\left(m^2+n^2\right)$$

$$m = 1, 2, 3\ldots\,,\;\; n = 1, 2, 3\ldots$$

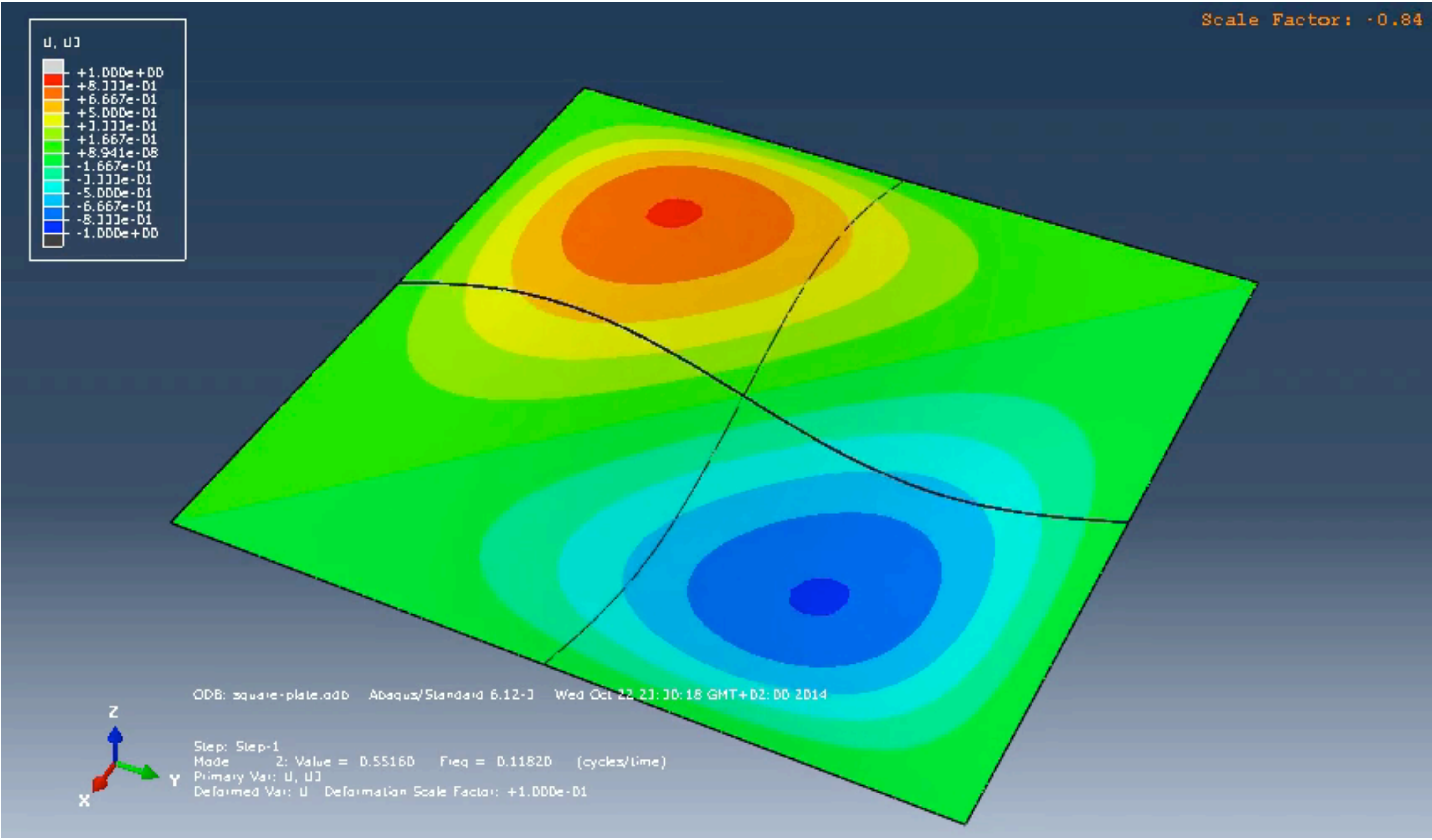


Simple support: all edges are forced to remain straight, i.e. no transverse displacement ($u_3 = 0$) while no normal rotation constraint is imposed at the ends (twisting rotation is 0) since edges remain straight)

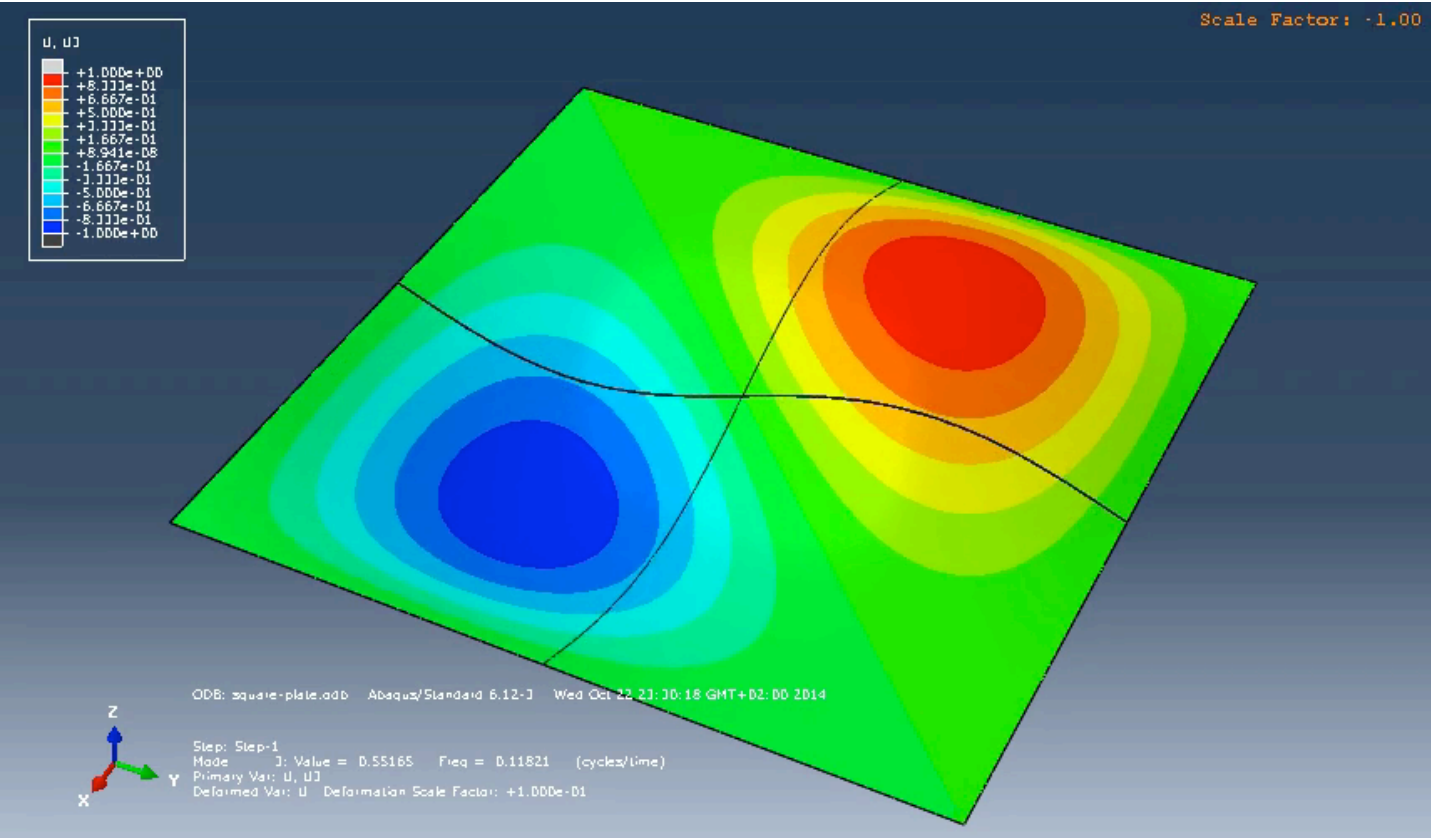
1[st] BENDING MODE OF SIMPLY SUPPORTED SQUARE PLATE



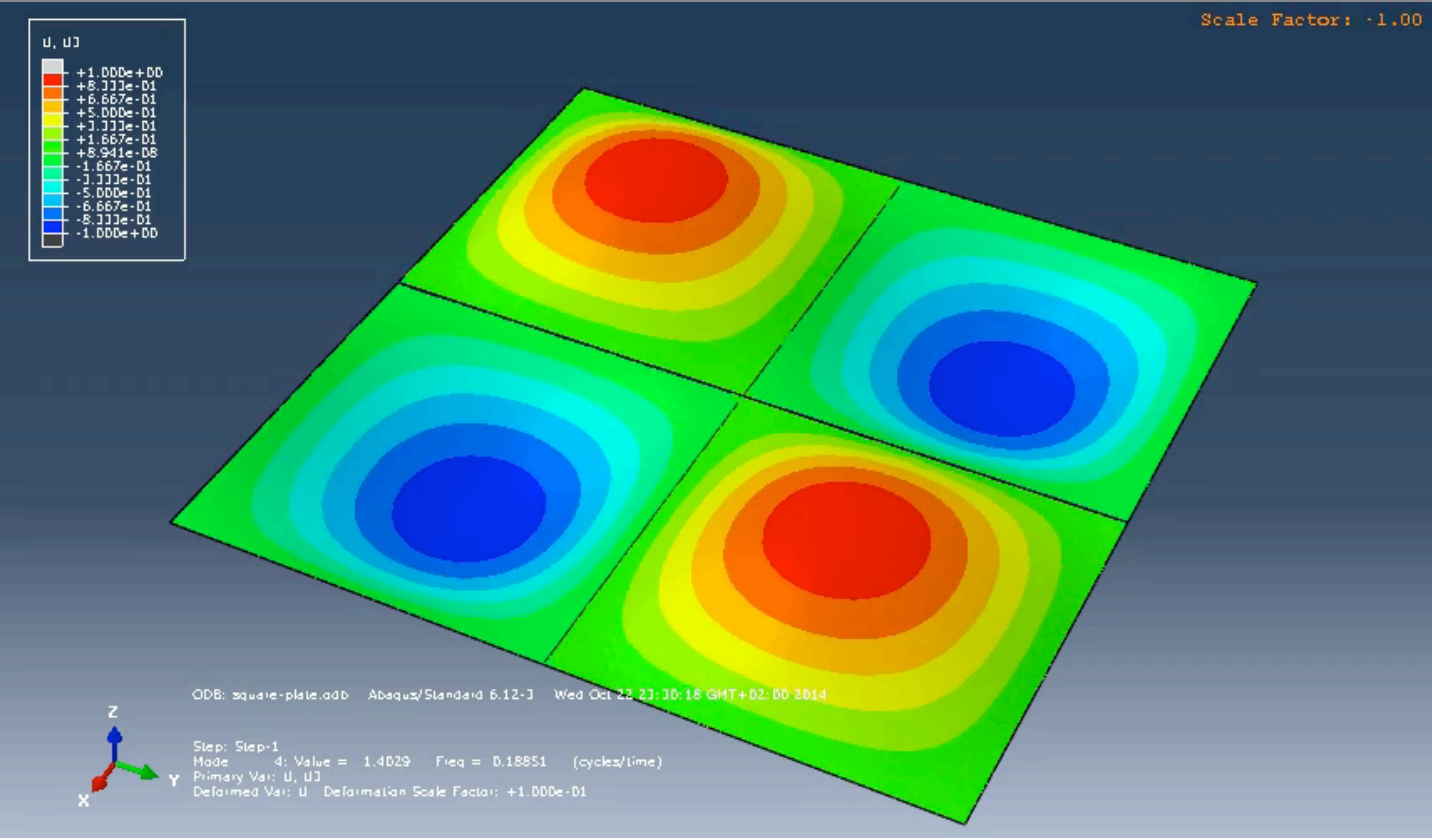
2nd BENDING MODE OF SIMPLY SUPPORTED SQUARE PLATE



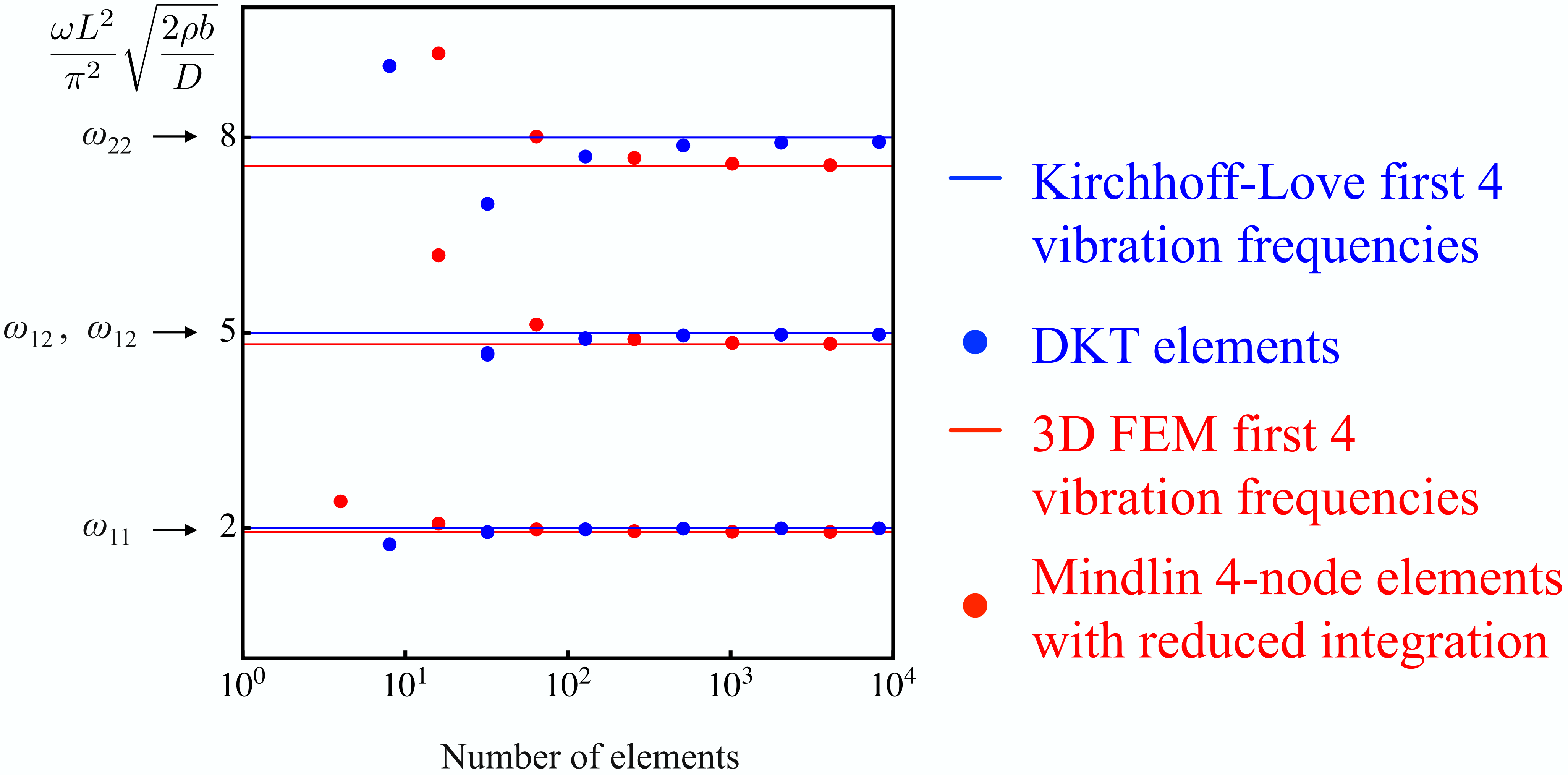
3rd BENDING MODE OF SIMPLY SUPPORTED SQUARE PLATE



4th BENDING MODE OF SIMPLY SUPPORTED SQUARE PLATE



Eigenvalues of square plate (2D plate & 3D FEM) – subspace method



SOME FINAL INFORMATION

FINAL EXAM: Wednesday, December 14, 2016 – Amphi Curie
9:00am-12:00 noon

(**NOTE**: Sample exam posted, along with your notes, at:
https://moodle.polytechnique.fr/

At your disposal if you have questions (nobody came during term…)

NEXT TERM (for those of you interested in solid mechanics)

MEC563: Stability of Solids; from Structures to Materials
(part of a core group of 3rd year courses that will give you the minimum background in this field: **MEC551**, **MEC553**, **MEC556**, **MEC557**, **MEC563**)

WHY STUDY STABILITY IN MECHANICS?

IN DESIGN WE GENERALLY ADDRESS TWO ISSUES:

- **CHECK OPERATING LOADS (STRESSES WITHIN ELASTIC LIMITS)**
- **DESIGN TO AVOID FAILURE (SAFETY AT EXTREME LOADS)**

FAILURE OF STRUCTURES FALLS INTO TWO BASIC TYPES:

- **FRACTURE (STRESS CONCENTRATION AT LOCAL FLAWS) – MEC551**
- **BUCKLING (OVERALL STRUCTURAL FAILURE DUE TO INSTABILITY) – MEC563**

REASON FOR BUCKLING INSTABILITY: NONLINEAR BEHAVIOR OF STRUCTURES

STUDY OF STABILITY IMPORTANT NOT ONLY FOR ENGINEERING STRUCTURES, BUT FOR A MUCH WIDER RANGE OF APPLICATIONS IN SOLIDS AND MATERIALS