TOPICS COVERED IN THIS LECTURE

1. PENALTY METHOD FOR 1D BEAMS – TIMOSHENKO BEAM THEORY

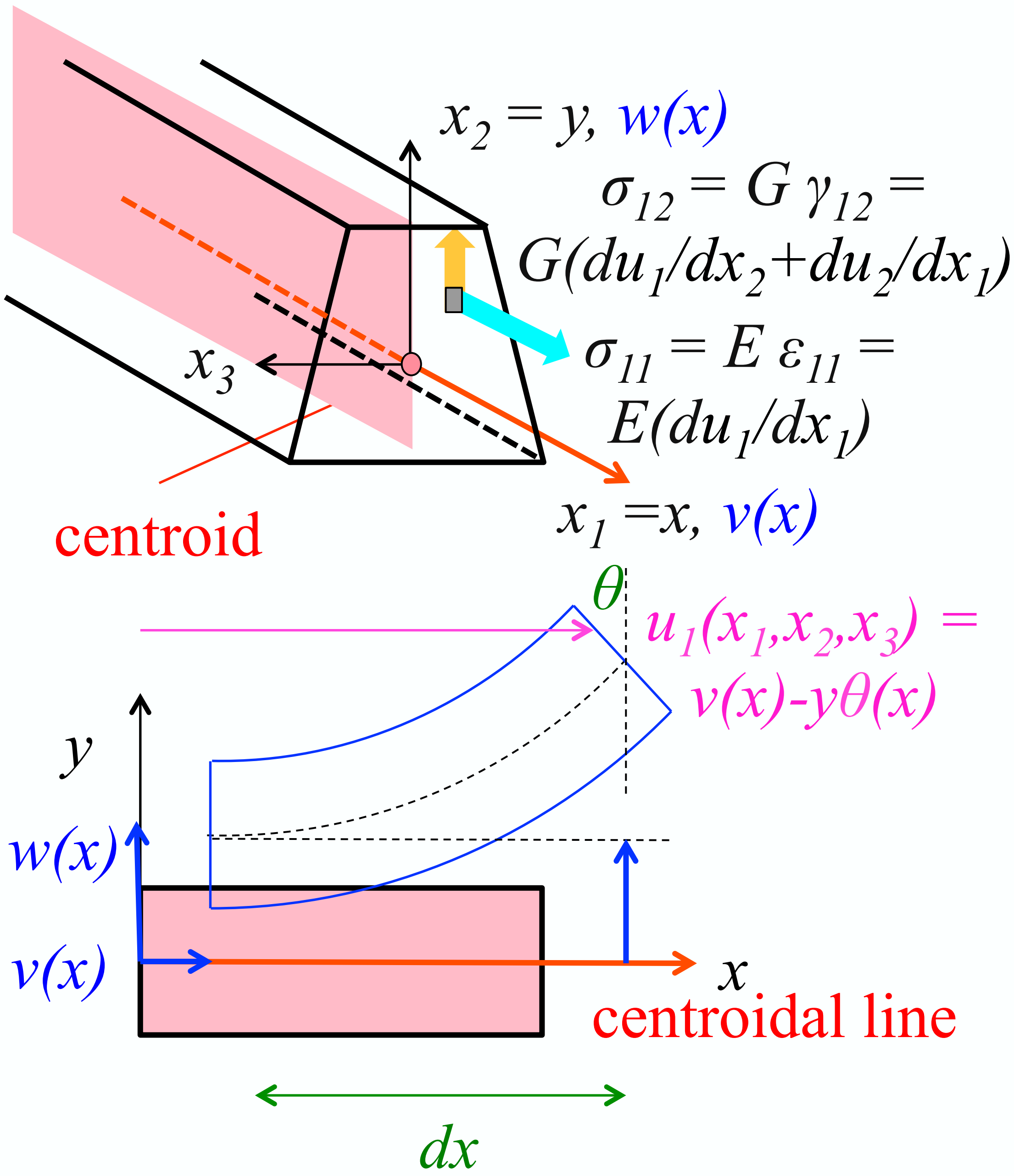
2. PENALTY METHOD FOR 2D PLATES – MINDLIN PLATE THEORY

3. PENALTY METHOD FOR INCOMPRESSIBLE LINEAR ELASTICITY

PENALTY METHODS FOR CONSTRAINED PROBLEMS

- Often in mechanics we are faced with constrained problems (exact or approximate) that require special handling…
- Constraints can come either from physics or from a mathematical relaxation of the problem. Here we use penalty type formulations to address the constraint issue
- One category of problems pertains to theories involving physically existing constraints, such as incompressibility.
- Another category pertains to theories involving energies with higher order gradients for which complicated special elements are needed! To avoid such elements, we use relaxed formulations (often motivated by physics) that again introduce constraints.

TIMOSHENKO BEAM THEORY (1D)



TIMOSHENKO BEAM THEORY

- Axial plus shear stresses (σ_{11}, σ_{12})
- Plane sections normal to centroidal line remain plane and rotate by an angle θ
- Small (infinitesimal) strain kinematics
- Linear elastic constitutive law (isotropic, can be generalized to transversely isotropic about centroidal line)

$$\mathcal{P}_{int} = \int_V \left[\frac{1}{2}\sigma_{11}\epsilon_{11} + \frac{1}{2}\sigma_{12}\gamma_{12}\right] dV$$

ADVANTAGE: THEORY USES ONLY FIRST ORDER GRADIENTS!

$$\mathcal{P}_{int} = \int_0^L \left[\int_A \frac{1}{2}E(\epsilon_{11})^2 + \frac{1}{2}G(\gamma_{12})^2\, dA\right] dx$$

$$\text{recall}: \epsilon_{11} = \frac{du_1}{dx_1} = \frac{dv}{dx} - y\frac{d\theta}{dx}$$

shear correction factor κ accounts for non-uniform shear stress distribution (κ=2/3 for rectangular section)

$$\text{recall}: \gamma_{12} = \frac{du_1}{dx_2} + \frac{du_2}{dx_1} = -\theta + \frac{dw}{dx}$$

axial energy, bending energy, shear energy

$$\mathcal{P}_{int} = \frac{1}{2}\int_0^L \left[EA\left(\frac{dv}{dx}\right)^2 + EI\left(\frac{d\theta}{dx}\right)^2 + \kappa GA\left(\frac{dw}{dx} - \theta\right)^2\right] dx$$

$$\mathcal{P}(u) = \mathcal{P}_{int} + \mathcal{P}_{ext} \ ; \quad u(x) \equiv (w(x), \theta(x)) \ ; \quad \text{(ignore axial displacement} \ - \ \text{decoupling)}$$

$$\mathcal{P}_{int} = \frac{1}{2} \int_0^L \left[EI \left(\frac{d\theta}{dx} \right)^2 + \kappa GA \left(\frac{dw}{dx} - \theta \right)^2 \right] dx$$

$$\mathcal{P}_{ext} = - \int_0^L [pw] \ dx - M(L)\theta(L) - V(L)w(L)$$

$$0 = \mathcal{P}_{,u}\delta u = \int_0^L \left[EI \frac{d\theta}{dx}\frac{d\delta\theta}{dx} + \kappa GA \left(\frac{dw}{dx} - \theta \right) \left(\frac{d\delta w}{dx} - \delta\theta \right) \right] dx$$

$$- \int_0^L [p\delta w] \ dx - M(L)\delta\theta(L) - V(L)\delta w(L)$$

moment equilibrium

transverse forces equilibrium

$$0 = - \int_0^L \left\{ \left[\frac{d}{dx} \left(EI \frac{d\theta}{dx} \right) + \kappa GA \left(\frac{dw}{dx} - \theta \right) \right] \delta\theta \right\} dx + \left[EI \frac{d\theta}{dx} - M \right]_L \delta\theta(L)$$

$$0 = - \int_0^L \left\{ \left[\frac{d}{dx} \left[\kappa GA \left(\frac{dw}{dx} - \theta \right) \right] + p \right] \delta w \right\} dx + \left[\kappa GA \left(\frac{dw}{dx} - \theta \right) - V \right]_L \delta w(L)$$

$$\frac{d}{dx}\left[EI\frac{d\theta}{dx}\right] + \kappa GA\left(\frac{dw}{dx} - \theta\right) = 0\,; \qquad EI\frac{d\theta}{dx} = M$$

$$\frac{d}{dx}\left[\kappa GA\left(\frac{dw}{dx} - \theta\right)\right] = 0\,; \qquad \kappa GA\left(\frac{dw}{dx} - \theta\right) = V$$

$$w(0) = 0\,, \quad \theta(0) = 0\,; \quad M(L) = 0\,, \quad V(L) = 1$$

Timoshenko term O(h^{-2}) negligible; Bernoulli term O(h^{-4}) dominant:

$$w(x) = \boxed{\frac{Fx}{\kappa GA}} + \boxed{\frac{F}{EI}\left[L\frac{x^2}{2} - \frac{x^3}{6}\right]}\,; \quad \boxed{\theta(x) = \frac{F}{EI}\left[Lx - \frac{x^2}{2}\right]}$$

Airy stress function has correct axial, shear forces and moment at ends

$$\phi(x_1,x_2) = \frac{h^2}{4}\frac{F}{L}\frac{x_2}{h}\left(\left(\frac{x_1}{L}-1\right)-3\frac{x_1}{L}\left(\frac{x_2}{h}\right)^2\right)\left(\frac{L}{h}\right)^2$$

Left: *(x_1=0)*

$N = 0$

$V = 1$

$M = L$

$$\sigma_{11} = \frac{\partial^2\phi}{\partial x_2 \partial x_2} = \frac{3}{2}\frac{x_2}{h}\frac{F}{L}\left(\frac{x_1}{L}-1\right)\left(\frac{L}{h}\right)^2$$

$$\sigma_{22} = \frac{\partial^2\phi}{\partial x_1 \partial x_1} = 0$$

$$\sigma_{12} = -\frac{\partial^2\phi}{\partial x_1 \partial x_2} = -\frac{3}{4}\frac{F}{L}\left(\left(\frac{x_2}{h}\right)^2-1\right)\frac{L}{h}$$

Right: *(x_1=L)*

$N = 0$

$V = 1$

$M = 0$

Exact (2D) plane stress solution of cantilever beam problem:

$$u_1 = \frac{F}{E}\left(\frac{3\eta^2}{4}(\chi_1 - 2)\chi_1\chi_2 + \boxed{\frac{2+\nu}{4}\chi_2\left(1-\chi_2^2\right)}\right)$$

2D theory correction

$$u_2 = \frac{F}{E}\left[\frac{\eta^3}{4}(3-\chi_1)\chi_1^2 + \boxed{\eta\left(1+\frac{5}{4}\nu\right)\chi_1} + \boxed{3\eta\nu(1-\chi_1)\chi_2^2}\right]$$

Euler–Bernoulli Beam Theory (1D):

$$u_2 = \frac{F}{E}\left(\frac{\eta^3}{4}(3-\chi_1)\chi_1^2\right)$$

Timoshenko correction

$\chi_1 = x_1/L$

$\chi_2 = x_2/h$

Timoshenko Beam Theory (1D):

$$u_2 = \frac{F}{E}\left(\frac{\eta^3}{4}(3-\chi_1)\chi_1^2 + \boxed{\eta\frac{1+\nu}{\kappa}\chi_1}\right)$$

$\eta = L/h >> 1$

slenderness

WHY TIMOSHENKO BEAM METHOD WORKS?

$$\mathcal{P} = \frac{EI}{2}\int_0^L \left[\left(\frac{d\theta}{dx}\right)^2 + \boxed{\frac{\kappa GA}{EI}\left(\frac{dw}{dx} - \theta\right)^2}\right] dx + \mathcal{P}_{ext}$$

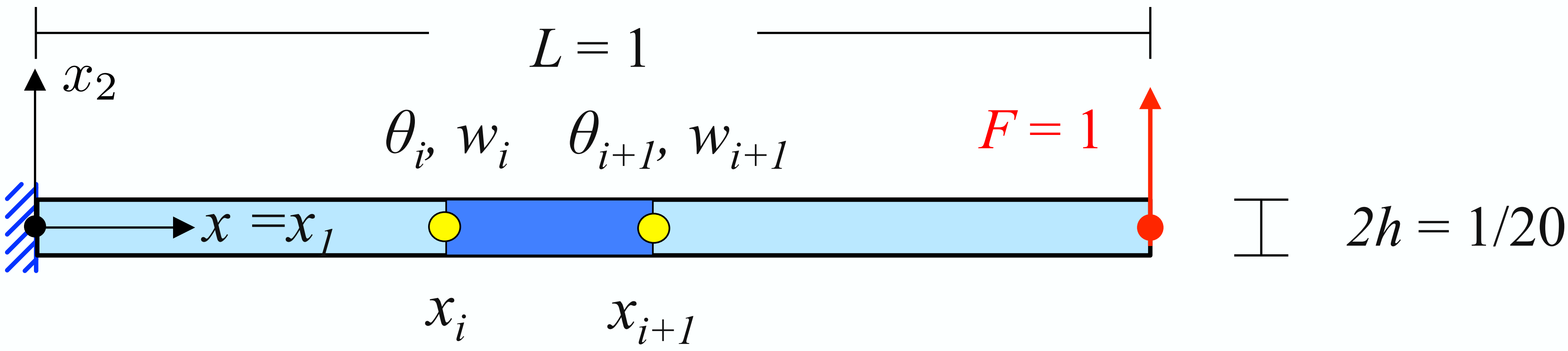
In minimizing the potential energy, since the coefficient of the shear contribution is large O(h^{-2}) the shear strain term has to be small to compensate, i.e. $\gamma = dw/dx - \theta$ must be of O(h^2) and hence $dw/dx \rightarrow \theta$

Methods in mechanics where we enforce a constraint by adding the constraint squared multiplied by a large number are termed penalty methods.

Their advantage is that they keep a positive internal energy and do not require the introduction of additional variables (i.e. Lagrange multipliers).

Where is the catch? Special care is needed to avoid locking phenomena!

FEM FOR TIMOSHENKO BEAM THEORY (1D)



Each element has 2 nodes and each node i has 2 d.o.f. (θ_i, w_i)

Since only 1st order gradients of the two independent variables $(\theta(x), w(x))$ appear, C^0 continuous shape functions are adequate; we employ isoparametric interpolation with piecewise linear shape functions as the simplest possible element in Timoshenko beam theory

$$w(\xi) = [N_1(\xi),\ 0,\ N_2(\xi),\ 0]\,\mathbf{q}_e = \mathbf{N}_w \mathbf{q}_e$$

$$\theta(\xi) = [0,\ N_1(\xi),\ 0,\ N_2(\xi)]\,\mathbf{q}_e = \mathbf{N}_\theta \mathbf{q}_e$$

$$x(\xi) = N_1(\xi) x_1 + N_2(\xi) x_2$$

isoparametric, linear interpolation!

$$\mathbf{q}_e^T = [q_1,\ q_2,\ q_3,\ q_4] = [w_1,\ \theta_1,\ w_2,\ \theta_2]$$

$$\frac{d\theta}{dx}(\xi) = \mathbf{B}_\theta \mathbf{q}_e\ , \qquad \frac{dw}{dx}(\xi) - \theta(\xi) = \mathbf{B}_\gamma \mathbf{q}_e$$

different orders of magnitude terms

$$\mathcal{P}^e_{int} = \frac{1}{2}\mathbf{q}_e^T \left[\int_{-1}^{+1} \underbrace{\left(EI \mathbf{B}_\theta^T \mathbf{B}_\theta \right.}_{O(h^4)} + \underbrace{\left. \kappa G A \mathbf{B}_\gamma^T \mathbf{B}_\gamma \right)}_{O(h^2)} \frac{l_e}{2}\, d\xi \right] \mathbf{q}_e$$

$$\mathcal{P}^e_{ext} = \int_{l_e} p\ w\ dx = \mathbf{q}_e^T \left[\int_{-1}^{+1} \mathbf{N}_w^T p\ \frac{l_e}{2}\, d\xi \right]$$

$$\left[\mathbf{K}_\theta + \frac{1}{\zeta^2}\mathbf{K}_\gamma\right]\mathbf{Q} = \mathbf{F}; \qquad \zeta \equiv \frac{h}{L} \ll 1 \text{ (aspect ratio)}$$

$$\left[\zeta^2\mathbf{K}_\theta + \mathbf{K}_\gamma\right]\mathbf{Q} = \zeta^2\mathbf{F} \implies \mathbf{Q} = \zeta^2\mathbf{K}_\gamma^{-1}\mathbf{F} + \ldots$$

$$\det[\mathbf{K}_\gamma] \neq 0 \implies \text{locking } (\mathbf{Q} \approx \mathbf{0})$$

$$\det[\mathbf{K}_\gamma] = 0 \implies \text{WORKS}$$

$$\det[\mathbf{K}_\gamma] = 0 \implies \text{underintegration!}$$

NOTE ON HOW TO AVOID LOCKING IN FEM

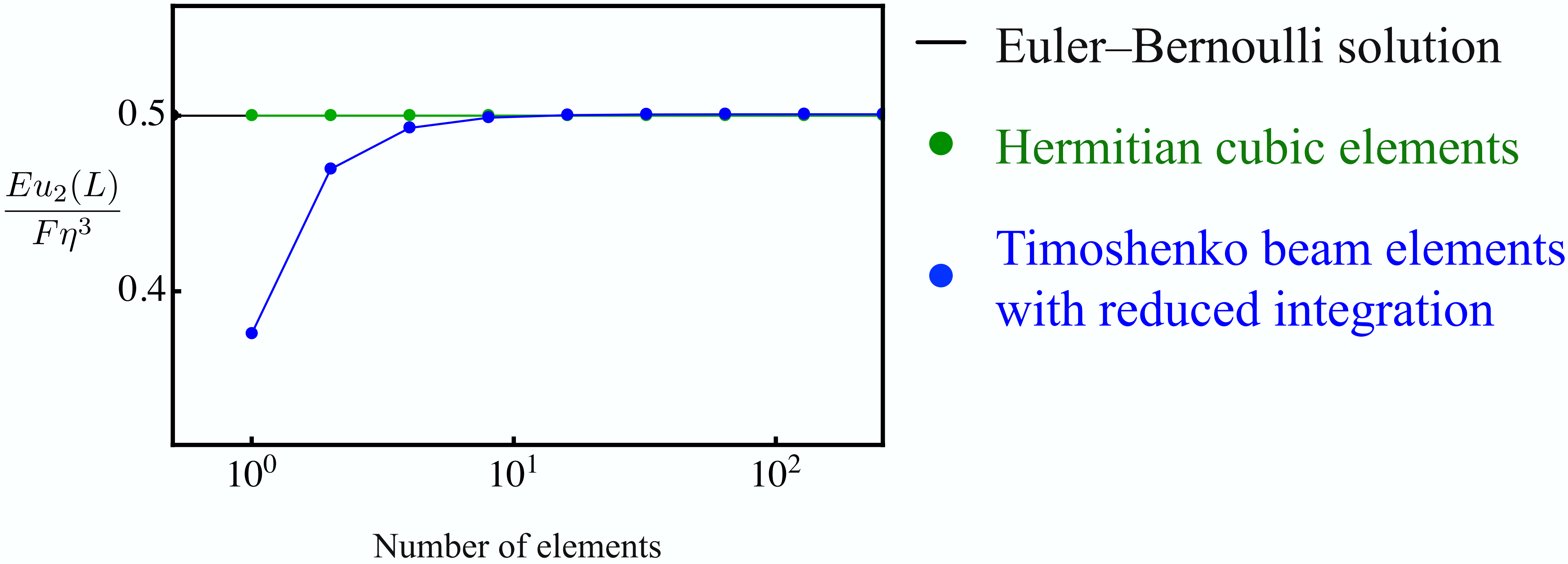
Since each constraint $f(w,\theta) = 0$ becomes an integrand penalty term of the form: $\zeta^{-1}[f(w,\theta)]^2$, every integration point adds to the stiffness matrix a rank-one term $\mathbf{Q}^T[\zeta^{-1}\mathbf{B}^T\mathbf{B}]\mathbf{Q}$.

Each numerical integration point increases the rank of the stiffness matrix $\mathbf{K}_\gamma$ by one (it corresponds to one constraint).

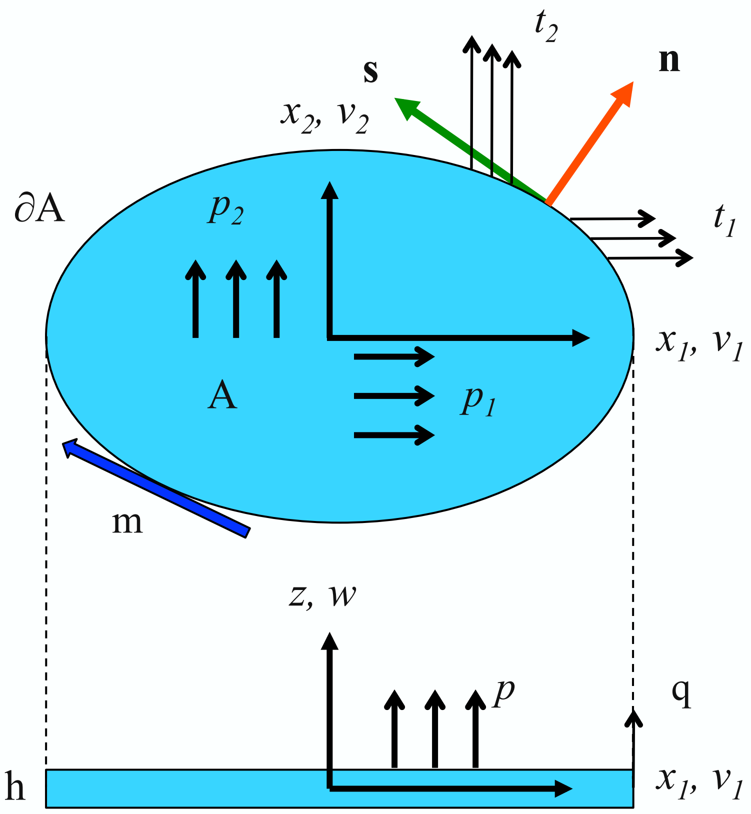
We need less constraints than available d.o.f. for $\mathbf{K}_\gamma$ and for most cases we must underintegrate to obtain a singular $\mathbf{K}_\gamma$

To check if your FEM discretization avoids locking, make sure that the total number of d.o.f. for the structure are more than the total number of numerical integration points for the constraint part of the energy

Shear locking in 1D beam bending



MINDLIN PLATE THEORY (2D)



MINDLIN PLATE THEORY:

- Plane stress state (only $\sigma_{\alpha\beta}$ – $\alpha, \beta = 1,2$)
- Cross-sections normal to the x_i axis rotate by small angle θ_i while mid-plane points move by (v_1, v_2, w)
- Small (infinitesimal) strain kinematics
- Linear elastic constitutive law (isotropic, can be generalized to transversely isotropic about normal direction)
- Reduces to Timoshenko beam theory for loading that is independent on x_2 (or x_1)

Note different order of magnitude terms

$$\mathcal{P}_{int}^{bend}(w,\theta_1,\theta_2) = \frac{1}{2}\int_A \left[\frac{E}{1-\nu^2}\frac{h^3}{12}K_{\alpha\beta}K_{\alpha\beta} + G\chi h\Gamma_\alpha\Gamma_\alpha\right] dA$$

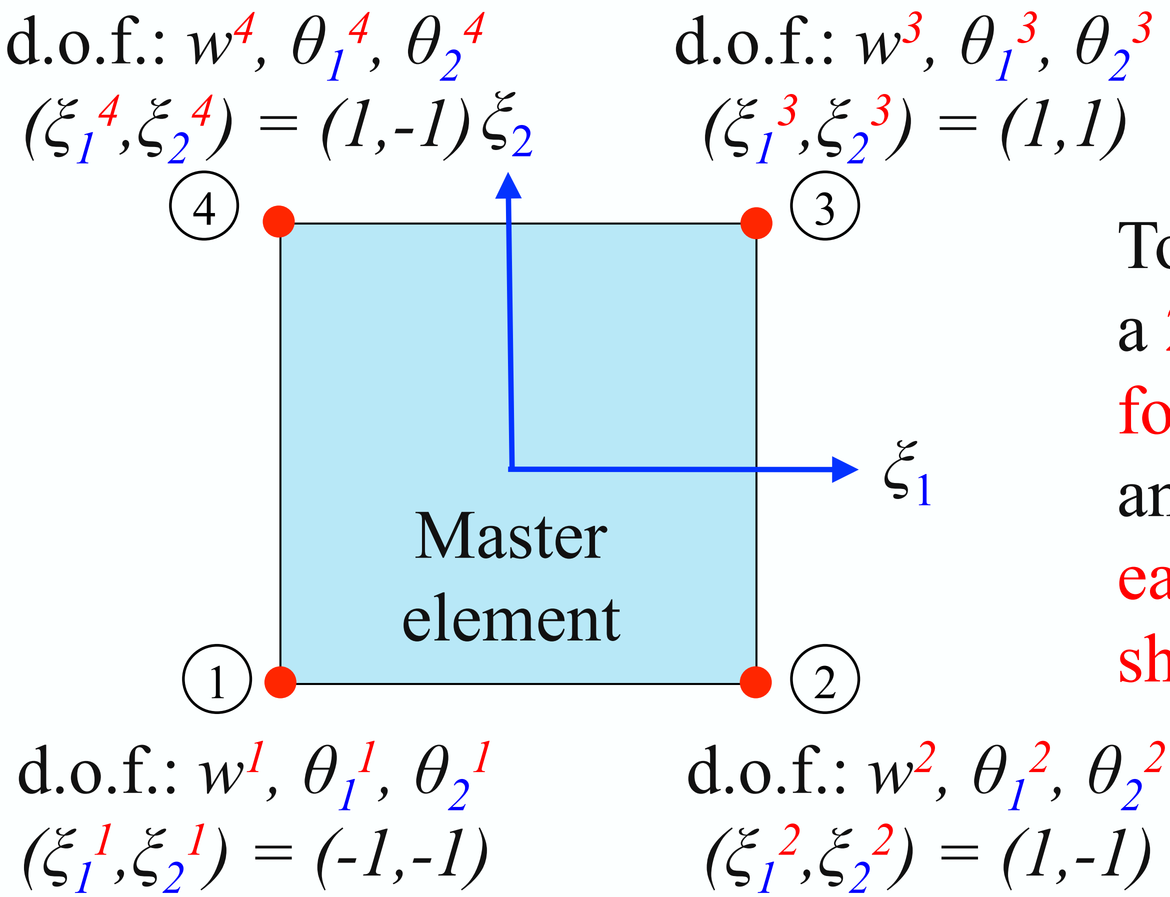
$$K_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial\theta_\alpha}{\partial x_\beta} + \frac{\partial\theta_\beta}{\partial x_\alpha}\right) \quad \text{curvature}$$

$$\Gamma_\alpha = \frac{\partial w}{\partial x_\alpha} - \theta_\alpha \quad \text{transverse shear strain}$$

Transverse shear energy is added; this is the penalty term that enforces the slope-rotation relation when *h* → *0*

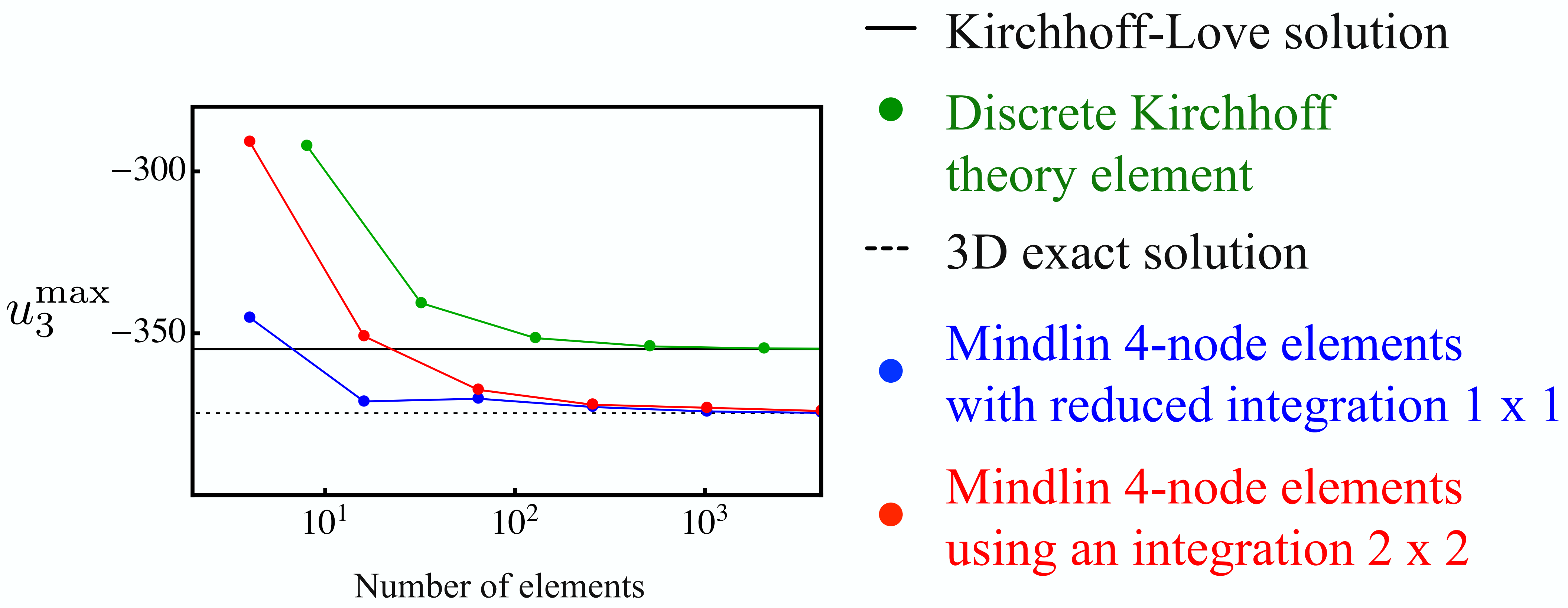
NOTE energy involves 1st order gradients of d.o.f.

Quad Mindlin theory elements: use bilinear interpolation for each one of the unknown fields: $w(x_1, x_2)$, $\theta_1(x_1, x_2)$, $\theta_2(x_1, x_2)$

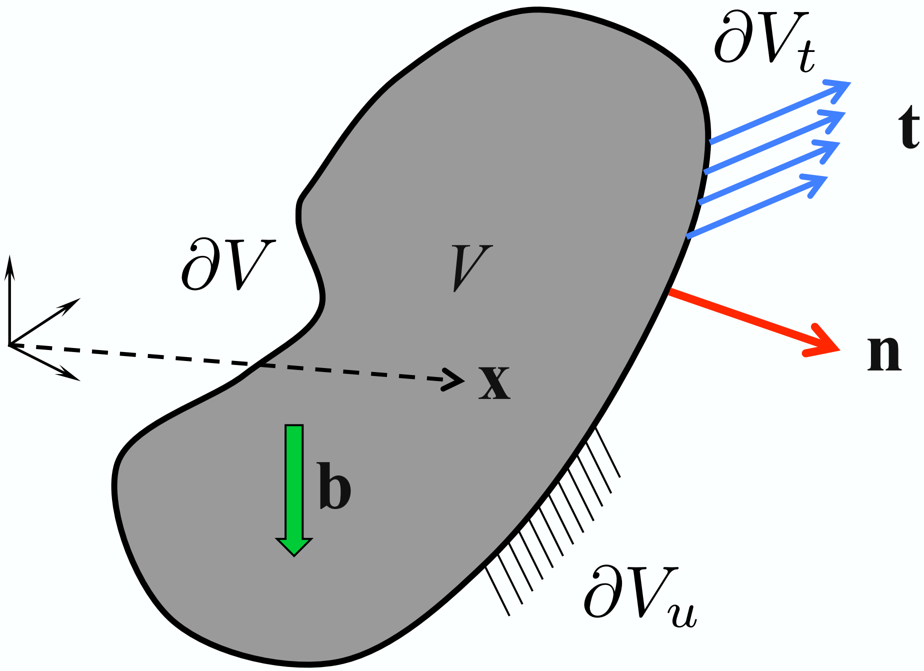


To avoid locking phenomena, use a 2x2 Gauss integration scheme for the bending part of the energy and 1 Gauss integration point for each one of the two transverse shear energy terms

Shear locking in 2D plate bending



INCOMPRESSIBLE LINEAR ELASTICITY (2D OR 3D)



Solid occupies domain: V

Domain boundary: ∂V

Body forces: $\mathbf{b}$

Surface traction: $\mathbf{t}$

Surface normal (outward): $\mathbf{n}$

Traction prescribed on: ∂V_t

Displacement prescribed on: ∂V_u

Position vector: $\mathbf{x}$

Energy density: $W(\boldsymbol{\varepsilon})$

Stress-strain: $\sigma_{ij} = \dfrac{\partial W}{\partial \epsilon_{ij}}$

(general nonlinear elastic material)

$$\mathcal{P} = \int_V W(\epsilon_{ij})\, dV - \int_V b_i u_i \, dV - \int_{\partial V_t} t_i u_i \, dS$$

$$W = \mu \epsilon_{ij}\epsilon_{ij} + \boxed{\frac{\lambda}{2}(\epsilon_{kk})^2}$$

← bulk component of energy

$$\mu = \frac{E}{2(1+\nu)} = G\,, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \;;\; \text{Lamé Constants}$$

$$\nu \to \frac{1}{2} \;;\; \text{incompressibility}$$

To model incompressible solids take: $\nu = 0.5 - \zeta$ ($\zeta << 1$) and use reduced integration for the bulk component of energy