TOPICS COVERED IN THIS LECTURE

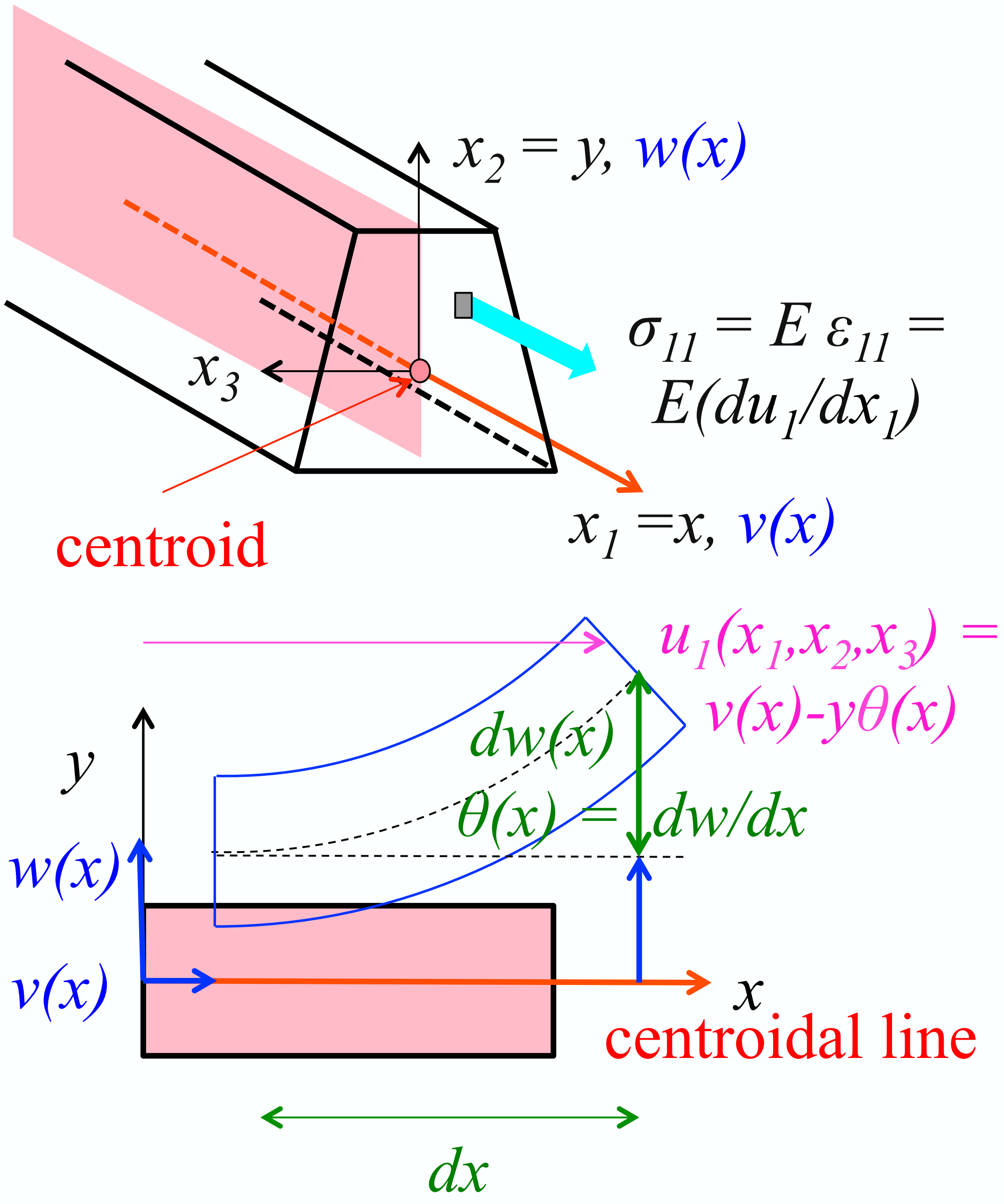
1. **1D BEAM THEORY**

2. **SHAPE FUNCTIONS NEEDED: HERMITIAN CUBICS**

3. **2D KIRCHOFF PLATE THEORY**

4. **SHAPE FUNCTIONS NEEDED: QUINTICS!**

1D (BERNOULLI-EULER-NAVIER) BEAM THEORY



BERNOULLI–EULER–NAVIER

- Uniaxial stress state (only $\sigma_{11} \neq 0$)
- Plane sections normal to initial centroidal line remain plane and normal to the deformed one
- Small (infinitesimal) strain kinematics
- Linear elastic constitutive law (isotropic, can be generalized to transversely isotropic about centroidal line)

DERIVATION OF 1D BEAM THEORY (IN-PLANE BENDING)

$$\mathcal{P}_{int} = \int_V \left[\frac{1}{2}\sigma_{11}\epsilon_{11}\right] dV = \int_0^L \left[\int_A \frac{1}{2}E(\epsilon_{11})^2 \, dA\right] dx$$

$$\text{recall : } \epsilon_{11} = \frac{du_1}{dx_1} \; , \; u_1 = v(x) - y\frac{d}{dx}w(x) \; ; \quad (x \equiv x_1, \; y \equiv x_2)$$

$$\mathcal{P}_{int} = \int_0^L \left[\int_A \frac{1}{2}E\left(\frac{dv}{dx} - y\frac{d^2w}{dx^2}\right)^2 dA\right] dx$$

section area

centroid def.

section moment of inertia

$$\text{recall : } \int_A dA = A \; , \quad \int_A y \, dA = 0 \; , \quad \int_A y^2 \, dA = I$$

axial energy

bending energy

$$\mathcal{P}_{int} = \int_0^L \left[\boxed{\frac{1}{2}EA\left(\frac{dv}{dx}\right)^2} + \boxed{\frac{1}{2}EI\left(\frac{d^2w}{dx^2}\right)^2}\right] dx$$

DERIVATION OF 1D BEAM THEORY (IN-PLANE BENDING)

$$\text{Uniaxial stress state :}\ \ \sigma_{11}(x,y) \ = \ E\,\epsilon_{11}(x,y)$$

$$\text{Strain distribution :}\ \ \epsilon_{11}(x,y) \ = \ \frac{du_1}{dx} = \epsilon(x) + y\,\kappa(x)$$

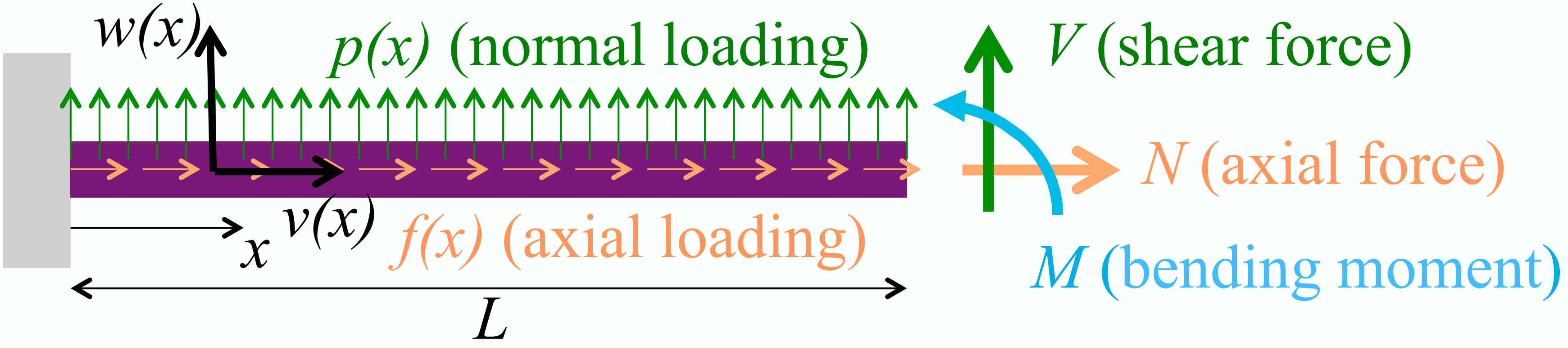
$$\text{Membrane strain :}\ \epsilon(x) \ = \ \frac{dv}{dx}$$

$$\text{Curvature strain :}\ \ \kappa(x) \ = \ -\frac{d^2w}{dx^2}$$

$$\text{Axial resultant :}\ \ N \ = \ \int_A \sigma_{11}\,dy \ = \ EA\,\epsilon(x)$$

$$\text{Moment resultant :}\ \ M \ = \ \int_A [\sigma_{11}\,y]\,dy \ = \ EI\,\kappa(x)$$

DERIVATION OF 1D BEAM THEORY (IN-PLANE BENDING)



$$\mathcal{P}_{ext} \quad = \quad -\int_0^L (fv + pw)\, dx - Nv(L) - Vw(L) - M\theta(L) \; ; \quad \left(\theta = \frac{dw}{dx}\right)$$

$$\boxed{\begin{aligned}\mathcal{P} \quad &= \quad \mathcal{P}_{int} + \mathcal{P}_{ext} = \int_0^L \left[\frac{1}{2}EA\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}EI\left(\frac{d^2w}{dx^2}\right)^2 - fv - pw\right] dx \\ &-Nv(L) - Vw(L) - M\frac{dw}{dx}(L)\end{aligned}}$$

Full decoupling: axial + bending

$$\delta\mathcal{P} \quad \equiv \quad \frac{d}{d\epsilon}\left[\mathcal{P}(\mathbf{u} + \epsilon\delta\mathbf{u})\right]_{\epsilon=0} = 0 \; ; \qquad \mathbf{u} \equiv (v(x), w(x))$$

equilibrium

DERIVATION OF 1D BEAM THEORY (IN-PLANE BENDING)

$$\delta\mathcal{P} = \int_0^L \left[EA\frac{dv}{dx}\frac{d\delta v}{dx} + EI\frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2} - f\delta v - pw\right]dx$$

$$-N\delta v(L) - V\delta w(L) - M\frac{d\delta w}{dx}(L) = 0$$

axial equilibrium

transverse equilibrium

$$\delta\mathcal{P} = \int_0^L \left\{\boxed{\left[-\frac{d}{dx}\left(EA\frac{dv}{dx}\right) - f\right]}\delta v + \boxed{\left[\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - p\right]}\delta w\right\}dx$$

$$+\boxed{\left[EA\frac{dv}{dx} - N\right]}\delta v(L) + \boxed{\left[EI\frac{d^2w}{dx^2} - M\right]}\frac{d\delta w}{dx}(L)$$

axial force

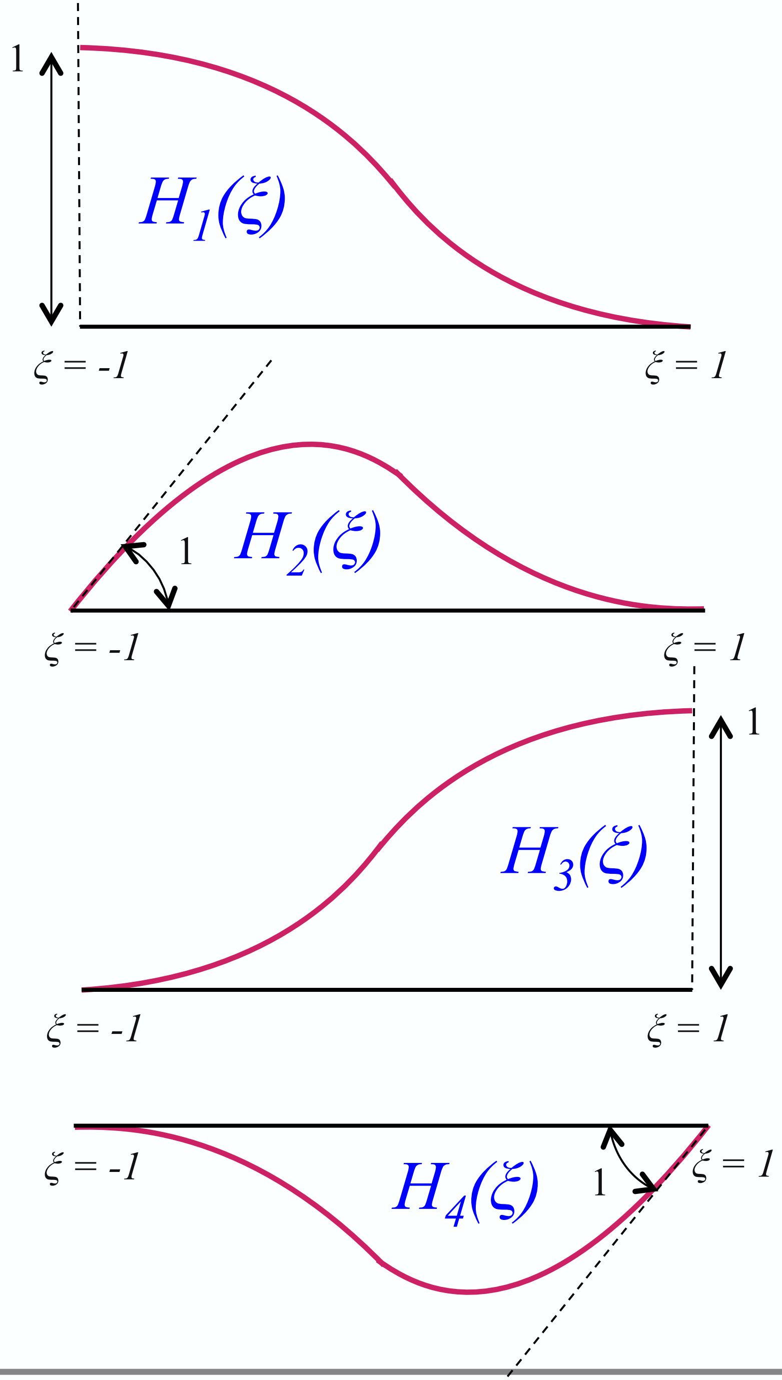
bending moment

$$+\boxed{\left[-\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) - V\right]}\delta w(L) = 0$$

shear force

1D BEAM C^1 SHAPE FUNCTIONS (HERMITIAN CUBICS)

FEM FOR BENDING OF 1D BEAM – HERMITIAN CUBICS



The bending energy – EI$(d^2w/dx^2)^2$ term – dictates C^1 continuity (i.e. continuous *dw/dx*) of the test function *w(x)* in the entire beam. Consequently we must ensure inter-element continuity of both *w(x)* and *dw/dx* at each boundary node. The simplest element functions that do this are Hermitian cubics

$$\begin{aligned}
H_1(\xi) &= \frac{1}{4}(1-\xi)^2(2+\xi) \\
H_2(\xi) &= \frac{1}{4}(1-\xi)^2(\xi+1) \\
H_3(\xi) &= \frac{1}{4}(1+\xi)^2(2-\xi) \\
H_4(\xi) &= \frac{1}{4}(1+\xi)^2(\xi-1)
\end{aligned}$$

FEM FOR BENDING OF 1D BEAM – HERMITIAN CUBICS

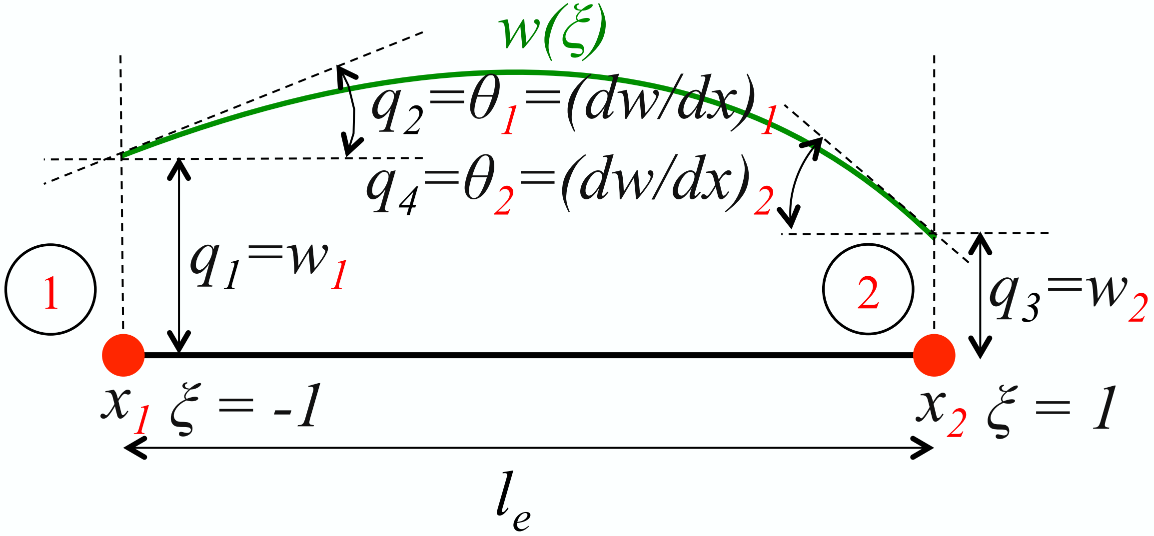
Sub-parametric representation:

$H_i(\xi)$: cubic – $(i=1,2,3,4)$; $N_j(\xi)$: linear – $(j=1,2)$

$$x(\xi) = N_1(\xi)\, x_1 + N_2(\xi)\, x_2$$

$$N_1(\xi) = (1-\xi)/2, \quad N_2(\xi) = (1+\xi)/2$$

$$w(\xi) = H_1(\xi)\, w_1 + H_2(\xi)\, w'_1 + H_3(\xi)\, w_2 + H_4(\xi)\, w'_2 \text{ (where } w' = dw/d\xi\text{)}$$



NOTE:

$w(\xi_1) = w_1, \quad w(\xi_2) = w_2$

$(dw/d\xi)(\xi_1) = w'_1, \ (dw/d\xi)(\xi_2) = w'_2$

$x(\xi_1) = x_1, \quad x(\xi_2) = x_2$

1D BEAM BENDING – STIFFNESS MATRIX AND FORCE VECTOR

Beam subjected to bending through a constant transverse load p

$$w(\xi) = \left[H_1(\xi),\ \frac{l_e}{2} H_2(\xi),\ H_3(\xi),\ \frac{l_e}{2} H_4(\xi) \right] \mathbf{q}_e = \mathbf{H}\mathbf{q}_e$$

$$\mathbf{q}_e^T = [q_1,\ q_2,\ q_3,\ q_4] = [w_1,\ \theta_1,\ w_2,\ \theta_2]$$

$$\frac{dw}{dx}(\xi) = \frac{2}{l_e} \frac{d\mathbf{H}}{d\xi} \mathbf{q}_e \ , \qquad \frac{d^2 w}{dx^2}(\xi) = \left(\frac{2}{l_e} \right)^2 \frac{d^2 \mathbf{H}}{d\xi^2} \mathbf{q}_e$$

element stiffness $\mathbf{k}_e$

$$\mathcal{P}_{int}^e = \frac{1}{2} \int_{l_e} EI \left(\frac{d^2 w}{dx^2} \right)^2 dx = \frac{1}{2} \mathbf{q}_e^T \boxed{\left[\int_{-1}^{+1} EI \left(\frac{d^2 \mathbf{H}}{d\xi^2} \right)^T \frac{d^2 \mathbf{H}}{d\xi^2} \left(\frac{2}{l_e} \right)^3 d\xi \right]} \mathbf{q}_e$$

$$\mathcal{P}_{ext}^e = \int_{l_e} p\ w\ dx = \mathbf{q}_e^T \boxed{\left[\int_{-1}^{+1} \mathbf{H}^T p\ \frac{l_e}{2}\ d\xi \right]}$$

element force $\mathbf{f}_e$

1D BEAM BENDING – STIFFNESS MATRIX AND FORCE VECTOR

Uniform section beam subjected to bending by a constant transverse load *p*.

NOTE*:* Stiffness matrix is exact in case when beam has only end loads, since $EI(d^4w/dx^4)=0$ gives a cubic for $w(x)$ which is represented exactly by the Hermitian interpolation functions

$$\mathbf{k}_e = \frac{EI}{(l_e)^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4(l_e)^2 & -6l_e & 2(l_e)^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2(l_e)^2 & -6l_e & 4(l_e)^2 \end{bmatrix}, \quad \mathbf{f}_e = p\begin{bmatrix} \dfrac{l_e}{2} \\ \dfrac{(l_e)^2}{12} \\ \dfrac{l_e}{2} \\ -\dfrac{(l_e)^2}{12} \end{bmatrix}$$

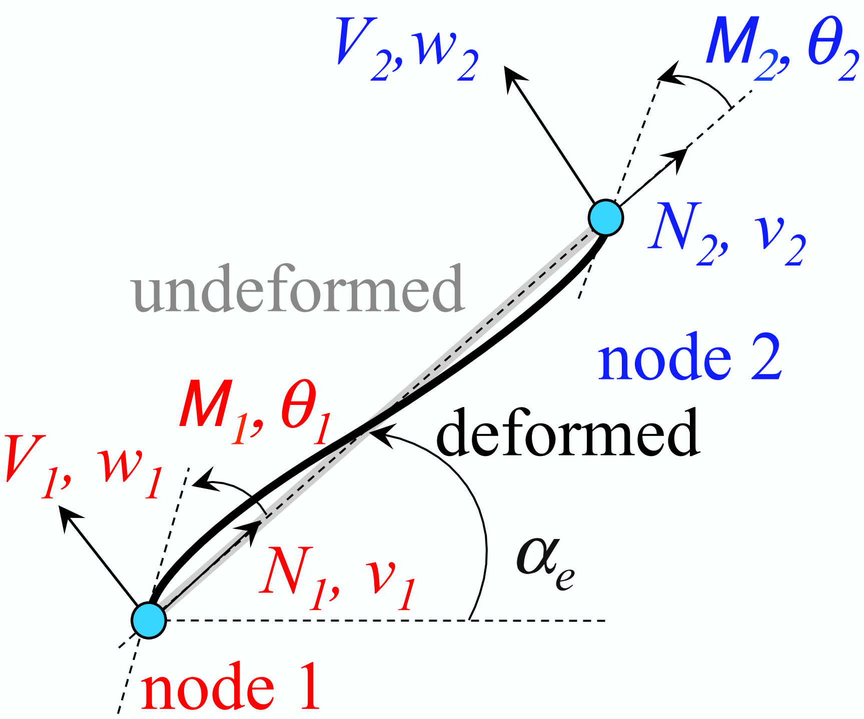
FRAME PROBLEMS

1D BEAM BENDING – STIFFNESS MATRIX AND FORCE VECTOR

General case for a planar frame considering axial and transverse loading.

$$\mathbf{k}_e = \begin{array}{r} \\ (\mathbf{K}_{11})_e \\ \\ \\ (\mathbf{K}_{21})_e \\ \\ \end{array} \left[\begin{array}{ccc|ccc} \frac{EA}{l_e} & 0 & 0 & -\frac{EA}{l_e} & 0 & 0 \\ 0 & \frac{12EI}{(l_e)^3} & \frac{6EI}{(l_e)^2} & 0 & -\frac{12EI}{(l_e)^3} & \frac{6EI}{(l_e)^2} \\ 0 & \frac{6EI}{(l_e)^2} & \frac{4EI}{l_e} & 0 & -\frac{6EI}{(l_e)^2} & \frac{2EI}{l_e} \\ \hline -\frac{EA}{l_e} & 0 & 0 & \frac{EA}{l_e} & 0 & 0 \\ 0 & -\frac{12EI}{(l_e)^3} & -\frac{6EI}{(l_e)^2} & 0 & \frac{12EI}{(l_e)^3} & -\frac{6EI}{(l_e)^2} \\ 0 & \frac{6EI}{(l_e)^2} & \frac{2EI}{l_e} & 0 & -\frac{6EI}{(l_e)^2} & \frac{4EI}{l_e} \end{array}\right] \begin{array}{l} \\ (\mathbf{K}_{12})_e \\ \\ \\ (\mathbf{K}_{22})_e \\ \\ \end{array}, \quad \mathbf{q}_e = \begin{array}{r} (\mathbf{q}_1)_e \\ \\ \\ (\mathbf{q}_2)_e \\ \\ \\ \end{array} \left[\begin{array}{c} v_1 \\ w_1 \\ \theta_1 \\ \hline v_2 \\ w_2 \\ \theta_2 \end{array}\right], \quad \mathbf{f}_e = \begin{array}{r} (\mathbf{f}_1)_e \\ \\ \\ (\mathbf{f}_2)_e \\ \\ \\ \end{array} \left[\begin{array}{c} N_1 \\ V_1 \\ M_1 \\ \hline N_2 \\ V_2 \\ M_2 \end{array}\right]$$

1D BEAM BENDING – STIFFNESS MATRIX AND FORCE VECTOR



v_1 , v_2 , axial displacements of beam

w_1 , w_2 , transverse displacements of beam

θ_1 , θ_2 , end rotations of beam

N_1 , N_2 , axial resultants of beam

V_1 , V_2 , shear resultants of beam

M_1 , M_2 , bending moments of beam

$$\boxed{\mathbf{f}_e = \mathbf{K}_e \, \mathbf{q}_e}$$

element force: $\mathbf{f}_e$

element displacement: $\mathbf{q}_e$

$\mathbf{f}_e$ and $\mathbf{q}_e$ must be expressed in global coordinates (recall truss problems)

1D BEAM BENDING – FRAME PROBLEM

SYSTEM IN MATRIX FORM BECOMES:

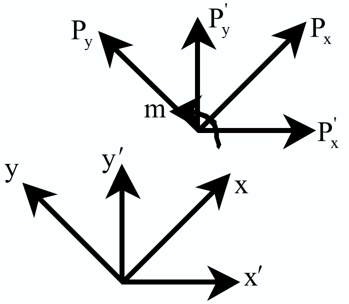
$$\left.\begin{matrix}\mathbf{P}_1 = \mathbf{K}_{11}\mathbf{d}_1 + \mathbf{K}_{12}\mathbf{d}_2 \\ \mathbf{P}_2 = \mathbf{K}_{21}\mathbf{d}_1 + \mathbf{K}_{22}\mathbf{d}_2\end{matrix}\right\} \quad \text{(i)} \quad \text{WHERE:}$$

$$\mathbf{P}_1 = \begin{bmatrix} P_{x1} \\ P_{y1} \\ m_1 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} P_{x2} \\ P_{y2} \\ m_2 \end{bmatrix}, \quad \mathbf{d}_1 = \begin{bmatrix} d_{x1} \\ d_{y1} \\ \theta_1 \end{bmatrix}, \quad \mathbf{d}_2 = \begin{bmatrix} d_{x2} \\ d_{y2} \\ \theta_2 \end{bmatrix},$$

$$\mathbf{K}_{11} = \begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L \end{bmatrix}, \quad \mathbf{K}_{22} = \begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix},$$

$$\mathbf{K}_{12} = \mathbf{K}_{21}^T = \begin{bmatrix} -EA/L & 0 & 0 \\ 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & -6EI/L^2 & 2EI/L \end{bmatrix}$$

CHANGE OF LOCAL TO GLOBAL COORDINATES FOR FORCE **P**



$$\boxed{\underset{\mathbf{P}'}{\begin{bmatrix} P'_x \\ P'_y \\ m' \end{bmatrix}} = \underset{\mathbf{T}}{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\mathbf{P}}{\begin{bmatrix} P_x \\ P_y \\ m \end{bmatrix}}}$$

rotation matrix

$$\mathbf{P}' = \mathbf{TP} \quad \text{(ii)}^*$$

CHANGE OF LOCAL TO GLOBAL COORDINATES FOR DISPLACEMENT, **d**

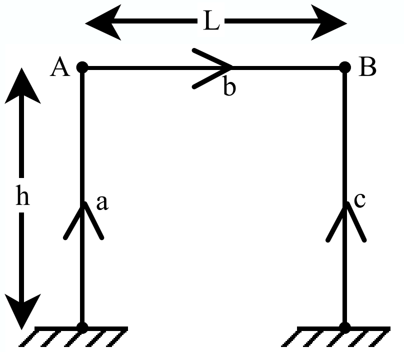
work $= \mathbf{P}^T\mathbf{d} = \mathbf{P}'^T\mathbf{d}'$, using (ii)*, $\quad \mathbf{P}^T\mathbf{d} = \mathbf{P}^T\mathbf{T}^T\mathbf{d}'$ for all **P**, **d**,

thus $\mathbf{d} = \mathbf{T}^T\mathbf{d}'$ (iii)*

element stiffness in rotated coordinates

SUBSTITUTE (ii)*, (iii)* INTO (i)* TO PUT THEM IN SAME FORM AS (1)

$$\boxed{\begin{matrix} \mathbf{P}_1 = \left(\mathbf{TK}_{11}\mathbf{T}^T\right)\mathbf{d}'_1 + \left(\mathbf{TK}_{12}\mathbf{T}^T\right)\mathbf{d}'_2 & \mathbf{K}'_{11} = \mathbf{TK}_{11}\mathbf{T}^T \;, & \mathbf{K}'_{12} = \mathbf{TK}_{12}\mathbf{T}^T \\ \mathbf{P}_2 = \left(\mathbf{TK}_{21}\mathbf{T}^T\right)\mathbf{d}'_1 + \left(\mathbf{TK}_{22}\mathbf{T}^T\right)\mathbf{d}'_2 & \mathbf{K}'_{21} = \mathbf{TK}_{21}\mathbf{T}^T \;, & \mathbf{K}'_{22} = \mathbf{TK}_{22}\mathbf{T}^T \end{matrix}}$$



$$\begin{bmatrix} \mathbf{P}_A \\ \mathbf{P}_B \end{bmatrix} = \begin{bmatrix} \mathbf{K}'_{22a} + \mathbf{K}'_{11b} & \mathbf{K}'_{12b} \\ \mathbf{K}'_{21b} & \mathbf{K}'_{22b} + \mathbf{K}'_{22c} \end{bmatrix} \begin{bmatrix} \mathbf{d}_A \\ \mathbf{d}_B \end{bmatrix}$$

ALL BEAMS SAME WITH AXIAL STIFFNESS: EA,
AND BENDING STIFFNESS: EI.

$$\mathbf{K}'_{22a} = \mathbf{K}'_{22c} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} EA/h & 0 & 0 \\ 0 & 12EI/h^3 & -6EI/h^2 \\ 0 & -6EI/h^2 & 4EI/h \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

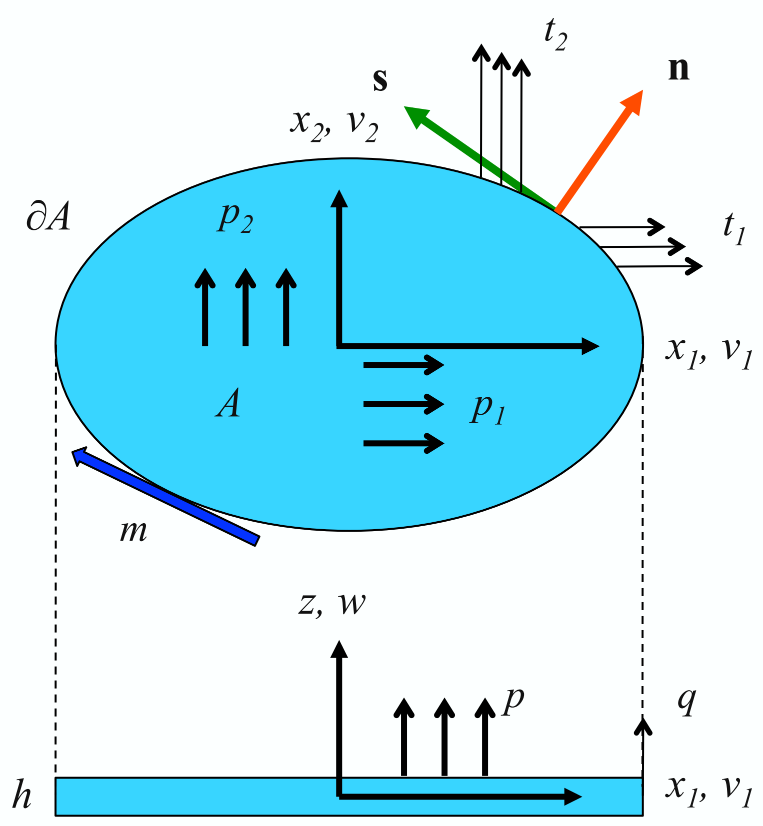
$$\left(\alpha_a = \alpha_c = \frac{\pi}{2}\right) \qquad = \begin{bmatrix} 12EI/h^3 & 0 & 6EI/h^2 \\ 0 & EA/h & 0 \\ 6EI/h^2 & 0 & 4EI/h \end{bmatrix}$$

$$\alpha_b = 0 \quad \Rightarrow \quad \mathbf{K}'_{11b} = \mathbf{K}_{11}, \quad \mathbf{K}'_{12b} = \mathbf{K}_{12}, \quad \mathbf{K}'_{21b} = \mathbf{K}_{21}, \quad \mathbf{K}'_{22b} = \mathbf{K}_{22}$$

$$\begin{bmatrix} P_{xA} \\ P_{yA} \\ m_A \\ P_{xB} \\ P_{yB} \\ m_B \end{bmatrix} = \begin{bmatrix} \frac{12EI}{h^3} + \frac{EA}{L} & 0 & \frac{6EI}{h^2} & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} + \frac{EA}{h} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{h^2} & \frac{6EI}{L^2} & \frac{4EI}{L} + \frac{4EI}{h} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{12EI}{h^3} + \frac{EA}{L} & 0 & \frac{6EI}{h^2} \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} + \frac{EA}{h} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & \frac{6EI}{h^2} & -\frac{6EI}{L^2} & \frac{4EI}{L} + \frac{4EI}{h} \end{bmatrix} \begin{bmatrix} d_{xA} \\ d_{yA} \\ \theta_A \\ d_{xB} \\ d_{yB} \\ \theta_B \end{bmatrix}$$

2D KIRCHHOFF PLATE THEORY

KIRCHHOFF PLATE THEORY:



- Plane stress state (only $\sigma_{\alpha\beta}$ – $\alpha, \beta = 1,2$)
- Normals to the undeformed middle plane remain normal to the deformed middle surface
- Small (infinitesimal) strain kinematics
- Linear elastic constitutive law (isotropic, can be generalized to transversely isotropic about normal direction)
- Reduces to Bernoulli-Euler-Navier for loading that is independent on x_2 (or x_1)

DERIVATION OF 2D KIRCHHOFF PLATE BENDING THEORY

$$\text{Plane stress state}: \ \sigma_{\alpha\beta}(x_1,x_2,z) = L_{\alpha\beta\gamma\delta}\,\epsilon_{\gamma\delta}(x_1,x_2,z); \ (\text{Greek indexes}: \ 1,2)$$

$$\text{Plane stress moduli}: \ L_{\alpha\beta\gamma\delta} = \frac{E}{1-\nu^2}\left[\frac{1-\nu}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})+\nu\delta_{\alpha\beta}\delta_{\gamma\delta}\right]$$

$$\text{Strain distribution}: \ \epsilon_{\alpha\beta}(x_1,x_2,z) = E_{\alpha\beta}(x_1,x_2)+zK_{\alpha\beta}(x_1,x_2)$$

$$\text{Membrane strains}: \ E_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta}+v_{\beta,\alpha}) \ ; \quad (f_{,\alpha}\equiv\partial f/\partial x_\alpha)$$

$$\text{Curvature strains}: \ K_{\alpha\beta} = -w_{,\,\alpha\beta}$$

$$\text{Membrane resultants}: \ N_{\alpha\beta} = \int_{-h/2}^{h/2}\sigma_{\alpha\beta}dz \ = \ hL_{\alpha\beta\gamma\delta}E_{\gamma\delta}$$

$$\text{Moment resultants}: \ M_{\alpha\beta} = \int_{-h/2}^{h/2}\sigma_{\alpha\beta}z\,dz \ = \ \frac{h^3}{12}L_{\alpha\beta\gamma\delta}K_{\gamma\delta}$$

2D PLATE THEORY REDUCES TO BEAM FOR IN-PLANE BEND.

$$\begin{aligned}
\text{Uniaxial stress state : } \sigma_{11}(x_1, x_2) &= E\,\epsilon_{11}(x_1, x_2); \\
\text{Strain distribution : } \epsilon_{11}(x_1, x_2) &= \epsilon(x_1) + x_2\,\kappa(x_1) \\
\text{Membrane strain : } \epsilon(x_1) &= \frac{dv}{dx_1} \\
\text{Curvature strain : } \kappa(x_1) &= -\frac{d^2 w}{d(x_1)^2} \\
\text{Axial resultant : } N &= \int_A \sigma_{11}\, dx_2 \;=\; EA\,\epsilon(x_1) \\
\text{Moment resultant : } M &= \int_A \sigma_{11} x_2\, dx_2 \;=\; EI\,\kappa(x_1)
\end{aligned}$$

DERIVATION OF 2D KIRCHHOFF PLATE BENDING THEORY

$$\text{Internal energy :}\quad \mathcal{P}_{int} = \int_A [\frac{1}{2}N_{\alpha\beta}E_{\alpha\beta} + \frac{1}{2}M_{\alpha\beta}K_{\alpha\beta})h]dA$$

$$\text{External energy :}\quad \mathcal{P}_{ext} = -\int_A [p_\alpha u_\alpha + pw]\,dA - \int_{\partial A} [t_\alpha u_\alpha + qw + m(-w,_n)]\,ds$$

$$\text{Potential energy :}\quad \mathcal{P} = \mathcal{P}_{int} + \mathcal{P}_{ext}$$

bending part

membrane part

$$\mathcal{P} = \frac{1}{2}\int_A [\boxed{L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}E_{\gamma\delta}} + \boxed{\frac{h^2}{12}K_{\alpha\beta}K_{\gamma\delta}})h]dA \; - \int_A [\boxed{p_\alpha u_\alpha} + \boxed{pw}]\,dA - \int_{\partial A} [\boxed{t_\alpha u_\alpha} + \boxed{qw + m(-w,_n}\,)]\,ds$$

membrane and bending parts of potential energy are completely decoupled

KIRCHHOFF PLATE BENDING FOR IN-PLANE LOADING

$$\mathcal{P}_{memb} = \int\limits_A \left[\frac{1}{2} \left(h L_{\alpha\beta\gamma\delta} E_{\alpha\beta} \; E_{\gamma\delta} \right) - p_\alpha u_\alpha \right] dA - \int\limits_{\partial A} \left[t_\alpha u_\alpha \right] ds$$

$$\begin{aligned} \delta\mathcal{P}_{memb} &= -\int\limits_A \left\{ \left[\left(h L_{\alpha\beta\gamma\delta} E_{\gamma\delta} \right)_{,\beta} + p_\alpha \right] \delta u_\alpha \right\} dA \\ &\quad + \int_{\partial A} \left\{ \left[t_a - h L_{\alpha\beta\gamma\delta} E_{\gamma\delta} n_\beta \right] \delta u_\alpha \right\} ds = 0 \end{aligned}$$

E-L : $\left(h L_{\alpha\beta\gamma\delta} E_{\gamma\delta} \right)_{,\beta} + p_\alpha = 0, \;\; \mathbf{x} \in A$

E.B.C. : $u_\alpha = \text{given} \; , \;\; \mathbf{x} \in \partial A_u \; ;$ part of boundary with known $\mathbf{u}$

N.B.C. : $t_a = h L_{\alpha\beta\gamma\delta} E_{\gamma\delta} n_\beta \; , \;\; \mathbf{x} \in \partial A_t \; ;$ part of boundary with known $\mathbf{t}$

NOTE: In-plane problem is 2D problem of isotropic, linear elasticity that we know how to solve with FEM (need C^0 inter-element continuity).

KIRCHHOFF PLATE BENDING FOR TRANSVERSE LOADING

$$\mathcal{P} \;=\; \int_A \left[\frac{1}{2}\left(\frac{h^3}{12} L_{\alpha\beta\gamma\delta} w_{,\alpha\beta}\, w_{,\gamma\delta}\right) - pw\right] dA$$

$$\delta\mathcal{P} \;=\; \int_A \left[\left(\frac{h^3}{12} L_{\alpha\beta\gamma\delta} w_{,\alpha\beta\gamma\delta} - pw\right) \delta w\right] dA + \int_{\partial A} M_{nn} \delta w_{,n}\, ds = 0$$

$$\text{E-L} \;:\; \frac{Eh^3}{12(1-\nu^2)} \nabla^4 w = p,\;\; \mathbf{x} \in A$$

$$\text{E.B.C.} \;:\; w = 0\,,\;\; \mathbf{x} \in \partial A$$

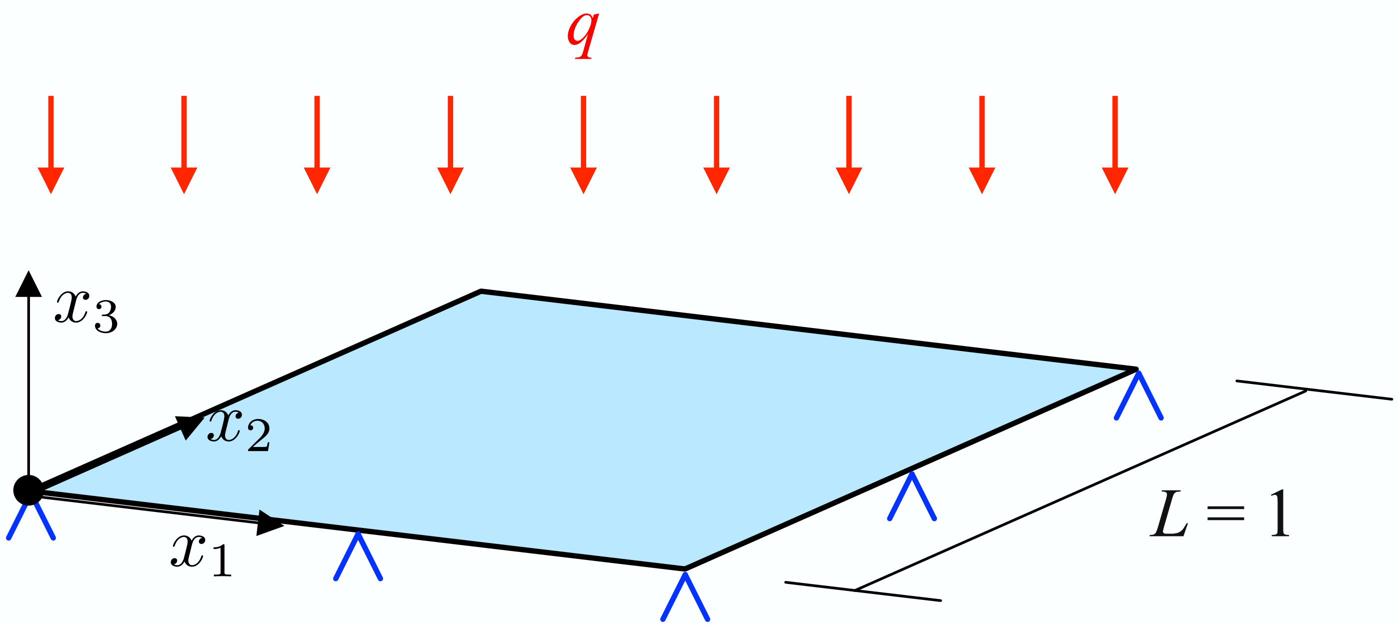
$$\text{N.B.C.} \;:\; w_{,11}(0, x_2) = w_{,11}(a_1, x_2) = w_{,22}(x_1, 0) = w_{,22}(x_1, a_2) = 0$$

NOTE: Since second order derivatives of the transverse displacement enter the bending energy, need C^1 inter-element continuity!

ANALYTICAL SOLUTION OF S.S. KIRCHHOFF PLATE

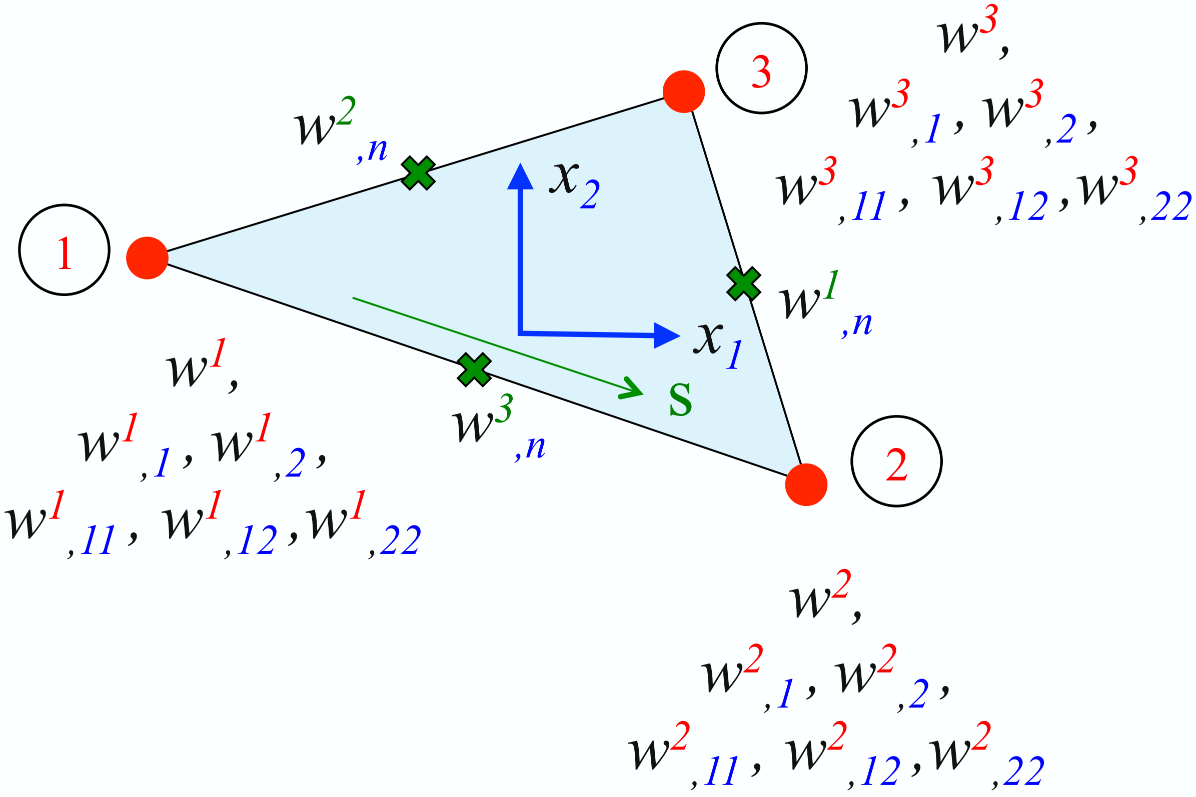
$$w(x_1, x_2) = \frac{16qL^4}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \dfrac{m\pi x_1}{L} \sin \dfrac{n\pi x_2}{L}}{mn\left(m^2+n^2\right)^2}$$

with $m = 1, 3, 5\ldots$ and $n = 1, 3, 5\ldots$



KIRCHHOFF PLATE BENDING FOR TRANSVERSE LOADING

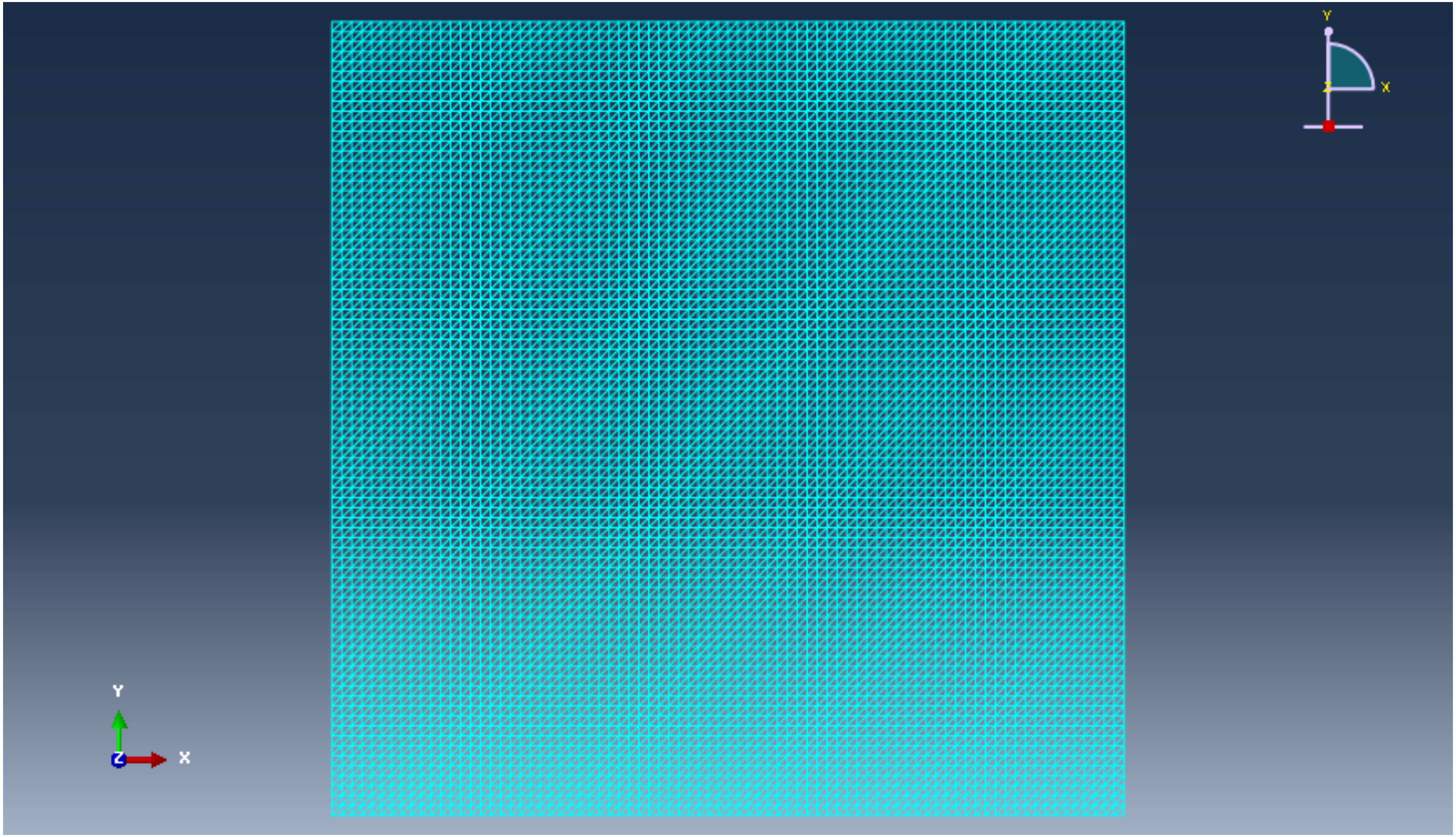
The simplest element that can satisfy C^1 inter-element continuity of $w(x_1,x_2)$ is the Clough Triangle that has 21 degrees of freedom an a full fifth order polynomial shape function



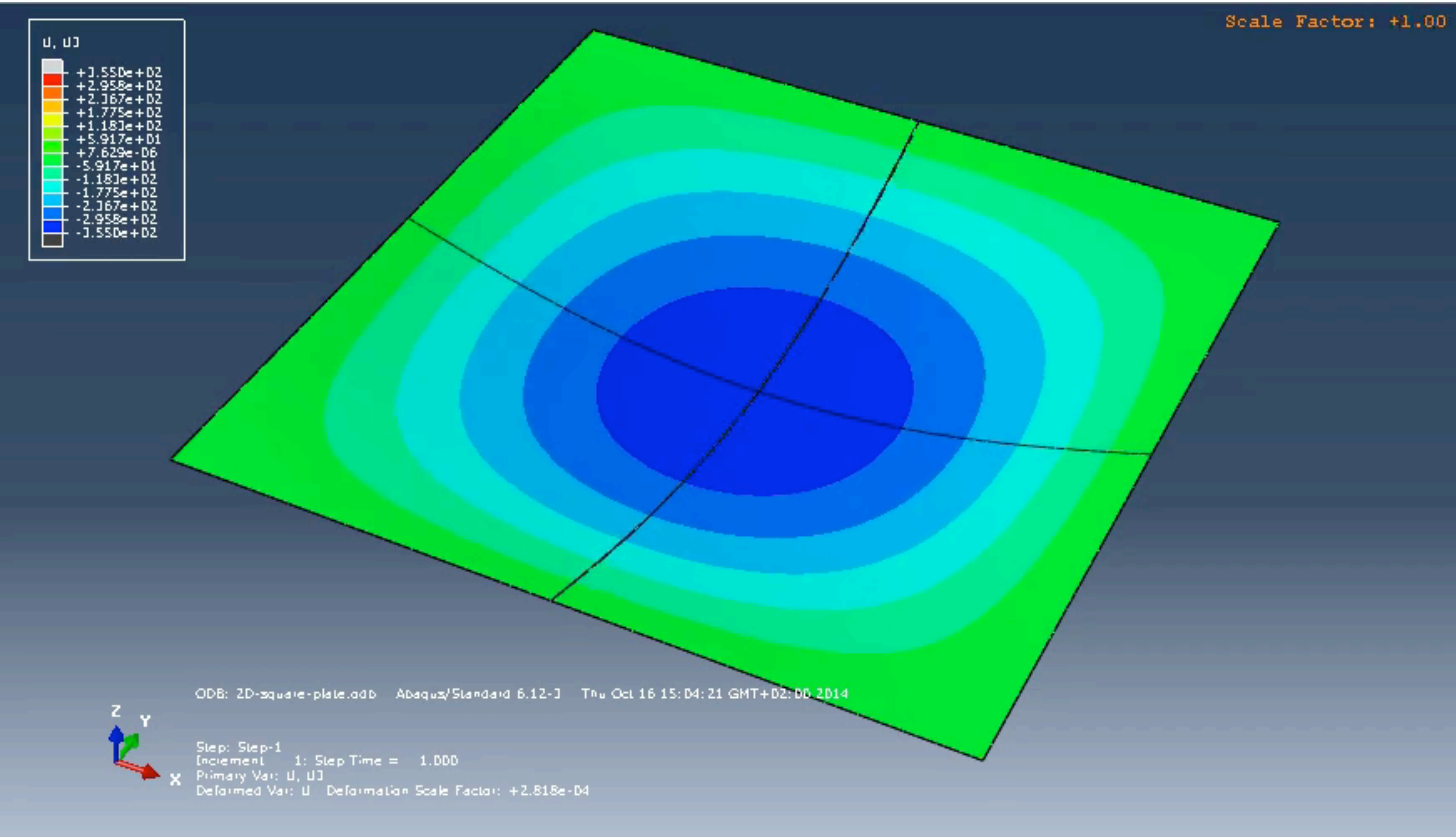
Inter-element continuity of $w(s)$ at each side: 5th order polynomial in s (length coordinate) share the same 6 constants: w, $w_{,s}$, $w_{,ss}$ at each end node

Inter-element continuity of $w_{,n}(s)$ at each side: 4th order polynomial in s share the same 5 constants: $w_{,n}$, $w_{,nn}$ at each end node and $w_{,n}$ at the mid-side node

SQUARE, SIMPLY SUPPORTED KIRCHHOFF PLATE



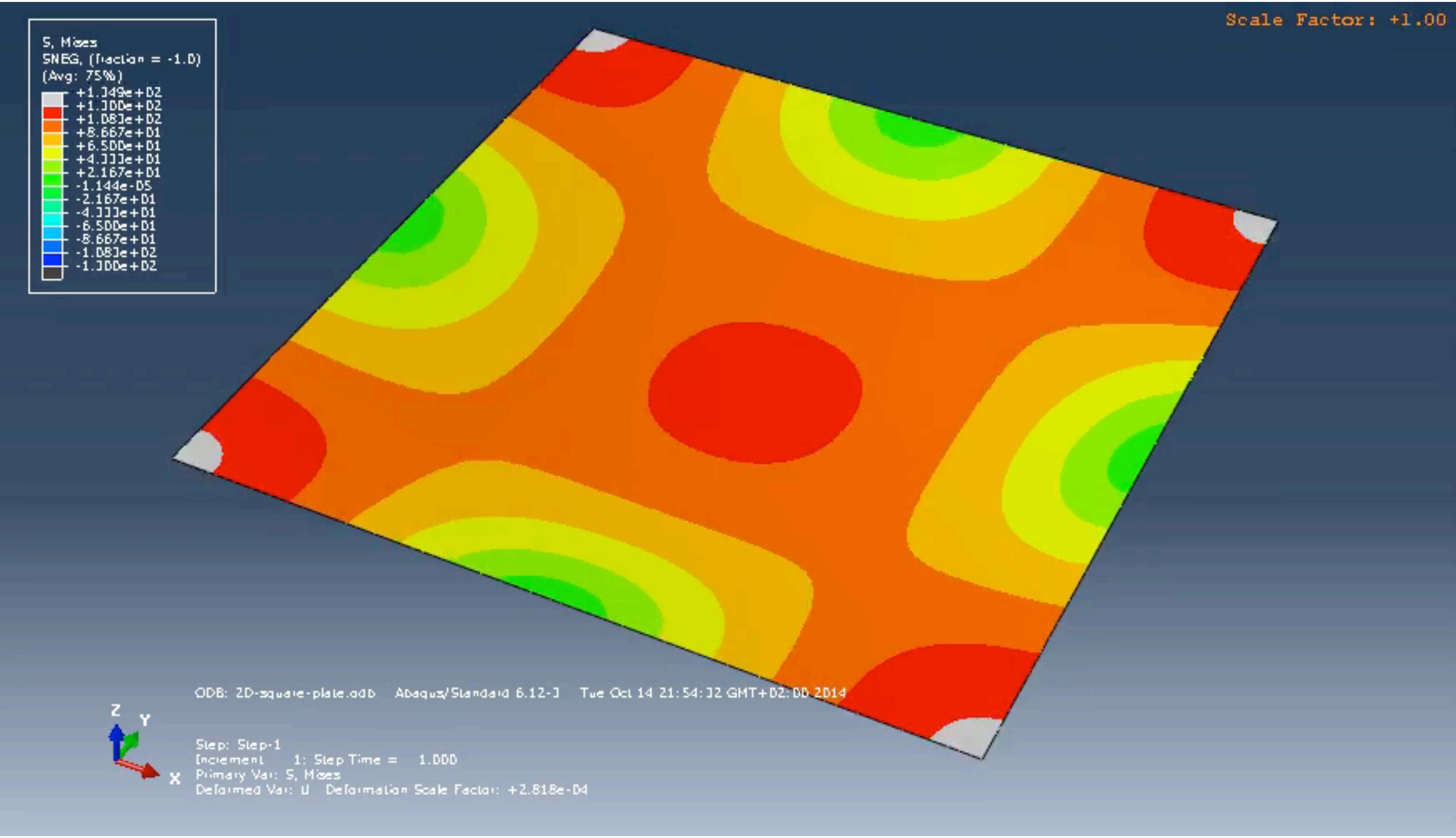
NORMAL DEFLECTION OF SQUARE, KIRCHHOFF PLATE



ODB: 2D-square-plate.odb   Abaqus/Standard 6.12-3   Thu Oct 16 15:04:21 GMT+02:00 2014

Step: Step-1
Increment   1: Step Time =   1.000
Primary Var: U, U3
Deformed Var: U   Deformation Scale Factor: +2.818e-04

TOP SURFACE VON-MISES STRESS FOR KIRCHHOFF PLATE



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ODB: 2D-square-plate.odb    Abaqus/Standard 6.12-3    Tue Oct 14 21:54:32 GMT+02:00 2014

Step: Step-1
Increment      1: Step Time =    1.000
Primary Var: S, Mises
Deformed Var: U   Deformation Scale Factor: +2.818e-04
```