TOPICS COVERED IN THIS LECTURE

1. ISOPARAMETRIC ELEMENTS IN 3D (TETRAHEDRA, BRICKS ETC.)

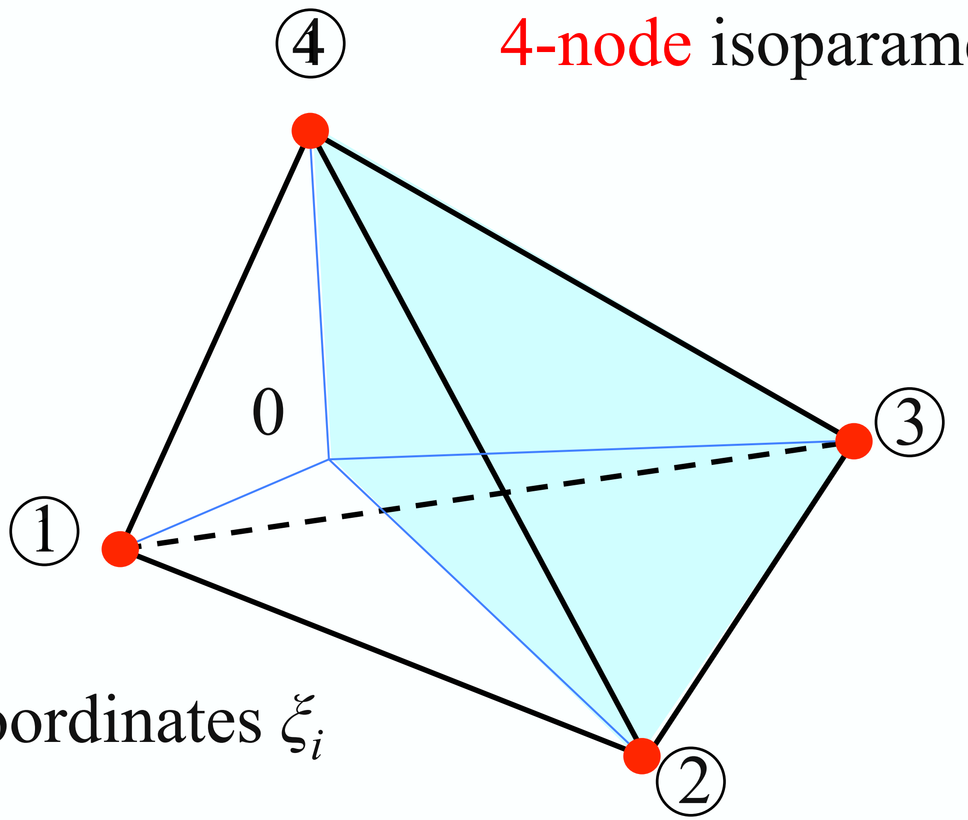
2. TORSION OF A BAR IN 3D AND MOTIVATION FOR STRESS-BASED APPROACH

3. STRESS-BASED FEM; PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY

4. FEM DISCRETIZATION USING COMPLEMENTARY ENERGY

ISOPARAMETRIC ELEMENTS IN 3D

4-node isoparametric tetrahedron



$$u_j(\boldsymbol{\xi}) = \sum_{I=1}^{4} N_I(\boldsymbol{\xi}) U_j^I$$

$$x_j(\boldsymbol{\xi}) = \sum_{I=1}^{4} N_I(\boldsymbol{\xi}) x_j^I$$

Coordinates ξ_i

$$N_I(\boldsymbol{\xi}^J) = \delta_{IJ}; \ (I, J = 1, \dots 4)$$

$\xi_1 = V_{0234}/V_{1234}$ – V_{0234} : *Volume 0234…*

$\xi_2 = V_{0134}/V_{1234}$

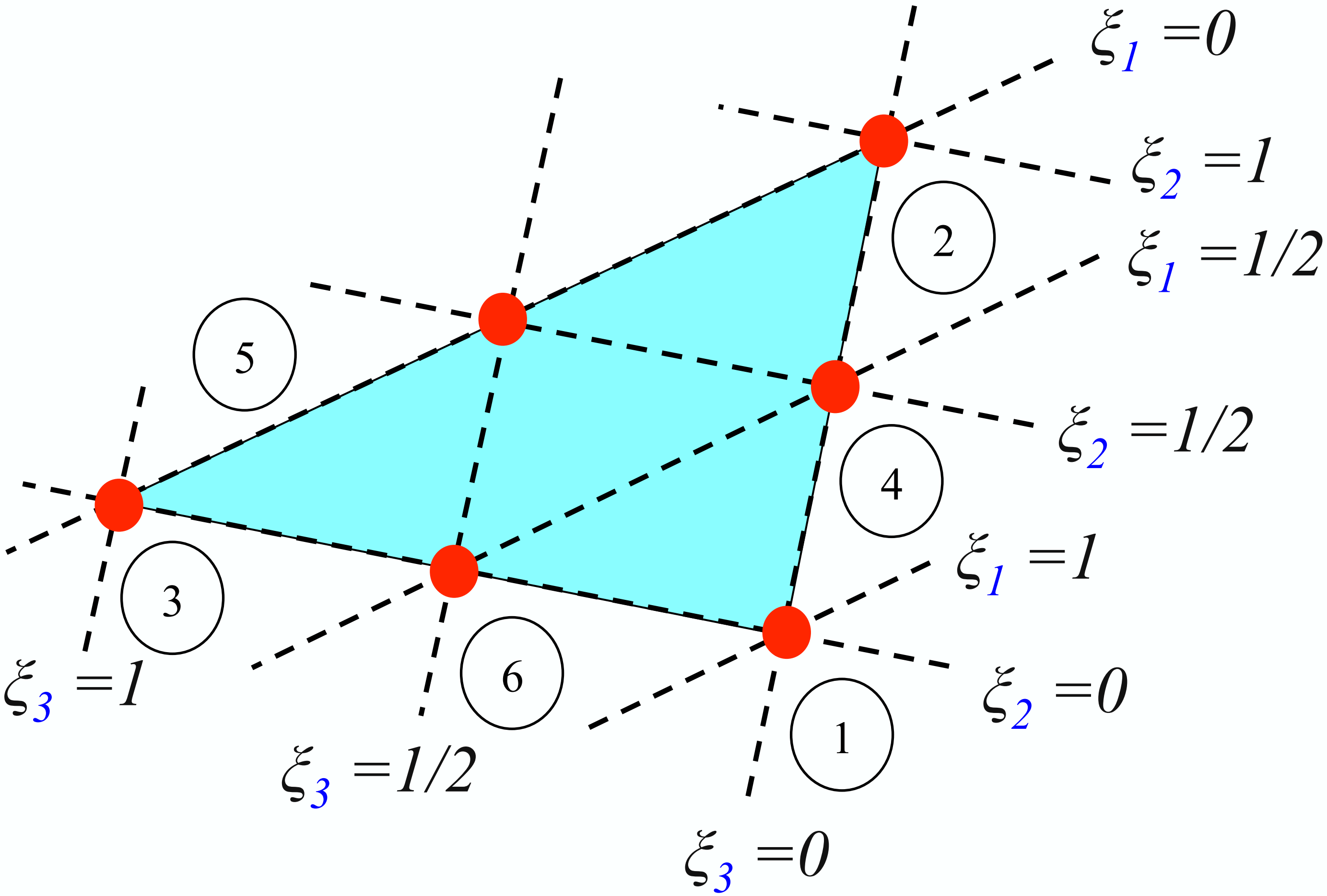
$\xi_3 = V_{0124}/V_{1234}$

$\xi_4 = V_{0123}/V_{1234}$

$$N_I(\boldsymbol{\xi}) = \xi_I; \ (I = 1, \dots 4)$$

Note: $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 1$, patch test automatically satisfied

Shape functions of node I are products of equations avoiding that node



$$N_1 = \xi_1(2\xi_1 - 1)$$

$$N_2 = \xi_2(2\xi_2 - 1)$$

$$N_3 = \xi_3(2\xi_3 - 1)$$

$$N_4 = 4\ \xi_1\xi_2$$

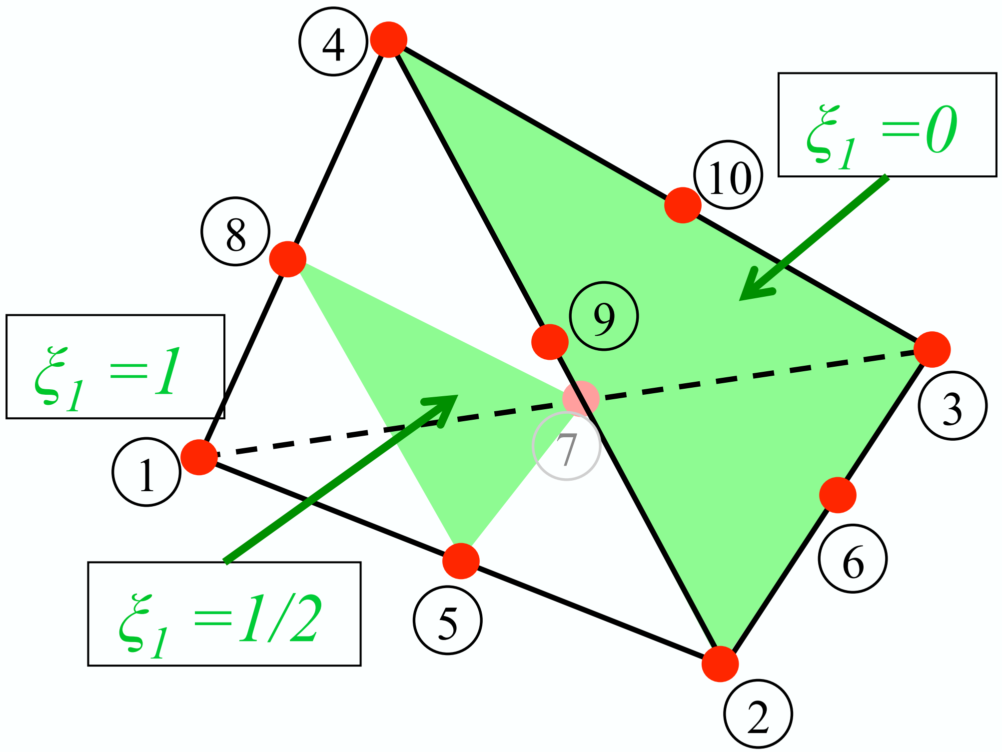
$$N_5 = 4\ \xi_2\xi_3$$

$$N_6 = 4\ \xi_3\xi_1$$

Triangular coordinates satisfy: $\xi_1+\xi_2+\xi_3 = 1$

10-node isoparametric tetrahedron in 3D similarly to 6-node triangle in 2D

Tetrahedral coordinates: $\xi_1+\xi_2+\xi_3+\xi_4 = 1$

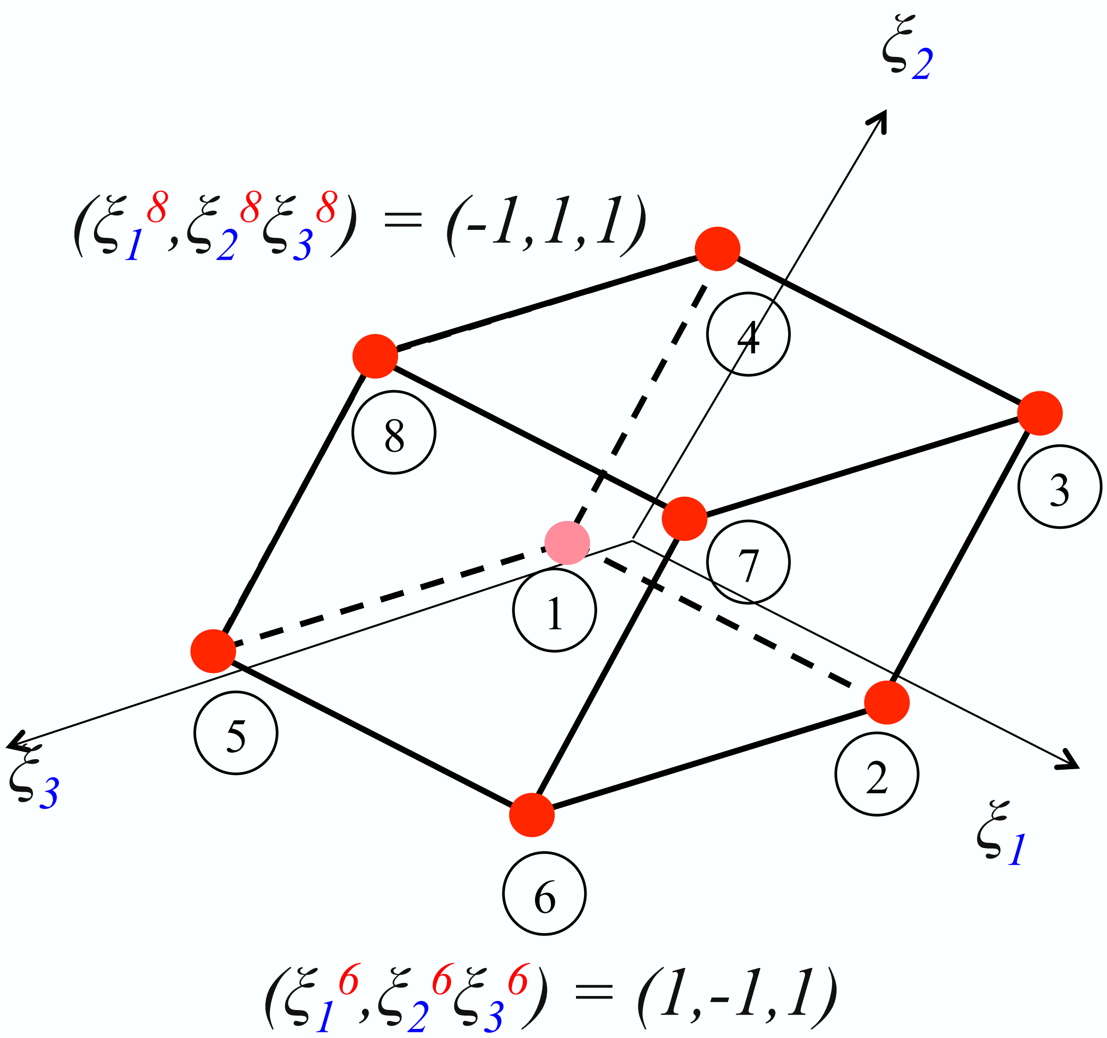


$$N_1 = \xi_1(2\xi_1 - 1)$$
$$N_2 = \xi_2(2\xi_2 - 1)$$
$$N_3 = \xi_3(2\xi_3 - 1)$$
$$N_4 = \xi_4(2\xi_4 - 1)$$
$$N_5 = 4\ \xi_1\xi_2$$
$$N_6 = 4\ \xi_2\xi_3$$
$$N_7 = 4\ \xi_3\xi_1$$
$$N_8 = 4\ \xi_1\xi_4$$
$$N_9 = 4\ \xi_2\xi_4$$
$$N_{10} = 4\ \xi_3\xi_4$$

Shape function of a node is product of equations of planes that do not contain that node; e.g. $N_{10} = 4\ \xi_3\ \xi_4$

Shape functions satisfy: $\Sigma N_I(\xi)=1$

8-node isoparametric hexadedron



Shape functions $N_I(\xi)$ $(I = 1,\ldots 8)$

$$N_I(\xi_1, \xi_2, \xi_3) =$$

$$= \frac{1}{8}(1 + \xi_1^I \xi_1)(1 + \xi_2^I \xi_2)(1 + \xi_3^I \xi_3)$$

Shape functions satisfy: $\sum N_I(\xi) = 1$

isoparametric hexadedron

$$\int_{V(\boldsymbol{\xi})} f(\xi_1, \xi_2, \xi_3)\, d\boldsymbol{\xi} \approx \sum_{I=1}^{n_I} \sum_{J=1}^{n_I} \sum_{K=1}^{n_I} w_I w_J w_K f(\xi_1^I, \xi_2^J, \xi_3^K)$$

1D Gaussian weights and points of [-1,1]

isoparametric tetrahedron

$$\int_{V(\boldsymbol{\xi})} f(\xi_1, \xi_2, \xi_3, \xi_4)\, d\boldsymbol{\xi} \approx \sum_{I=1}^{n_I} w_I f(\xi_1^I, \xi_2^I, \xi_3^I, \xi_4^I)$$

3D Gaussian weights and points calculated in master element

$n_I = 1$ – accuracy $O(h^2)$

$w_1 = 1$, $\xi^1 = (1/4, 1/4, 1/4, 1/4)$

$n_I = 5$ – accuracy $O(h^4)$

$w_1 = -4/5$, $\xi^1 = (1/4, 1/4, 1/4, 1/4)$

$w_{2,3,4,5} = 9/20$, $\xi^2 = (1/2, 1/6, 1/6, 1/6)$, others by cyclic symmetry

TORSION OF A BAR IN 3D; DISPLACEMENRT APPROACH

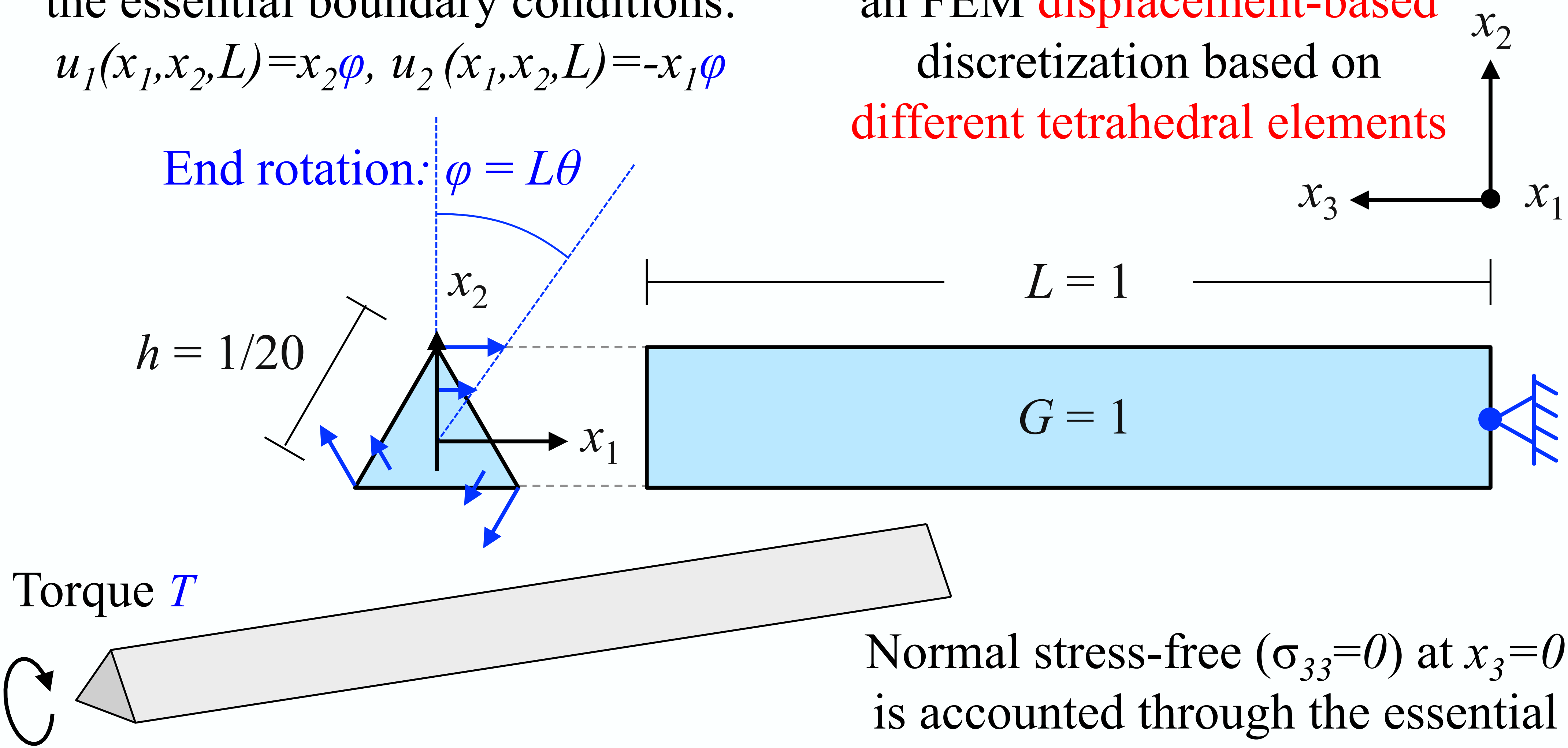
- There are problems in elasticity that are more easily solved with a stress-based formulation (and not with the displacement-based formulation that we have seen so far).

- One such case is the torsion of a prismatic bar with arbitrary section; away from the two ends (where a boundary layer may develop) the stresses are independent of the axial coordinate (depend solely on the two in-plane coordinates); the displacements depend on all three coordinates

- Here we solve the torsion problem of a triangular section bar using a displacement-based FEM formulation and 3D tetrahedral elements; we compare the solution to the analytical one (obtained using the stress-based formulation) and also show the development of a boundary layer at one end in the case where warping is prevented (full clamping).

End rotation at $x_3=L$ imposed through the essential boundary conditions:
$u_1(x_1,x_2,L)=x_2\varphi,\ u_2(x_1,x_2,L)=-x_1\varphi$

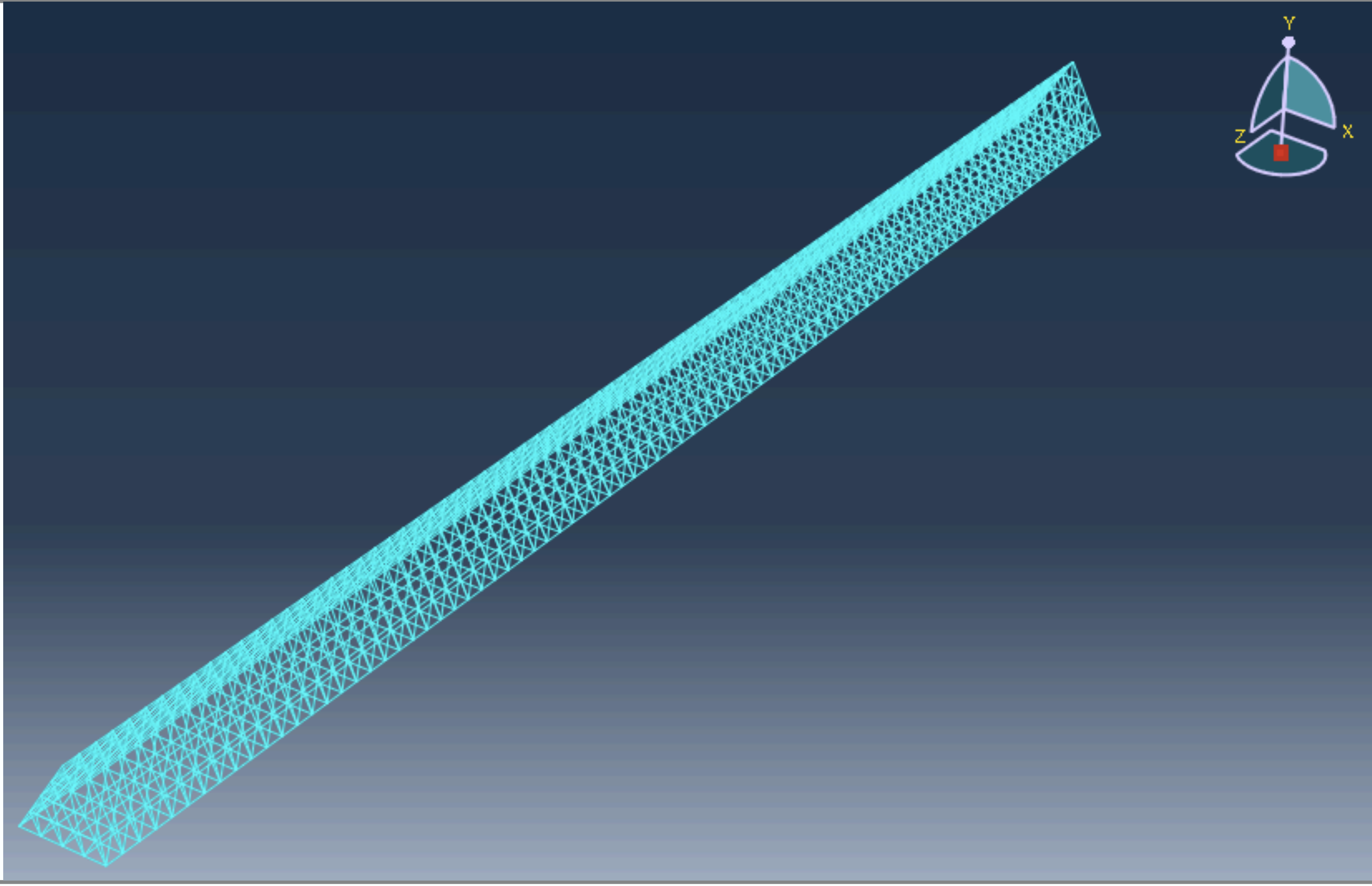
Prismatic bar analyzed using an FEM displacement-based discretization based on different tetrahedral elements



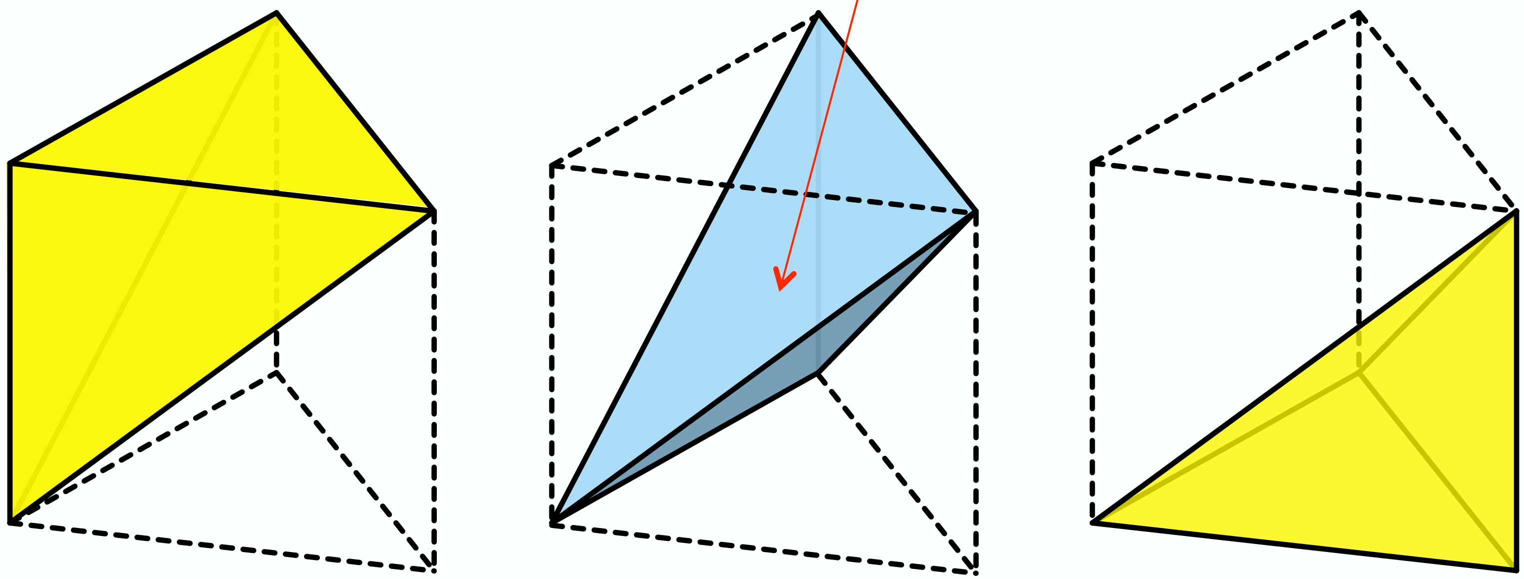
Normal stress-free ($\sigma_{33}=0$) at $x_3=0$ is accounted through the essential boundary conditions:
$u_1(x_1,x_2,0)=u_2(x_1,x_2,0)=0$

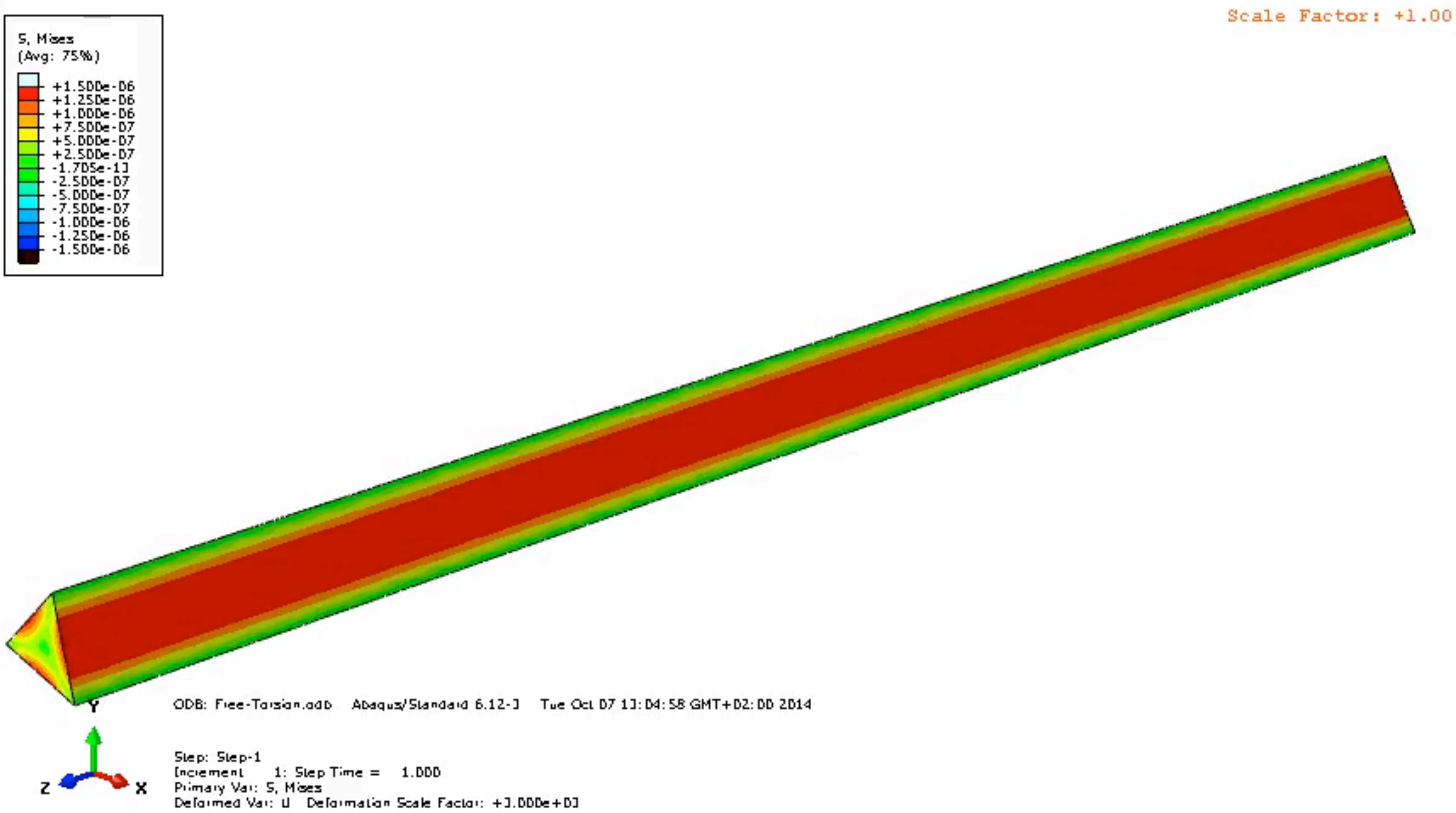
Prismatic bar subjected to rate of twist θ

PRISM'S TETRAHEDRAL MESH – 3979 ELEMENTS, 1259 NODES



NOTE on meshing: a straightforward division of the pentahedral prism in three tetrahedra gives one ill-conditioned element; as a result the code automatically changes meshes to have well-behaved elements (with gradients of the same order in all directions). This explains the rather bizzare number of total elements in the 3D prism





Scale Factor: +1.00
S, Mises
(Avg: 75%)
+1.500e-06
+1.250e-06
+1.000e-06
+7.500e-07
+5.000e-07
+2.500e-07
-1.705e-13
-2.500e-07
-5.000e-07
-7.500e-07
-1.000e-06
-1.250e-06
-1.500e-06
ODB: Free-Torsion.odb   Abaqus/Standard 6.12-1   Tue Oct 07 13:04:58 GMT+02:00 2014
Step: Step-1
Increment   1: Step Time =   1.000
Primary Var: S, Mises
Deformed Var: U   Deformation Scale Factor: +1.000e+01
Y
Z
X

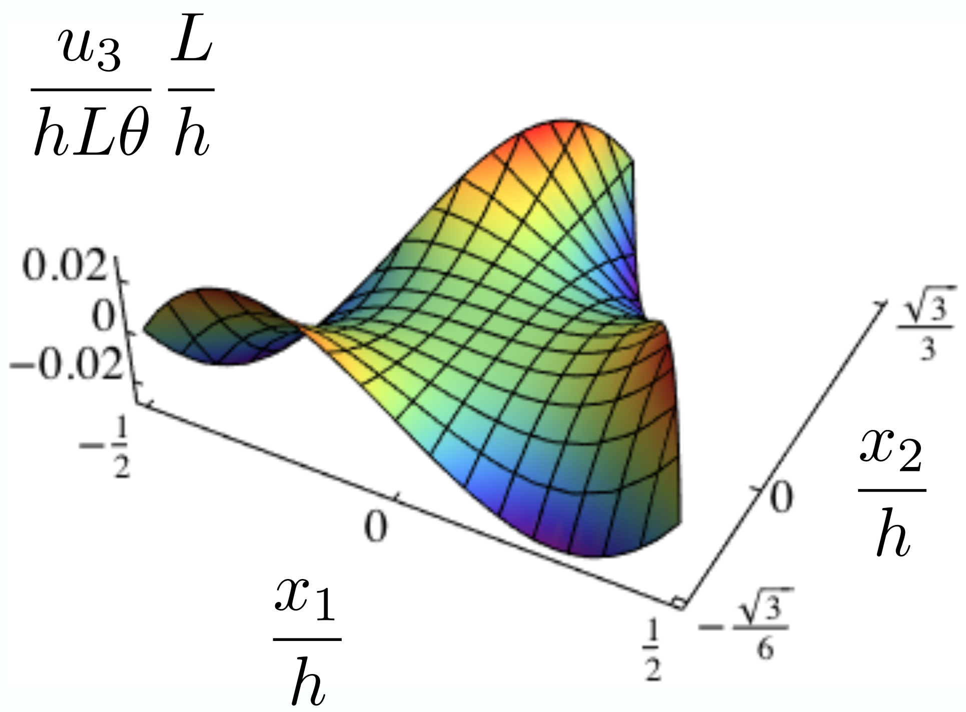
(non-dimensionalized) warping (i.e. $u_3(x_1, x_2)$) of each cross-section

Global displacement field (analytical)

$$\frac{u_1}{hL\theta} = \frac{x_3}{L}\frac{x_2}{h}$$

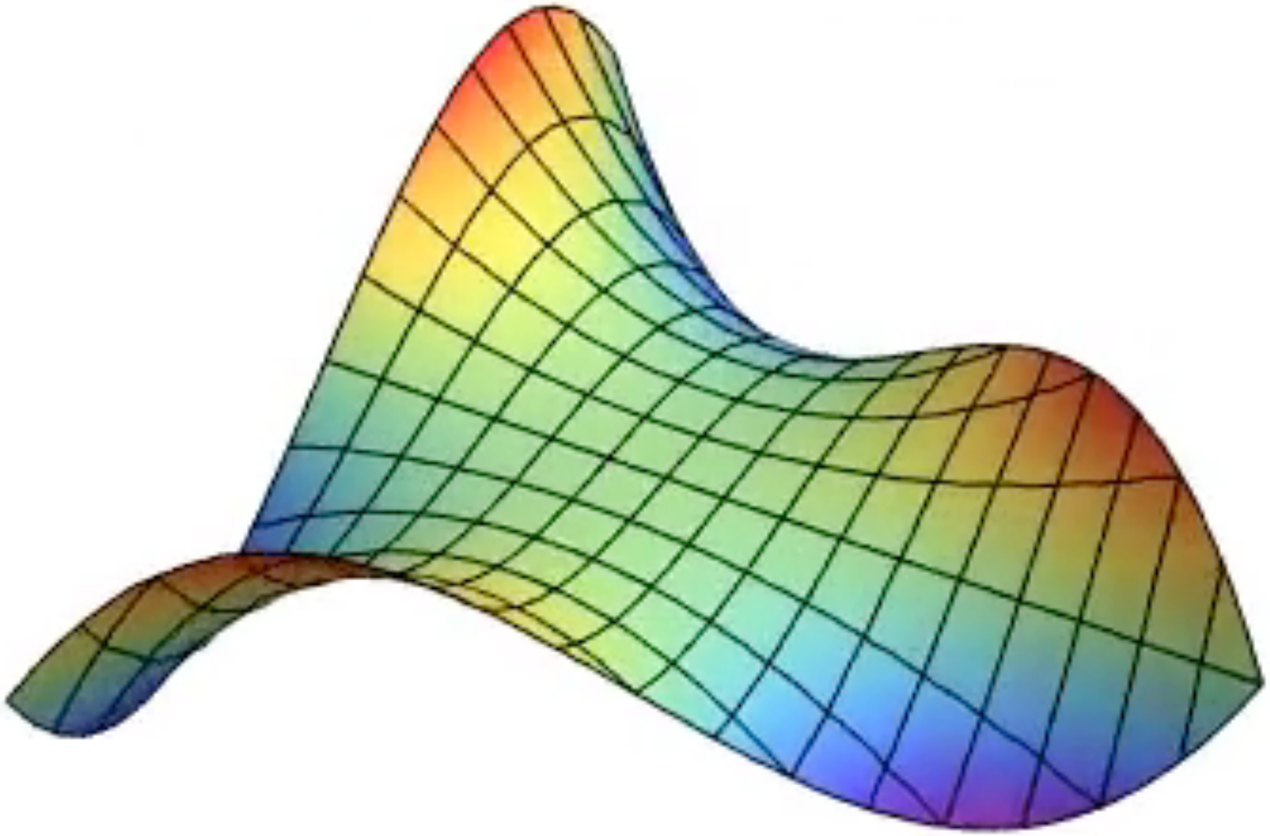
$$\frac{u_2}{hL\theta} = -\frac{x_3}{L}\frac{x_1}{h}$$

$$\frac{u_3}{hL\theta} = \sqrt{3}\frac{h}{L}\left[\frac{1}{3}\left(\frac{x_1}{h}\right)^3 - \frac{x_1}{h}\left(\frac{x_2}{h}\right)^2\right]$$

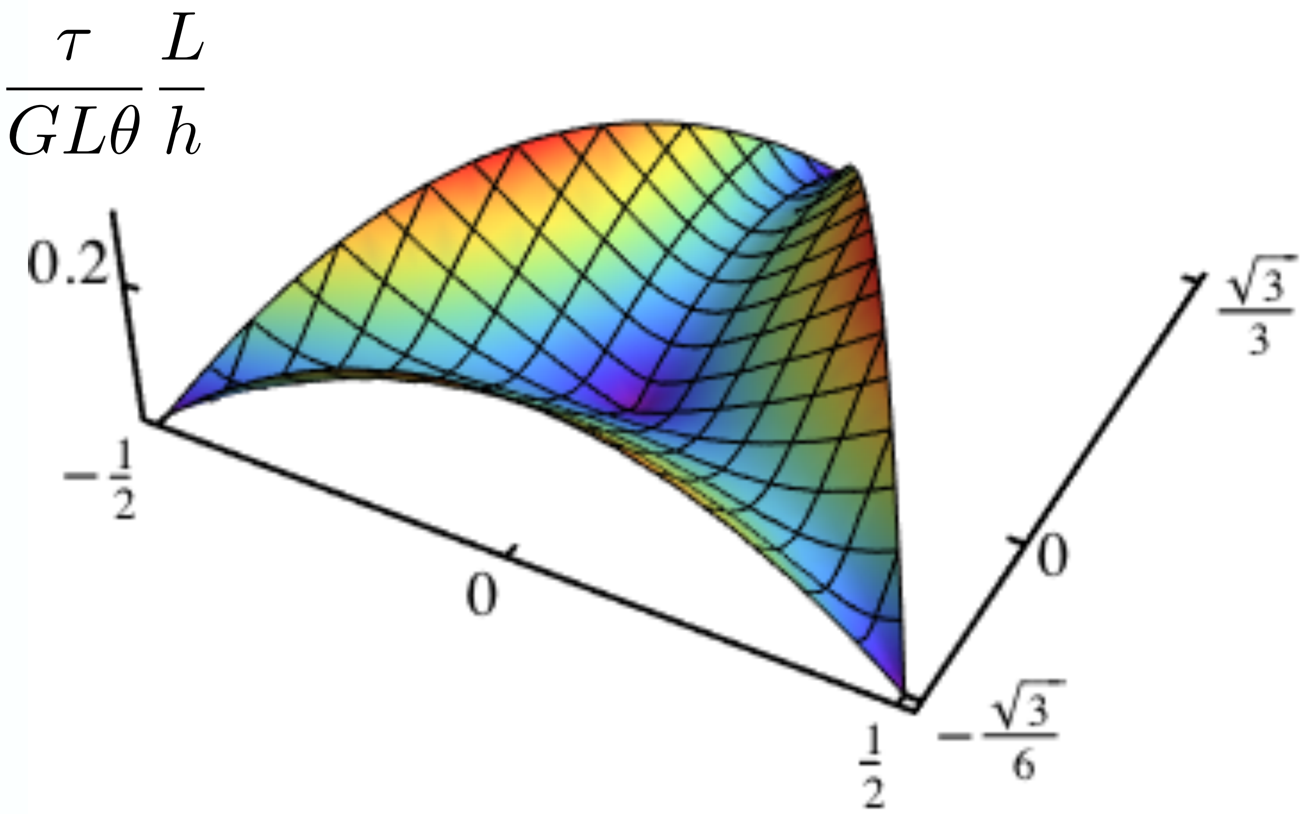


Notice the C_3 symmetry of the warping (i.e. $u_3(x_1, x_2)$) distribution;
Warping is zero at middle of the section and of the edges

WARPING (AXIAL DISPLAC. AT EACH SECTION – $u_3(x_1,x_2)$)

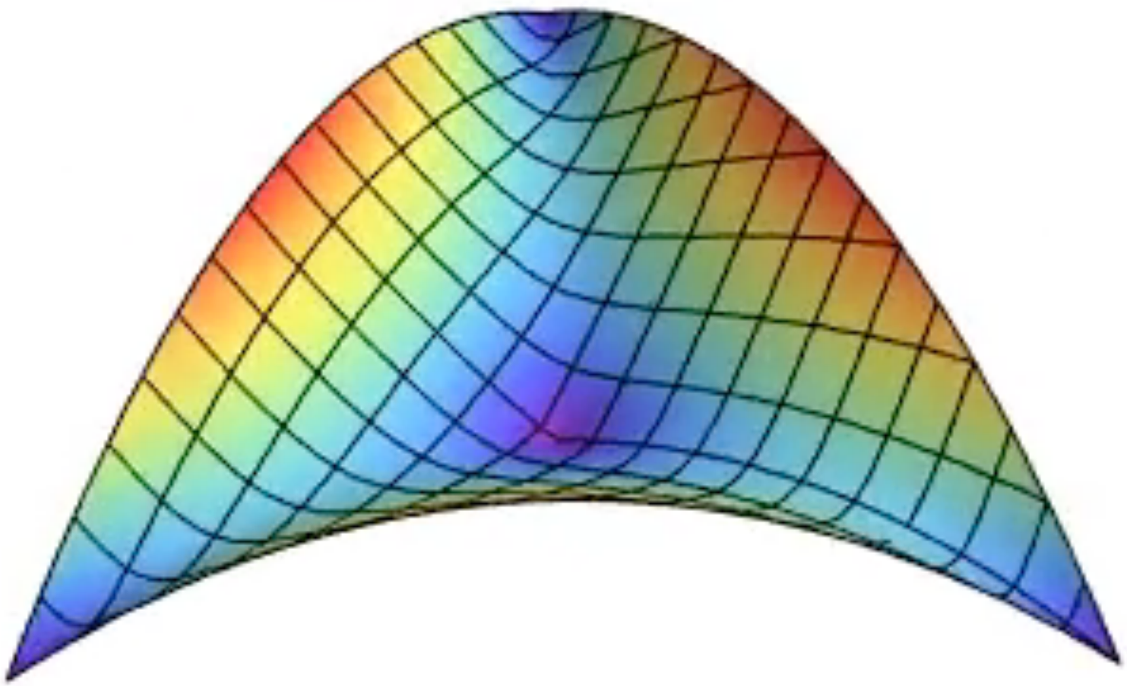


(non-dimensionalized) shear stress $\tau = [(\sigma_{31})^2 + (\sigma_{32})^2]^{1/2}$ on the cross-section



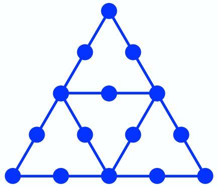
Notice the C_3 symmetry of the stress distribution;
Stress is zero at middle and at corners and maximum at middle of edges

STRESS STATE AT EACH SECTION (ONLY SHEAR EXISTS)

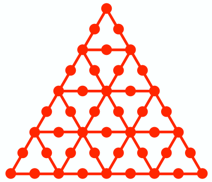


analytical solution $\left.\frac{\tau}{GL\theta}\frac{L}{\theta}\right|_{x_1=0} = \frac{\sqrt{3}}{2}\left|\frac{x_2}{h}\left(\frac{x_2}{h}-\frac{\sqrt{3}}{3}\right)\right|$

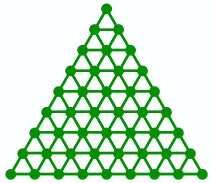
2 layers of quadratic elements

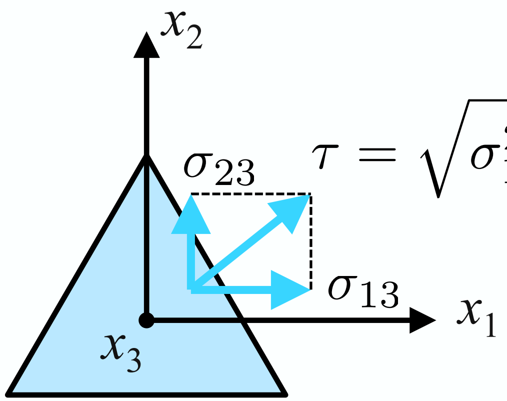


4 layers of quadratic elements

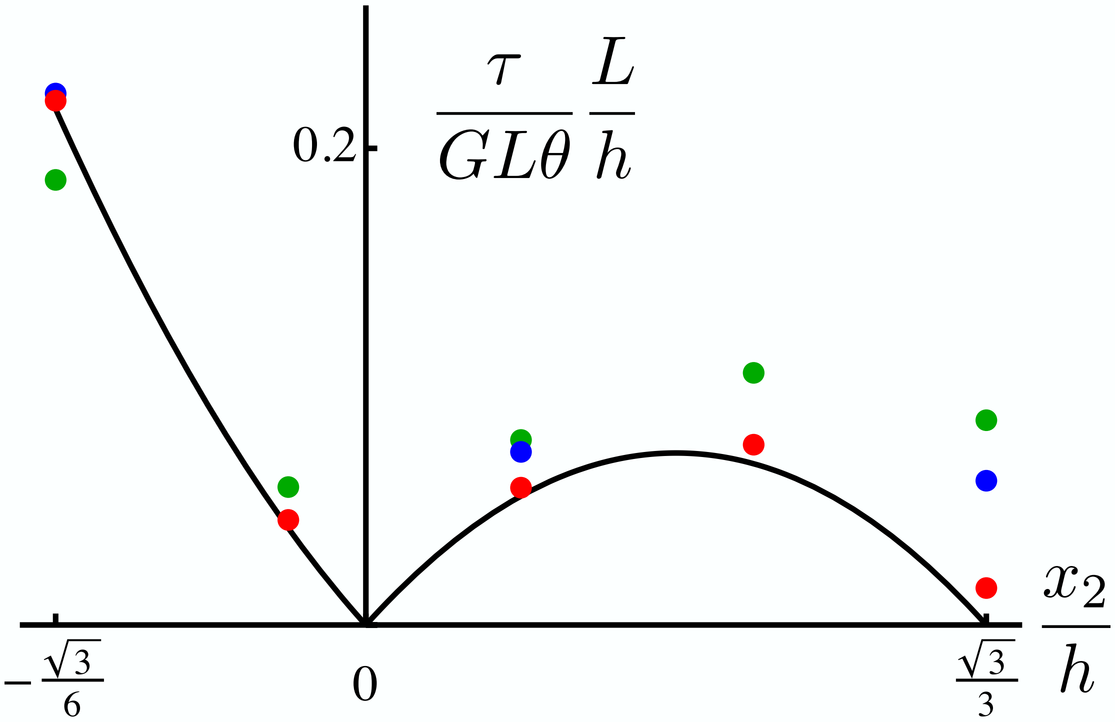


8 layers of linear elements



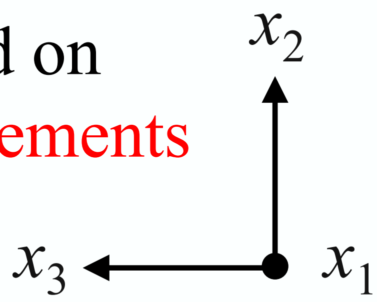


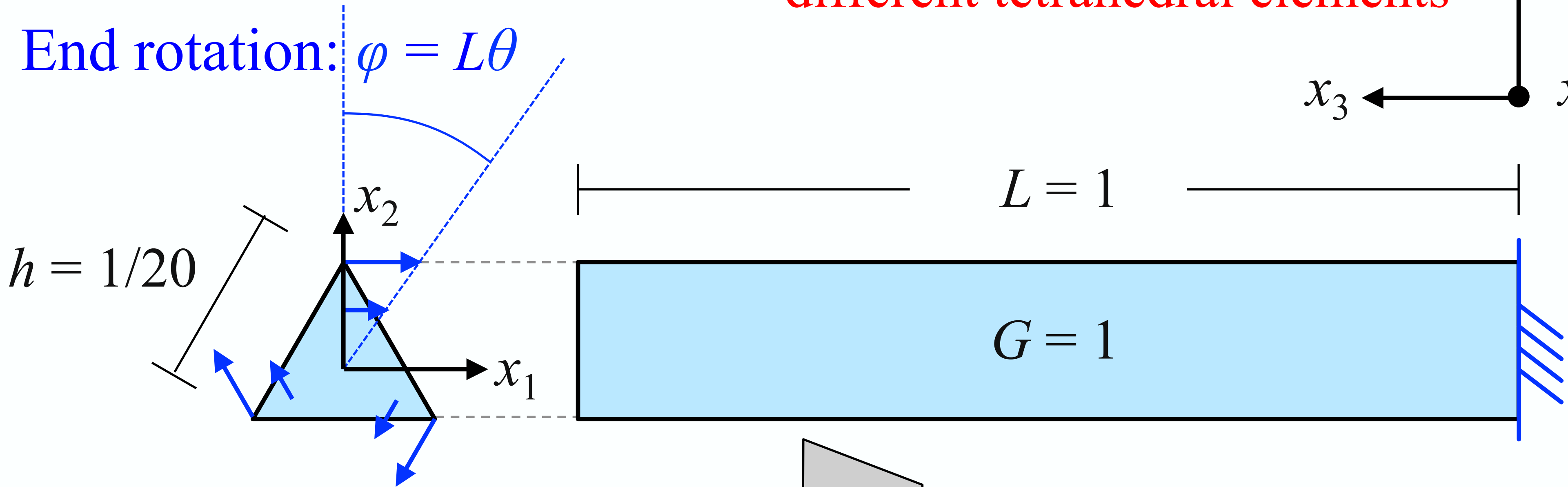
$\tau = \sqrt{\sigma_{13}^2 + \sigma_{23}^2}$

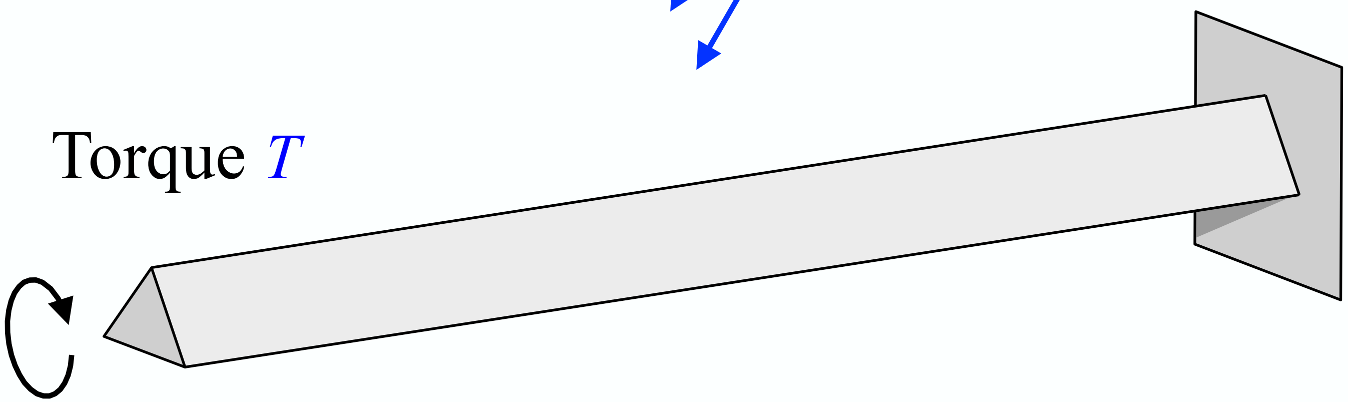


End rotation at $x_3=L$ imposed through the essential boundary conditions:
$u_1(x_1,x_2,L)=x_2\varphi,\ u_2(x_1,x_2,L)=-x_1\varphi$

Prismatic bar analyzed using an FEM displacement-based discretization based on different tetrahedral elements





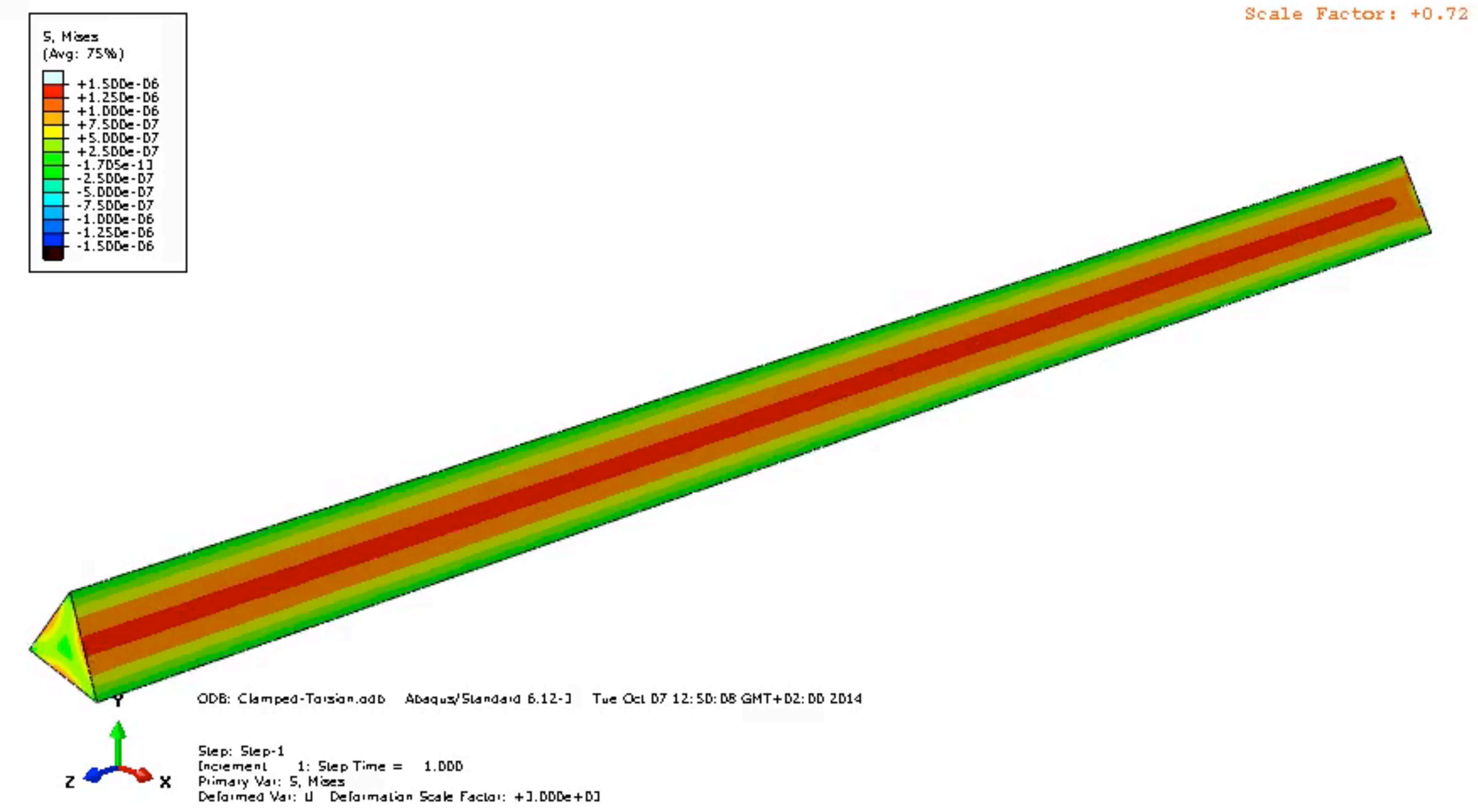


Clamped support at $x_3=0$:
$u_i(x_1,x_2,0)=0\ (i = 1,2,3)$,

Prismatic bar subjected to rate of twist θ

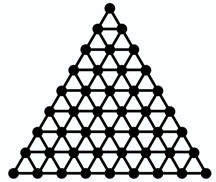
Boundary layer develops!

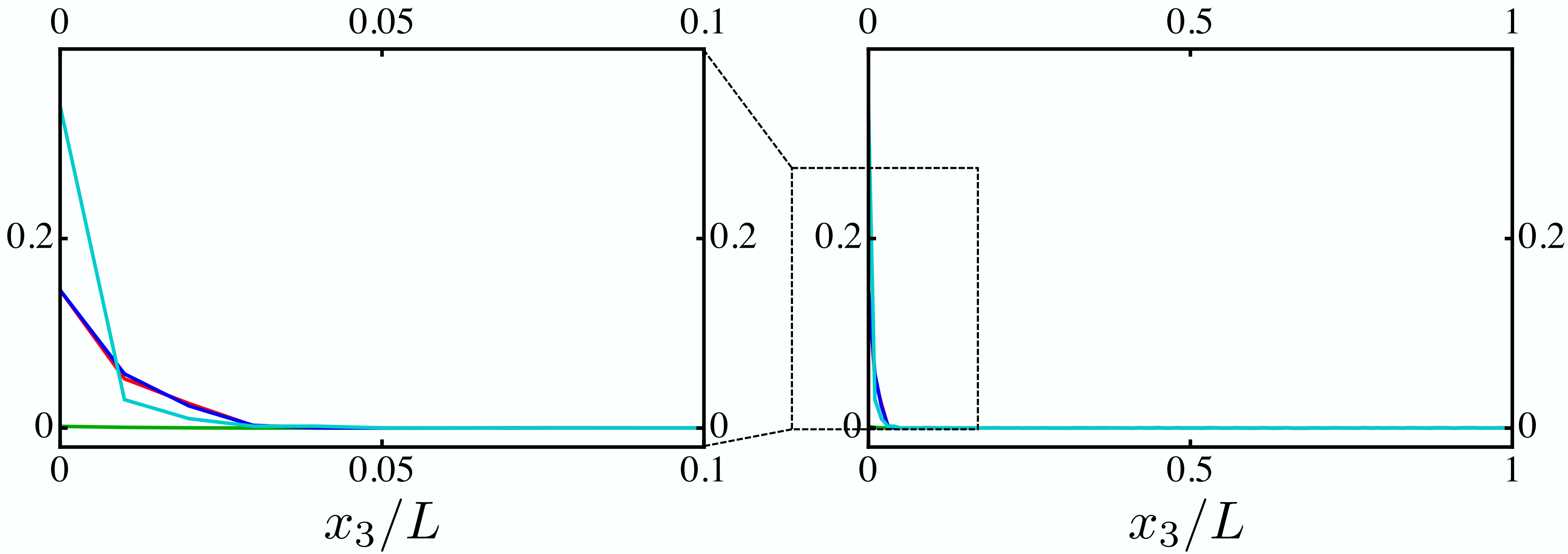
TORSION OF A TRIANGULAR SECTION BAR CLAMPED AT $x_3=0$



$\frac{\sigma_{11}}{GL\theta}\frac{L}{h}$ $\frac{\sigma_{22}}{GL\theta}\frac{L}{h}$ $\frac{\sigma_{12}}{GL\theta}\frac{L}{h}$ $\frac{\sigma_{33}}{GL\theta}\frac{L}{h}$

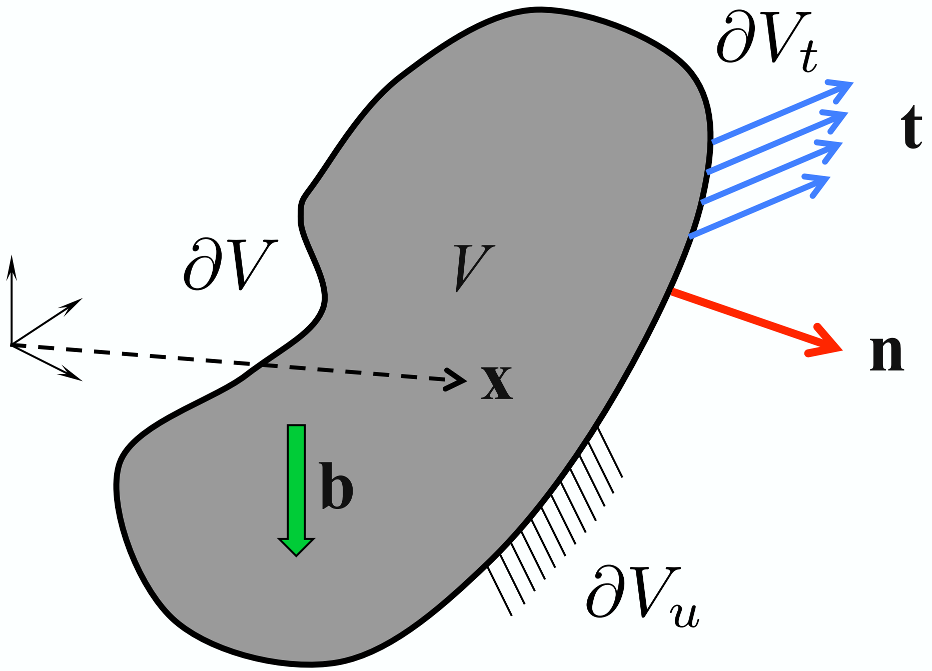
8 layers of linear elements





Since warping in prevented at $x_3=0$, normal (and a tiny in-plane shear) stresses develop in a boundary layer of the order of the section thickness

PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY



Solid occupies domain: V

Domain boundary: ∂V

Body forces: **b**

Surface traction: **t**

Surface normal (outward): **n**

Traction prescribed on: ∂V_t

Displacement prescribed on: ∂V_u

Position vector: **x**

Complementary energy density: $W^*(\boldsymbol{\sigma})$

Strain-stress: $\epsilon_{ij} = \dfrac{\partial W^*}{\partial \sigma_{ij}}$

(general nonlinear elastic material)

$$\text{Complementary} \quad : \quad \mathcal{P}^* = \mathcal{P}^*_{int} + \mathcal{P}^*_{ext}$$

$$\text{Internal} \quad : \quad \mathcal{P}^*_{int} = \int_V W^*(\sigma_{ij})\, dV \; ; \qquad \epsilon_{ij} = \frac{\partial W^*}{\partial \sigma_{ij}}$$

$$\text{External} \quad : \quad \mathcal{P}^*_{ext} = -\int_V b_i u_i \, dV - \int_{\partial V_u} t_i u_i \, dS$$

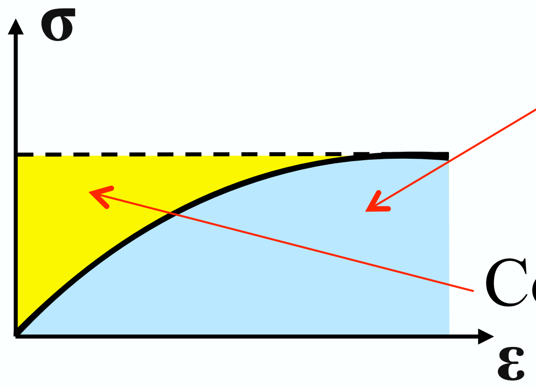
$$\mathcal{P}^*(\boldsymbol{\sigma} + \epsilon \delta \boldsymbol{\sigma}) \geq \mathcal{P}^*(\boldsymbol{\sigma}) \; ;$$

Equilibrium at minimum over all statically admissible stress fields

$$\nabla \bullet \delta\boldsymbol{\sigma}(\mathbf{x}) = 0, \;\; \forall \; \mathbf{x} \in V$$
$$\mathbf{n} \bullet \delta\boldsymbol{\sigma}(\mathbf{x}) = 0, \;\; \forall \; \mathbf{x} \in \partial V_t$$

$$\frac{d}{d\epsilon}\left[\mathcal{P}^*(\boldsymbol{\sigma} + \epsilon \delta \boldsymbol{\sigma})\right]_{\epsilon=0} = 0 \; ; \qquad \text{extremum (1)}$$

$$\frac{d^2}{d\epsilon^2}\left[\mathcal{P}^*(\boldsymbol{\sigma} + \epsilon \delta \boldsymbol{\sigma})\right]_{\epsilon=0} > 0 \; ; \qquad \text{minimum (2)}$$



Potential energy: $W(\boldsymbol{\epsilon}) = \int_{\mathbf{0}}^{\boldsymbol{\epsilon}} \sigma_{ij}\, d\epsilon_{ij}$

Complementary energy: $W^*(\boldsymbol{\sigma}) = \int_{\mathbf{0}}^{\boldsymbol{\sigma}} \epsilon_{ij}\, d\sigma_{ij}$

Equilibrium : $\dfrac{\partial \sigma_{ij}}{\partial x_i} + b_j = 0\ ,\ \ \mathbf{x} \in V$

Boundary : $t_j = n_i\, \sigma_{ij}\ ,\ \ \mathbf{x} \in \partial V_t$

static admissibility condition

Constitutive (Linear) : $\epsilon_{ij} = L^{-1}_{ijkl} \sigma_{kl}$

Complementary (Linear) : $W^* = \int_{\mathbf{0}}^{\boldsymbol{\sigma}} \epsilon_{ij}\, d\sigma_{ij} = \frac{1}{2} L^{-1}_{ijkl}\, \sigma_{ij}\, \sigma_{kl}$

TORSION OF A BAR IN 3D; STRESS-BASED APPROACH

$$\mathcal{P}^*_{int} = \int_V W^*(\sigma_{ij})\, dV = \int_V \frac{1}{2G}\left[(\sigma_{32})^2 + (\sigma_{31})^2\right]\, dV$$

$$= L\int_A \frac{1}{2G}\left[\left(\frac{\partial\psi}{\partial x_1}\right)^2 + \left(\frac{\partial\psi}{\partial x_2}\right)^2\right]\, dA$$

$$\mathcal{P}^*_{ext} = -\int_{\partial V_u} t_i u_i\, dS = -\int_A \left[\sigma_{31}u_1 + \sigma_{32}u_2\right]\, dA$$

$$= -\int_A \left[\sigma_{31}(\phi x_2) + \sigma_{32}(-\phi x_1)\right]\, dA = -T\phi = -TL\theta$$

$$\text{Torque } T = \int_A \left[\left(\frac{\partial\psi}{\partial x_2}\right)x_2 + \left(\frac{\partial\psi}{\partial x_1}\right)x_1\right]\, dA = 2\int_A \psi\, dA$$

Torsion for bar of arbitrary section A using (Prandlt) stress potential ψ:

$$\mathcal{P}^*_{int} = L \int_A \frac{1}{2G} \left[\left(\frac{\partial \psi}{\partial x_1} \right)^2 + \left(\frac{\partial \psi}{\partial x_2} \right)^2 \right] dA$$

$$\mathcal{P}^*_{ext} = -2L\theta \int_A \psi \, dA$$

$$\text{Admissibility} \; : \quad \psi = 0 \quad \forall \mathbf{x} \in \partial A \quad (\sigma_{31} n_1 + \sigma_{32} n_2 = 0)$$

$$\text{Euler} - \text{Lagrange} \; : \quad \boxed{\frac{\partial^2 \psi}{\partial x_1 \partial x_1} + \frac{\partial^2 \psi}{\partial x_2 \partial x_2} - 2G\theta = 0} \quad \forall \mathbf{x} \in A$$

Prandlt function $\psi(x_1, x_2) = \frac{\sqrt{3}\theta G}{12h}\left(\sqrt{3}h\left(x_1^2 + x_2^2\right) + 6x_1^2 x_2 - 2x_2^3\right)$

$$\sigma_{31}(x_1, x_2) = \frac{\partial \psi}{\partial x_2} = Gh\theta\left[\frac{\sqrt{3}}{2}\left(\left(\frac{x_1}{h}\right)^2 - \left(\frac{x_2}{h}\right)^2\right) + \frac{x_2}{2h}\right]$$

$$\sigma_{32}(x_1, x_2) = -\frac{\partial \psi}{\partial x_1} = -Gh\theta\left(\frac{x_1}{2h} + \sqrt{3}\frac{x_1 x_2}{h^2}\right)$$

Shear stress $\tau(x_1, x_2) \equiv \sqrt{\sigma_{31}^2 + \sigma_{32}^2}$

$$\text{NOTE}: \quad \sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{33} = 0$$

FEM DISCRETIZATION USING COMPLEMENTARY ENERGY

We discretize the (Prandlt) stress potential $\psi(x_1,x_2)$ over section A

$$\mathcal{P}^* = L \int_A \left\{ \frac{1}{2G}\left[\left(\frac{\partial\psi}{\partial x_1}\right)^2 + \left(\frac{\partial\psi}{\partial x_2}\right)^2\right] - 2\theta\psi \right\} dA$$

$$\frac{1}{L}\delta\mathcal{P}^* = \int_A \left\{ \frac{1}{G}\left[\frac{\partial\psi}{\partial x_1}\frac{\partial\delta\psi}{\partial x_1} + \frac{\partial\psi}{\partial x_2}\frac{\partial\delta\psi}{\partial x_2}\right] - 2\theta\delta\psi \right\} dA = 0$$

discretization : $\psi = \mathbf{N}\mathbf{q}_e, \quad \mathbf{q}_e = [\Psi^1, \Psi^2, \Psi^3, \cdots]^T$

discretization : $\left[\frac{\partial\psi}{\partial x_1}, \frac{\partial\psi}{\partial x_1}\right]^T = \mathbf{B}\mathbf{q}_e$

element : $\mathbf{k}_e = \frac{1}{G}\int_{A_e} \left[\mathbf{B}^T\mathbf{B}\right] dA, \ \mathbf{f}_e = 2\theta \int_{A_e} \left[\mathbf{N}^T\right] dA$

ANNOUNCEMENT

PROGRAM CHANGE:

NOV 9: Lecture 6 at 10:45 in PC64

NOV 9: Lecture 7 at 16:15 IN PC65

NOV 16: PC 6 at 10:45 in Salle Info n°31

NOV 16: PC 7 at 16:15 in Salle Info n°31