TOPICS COVERED IN THIS LECTURE

1. CONVERGENCE REQUIREMENTS, PATCH TEST & OTHER REMARKS

2. SINGULARITIES AT NOTCHES (AND CRACKS) IN 2D

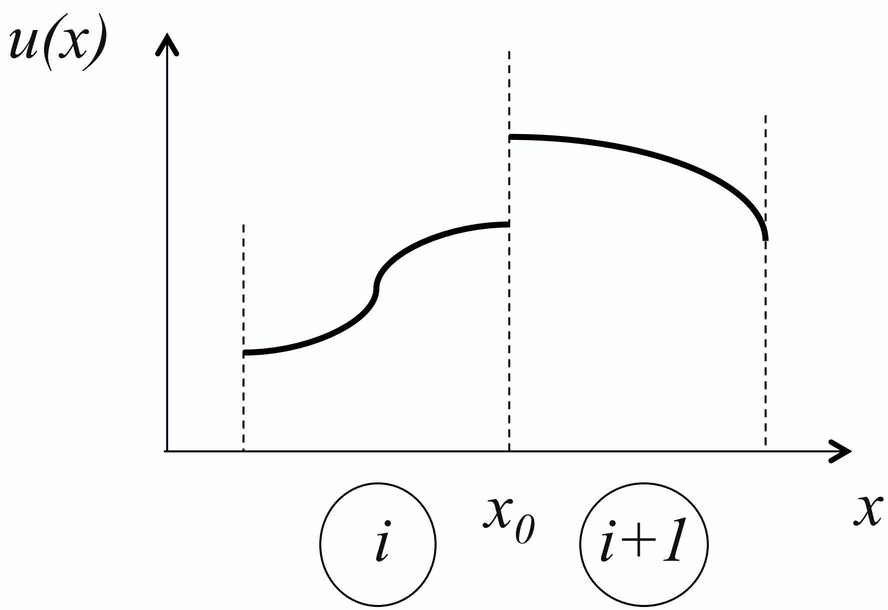
3. STRESS-INTENSITY FACTORS VIA INTERPOLATION AND SPECIAL ELEMENTS

4. STRESS-INTENSITY FACTORS USING ENERGY METHODS

CONVERGENCE REQUIREMENTS & OTHER REMARKS

Sufficient conditions for convergence in linear elasticity problems solved using the FEM are C^0 continuity and completeness of interpolation functions, i.e. their capability of representing exactly constant strain solutions (or equivalently any linear displacement field)

- Inter-element continuity needed for finite energy of the approximation



$$\frac{du}{dx} = \delta(x - x_0) + \cdots$$

$$\int_x \left(\frac{du}{dx}\right)^2 dx = \int_x \delta^2 \, dx = \infty$$

As mesh size (max size of element) $h \rightarrow 0$, the strains in each element are almost constant. Consequently FEM interpolation must be able to represent any constant strain field and shape functions must be able to represent exactly an arbitrary linear displacement field (property known under the name: patch test)

$$\text{linear field}: u_i(\boldsymbol{\xi}) = \alpha_{ij} x_j(\boldsymbol{\xi}) + \beta_i \ , \ U_i^J = \alpha_{ij} x_j^J + \beta_i$$

$$\text{interpolation}: u_i(\boldsymbol{\xi}) = \sum_J N_J(\boldsymbol{\xi}) U_i^J = \sum_J N_J(\boldsymbol{\xi})(\alpha_{ij} x_j^J + \beta_i)$$

$$\text{interpolation}: x_i(\boldsymbol{\xi}) = \sum_J N_J(\boldsymbol{\xi}) x_i^J$$

$$\text{combine}: u_i(\boldsymbol{\xi}) = \alpha_{ij} x_j(\boldsymbol{\xi}) + \beta_i [\sum_J N_J(\boldsymbol{\xi})] \Longrightarrow \boxed{\sum_J N_J(\boldsymbol{\xi}) = 1}$$

REMARKS ON RIGID BODY MOTION

Satisfying the patch test requirement, includes accounting for rigid body translations and infinitesimal rotations.

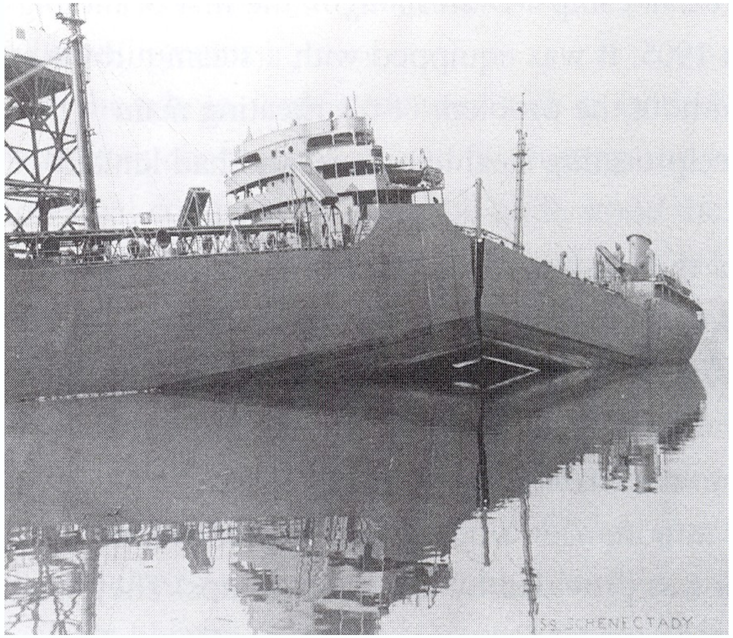
Recall that FEM discretization can include arbitrary linear displacement fields: $u_i = \alpha_{ij} x_j + \beta_i$ and thus α_{ij} can include infinitesimal rotations (antisymmetric tensors $\Omega_{ij} = -\Omega_{ji}$) and β_i accounts for translations

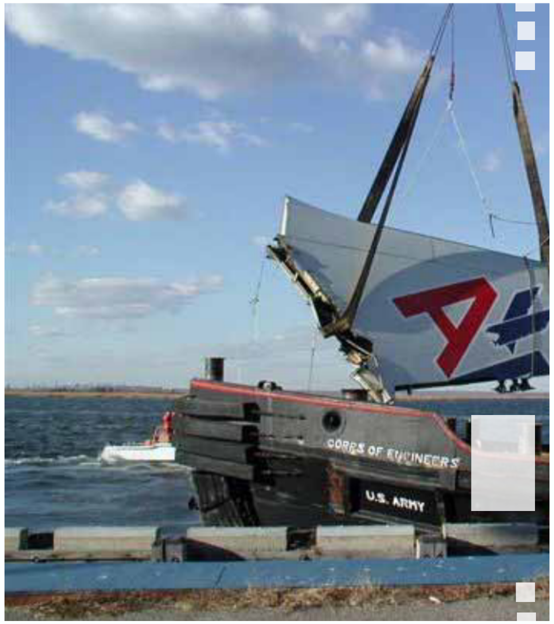
NOTE: Small strain linear elasticity is invariant under infinitesimal strains but is not invariant under finite strains!

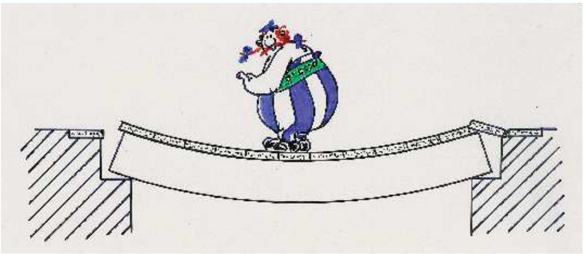
For displacement field $u_i = [R_{ij} - \delta_{ij}]x_j$, you can easily see that the strains are $\varepsilon_{ij} = [R_{ij} + R_{ji}]$ that does not vanish for a finite rotation R_{ij}

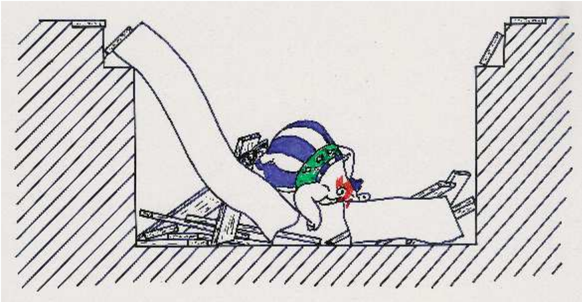
NOTE: Finite strain theories are needed to correct this problem!

SINGULARITIES OF NOTCHES (AND CRACKS) IN 2D









Fracture plays an important role in engineering design. Inevitable flaws (in the form of cracks) can grow uncontrollably, if they exceed a certain size

Pictures above are: from failure of *US Schenectady* (1943), *de Haviland Comet* (1950's) and *Flight AA587* (2001)

- Linear Elastic Fracture Mechanics (LEFM) is a successful theory for designing against inevitable flaws in structures

- Stresses are infinite at ends of flaws (crack tips) but their known singularity has an amplitude called stress intensity factor

- The stress intensity factor is related to the energy required to advance the crack, a measurable material property (energy release rate). When the stress intensity factor is lower than the one corresponding to the critical energy release rate the crack does not advance

- In LEFM we calculate using FEM the stress intensity factor for a given structure and loading and check if it is less than its critical value

- LEFM is a successful theory for brittle solids where the process zone ahead of the crack tip is small compared to the flaw dimensions

In the absence of body forces, boundary value problems in 2D isotropic linear elasticity (plane stress or plane strain) and be found using the Airy stress function φ (equilibrium automatically satisfied).

$$\text{2D elasticity} \;:\; \text{must find a biharmonic function } \phi \;:\; \nabla^4\phi = 0$$

$$\text{cartesian} \;:\; \nabla^4\phi = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\left(\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2}\right) = 0$$

$$\text{cartesian} \;:\; \sigma_{11} = \frac{\partial^2\phi}{\partial x_2 \partial x_2}\,,\;\; \sigma_{22} = \frac{\partial^2\phi}{\partial x_1 \partial x_1}\,,\;\; \sigma_{12} = -\frac{\partial^2\phi}{\partial x_1 \partial x_2}$$

$$\text{polar} \;:\; \nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2}\right) = 0$$

$$\text{polar} \;:\; \sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2}\,,\;\; \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}\,,\;\; \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial \theta}\right)$$

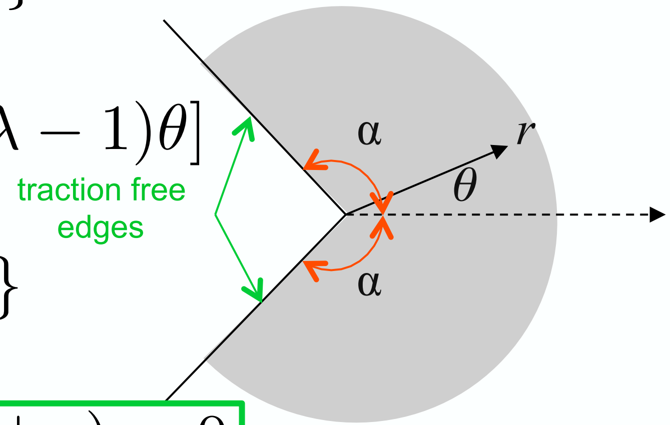
Interested in finding leading order terms near tip of elastic wedge:

$$\phi = r^{\lambda+1} f(\theta) \implies \left(\frac{d^2}{d\theta^2} + (\lambda-1)^2\right)\left(\frac{d^2}{d\theta^2} + (\lambda+1)^2\right) f(\theta) = 0$$

$$f(\theta) = A_1 \cos[(\lambda+1)\theta] + A_2 \cos[(\lambda-1)\theta] + A_3 \sin[(\lambda+1)\theta] + A_4 \sin[(\lambda-1)\theta]$$

$$\sigma_{r\theta} = \lambda r^{\lambda-1}\{A_1(\lambda+1)\sin[(\lambda+1)\theta] + A_2(\lambda-1)\sin[(\lambda-1)\theta]$$
$$-A_3(\lambda+1)\cos[(\lambda+1)\theta] - A_4(\lambda-1)\cos[(\lambda-1)\theta]\}$$

$$\sigma_{\theta\theta} = \lambda r^{\lambda-1}\{A_1(\lambda+1)\cos[(\lambda+1)\theta] + A_2(\lambda+1)\cos[(\lambda-1)\theta]$$
$$+A_3(\lambda+1)\sin[(\lambda+1)\theta] + A_4(\lambda+1)\sin[(\lambda-1)\theta]\}$$



λ found from traction free edges : $\sigma_{r\theta}(r, \pm\alpha) = \sigma_{\theta\theta}(r, \pm\alpha) = 0$

NOTCH PROBLEM – SINGULARITY NEAR TIP

From stresses one can then calculate the corresponding displacements:

$$\epsilon_{rr} = \frac{1}{8G}[(\kappa+1)\sigma_{rr} - (3-\kappa)\sigma_{\theta\theta}] = \frac{\partial u_r}{\partial r}$$

$$\epsilon_{\theta\theta} = \frac{1}{8G}[(\kappa+1)\sigma_{\theta\theta} - (3-\kappa)\sigma_{rr}] = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\epsilon_{r\theta} = \frac{1}{2G}\sigma_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

NOTE: only displacement expressions depend on plane stress or strain (via Kolosov's constant κ);

$$\text{Kolosov's constant}: \ \kappa = 3 - 4\nu \ \text{plane strain}; \ \kappa = (3-\nu)/(1+\nu) \ \text{plane stress}$$

$$2Gu_r = r^\lambda \{-A_1(\lambda+1)\cos[(\lambda+1)\theta] + A_2(\kappa-\lambda)\cos[(\lambda-1)\theta]$$

$$-A_3(\lambda+1)\sin[(\lambda+1)\theta] + A_4(\kappa-\lambda)\sin[(\lambda-1)\theta]$$

$$2Gu_\theta = r^\lambda \{A_1(\lambda+1)\sin[(\lambda+1)\theta] + A_2(\kappa+\lambda)\sin[(\lambda-1)\theta]$$

$$-A_3(\lambda+1)\cos[(\lambda+1)\theta] - A_4(\kappa+\lambda)\cos[(\lambda-1)\theta]$$

$$\begin{bmatrix} (\lambda+1)\sin[(\lambda+1)\alpha] & (\lambda-1)\sin[(\lambda-1)\alpha] \\ (\lambda+1)\cos[(\lambda+1)\alpha] & (\lambda+1)\cos[(\lambda-1)\alpha] \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

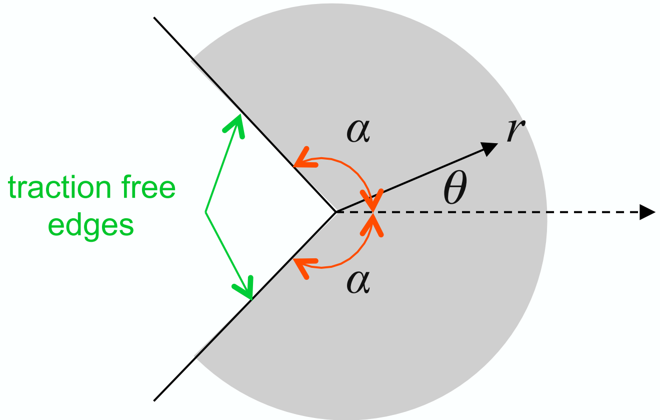
symmetric solution : $A_1 \neq 0,\ A_2 \neq 0,\ (A_3 = A_4 = 0)$

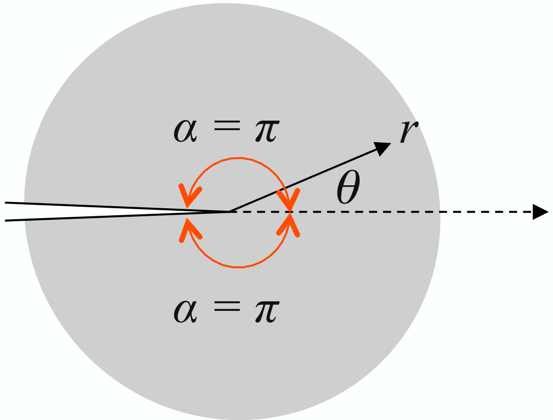
$\lambda \sin(2\alpha) + \sin(2\lambda\alpha) = 0,$ solution : λ_S

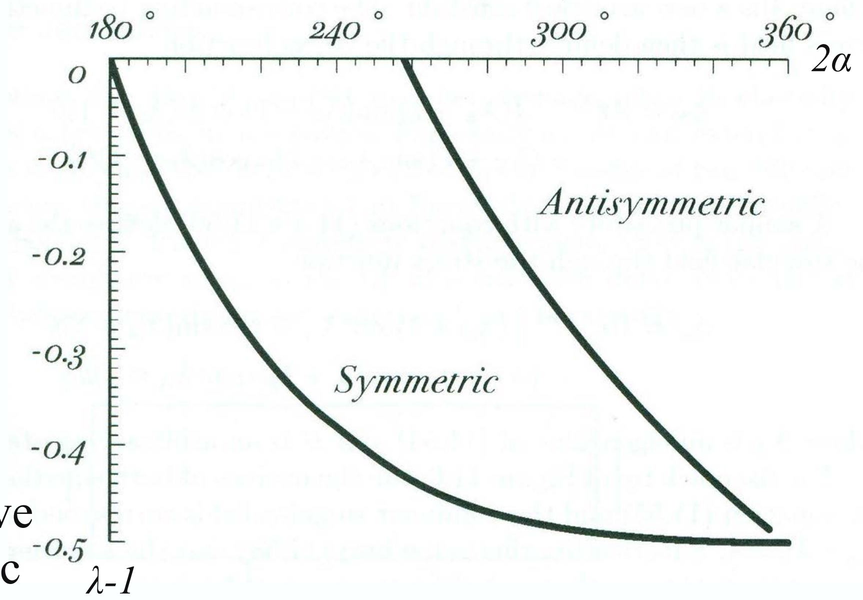
$$\begin{bmatrix} (\lambda+1)\cos[(\lambda+1)\alpha] & (\lambda-1)\cos[(\lambda-1)\alpha] \\ (\lambda+1)\sin[(\lambda+1)\alpha] & (\lambda+1)\sin[(\lambda-1)\alpha] \end{bmatrix} \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

antisymmetric solution : $(A_1 = A_2 = 0),\ A_3 \neq 0,\ A_4 \neq 0$

$\lambda \sin(2\alpha) - \sin(2\lambda\alpha) = 0,$ solution : λ_A







For $\pi/2 < \alpha < \pi$ (notch) we have symmetric and anti-symmetric solutions with $\lambda_A > \lambda_S$

For $\alpha = \pi$ (crack) we still have symmetric and anti-symmetric solutions with $\lambda_A = \lambda_S = 0.5$

Solving systems for symmetric (A_1, A_2) and anti-symmetric (A_3, A_4) case, we get the corresponding Airy stress functions φ_S and φ_A. The unknown coefficients A and B will be related with stress intensity factors K that depends on the geometry and loading of the structure.

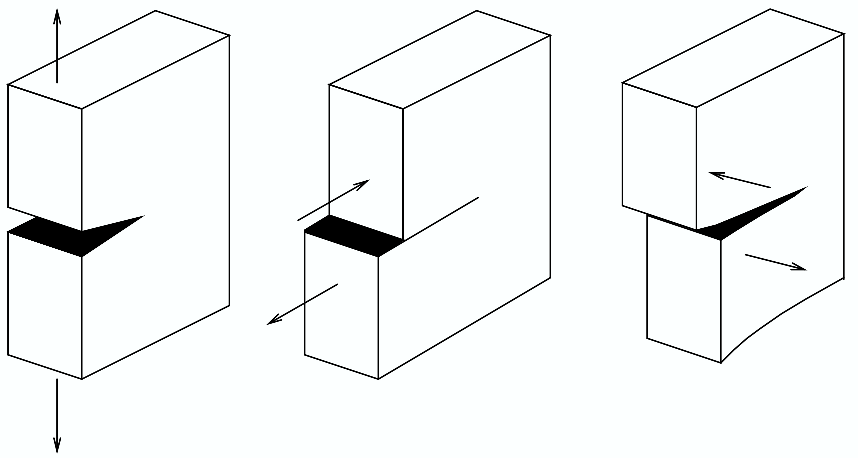
$$\text{symmetric}: \quad A_1 = A(\lambda_S - 1)\sin[(\lambda_S - 1)\alpha], \quad A_2 = -A(\lambda_S + 1)\sin[(\lambda_S + 1)\alpha],$$

$$\phi_S = A r^{\lambda_S + 1}\{(\lambda_S - 1)\sin[(\lambda_S - 1)\alpha]\cos[(\lambda_S + 1)\theta] - (\lambda_S + 1)\sin[(\lambda_S + 1)\alpha]\cos[(\lambda_S - 1)\theta]\}$$

$$\text{antisym.}: \quad A_3 = B(\lambda_A + 1)\sin[(\lambda_A - 1)\alpha], \quad A_4 = -B(\lambda_A + 1)\sin[(\lambda_A + 1)\alpha],$$

$$\phi_A = B r^{\lambda_A + 1}\{(\lambda_A - 1)\sin[(\lambda_A - 1)\alpha]\sin[(\lambda_A + 1)\theta] - (\lambda_A + 1)\sin[(\lambda_A + 1)\alpha]\sin[(\lambda_A - 1)\theta]\}$$

- K_I and K_{II} are called the stress-intensity factors in Mode-I and Mode-II loading.
- Crack propagates if they exceed a critical value (fracture toughness)



Mode I: symmetric solution (2D)

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{5}{4}\cos\left(\frac{\theta}{2}\right) - \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\cos\left(\frac{\theta}{2}\right) + \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\left(\frac{\theta}{2}\right) + \frac{1}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

Mode II: anti-symmetric solution (2D)

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{5}{4}\sin\left(\frac{\theta}{2}\right) + \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{\theta\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{3}{4}\sin\left(\frac{\theta}{2}\right) - \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\left(\frac{\theta}{2}\right) + \frac{3}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

The cartesian components of stresses and displacements near the crack tip are:

Mode I: symmetric solution (2D)

$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}}\left[\cos(\tfrac{1}{2}\theta)\left\{1-\sin(\tfrac{1}{2}\theta)\sin(\tfrac{3}{2}\theta)\right\}\right]$$

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}}\left[\cos(\tfrac{1}{2}\theta)\left\{1+\sin(\tfrac{1}{2}\theta)\sin(\tfrac{3}{2}\theta)\right\}\right]$$

$$\sigma_{12} = \frac{K_I}{\sqrt{2\pi r}}\left[\cos(\tfrac{1}{2}\theta)\sin(\tfrac{1}{2}\theta)\cos(\tfrac{3}{2}\theta)\right]$$

$$u_1 = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\left[\cos(\tfrac{1}{2}\theta)\left\{\kappa-1+2\sin^2(\tfrac{1}{2}\theta)\right\}\right]$$

$$u_2 = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\left[\sin(\tfrac{1}{2}\theta)\left\{\kappa+1-2\cos^2(\tfrac{1}{2}\theta)\right\}\right]$$

Mode II: anti-symmetric solution (2D)

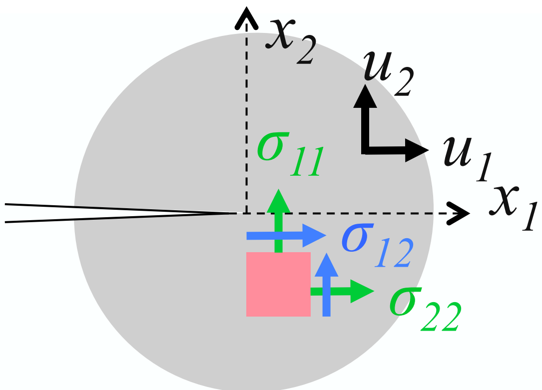
$$\sigma_{11} = \frac{K_{II}}{\sqrt{2\pi r}}\left[-\sin(\tfrac{1}{2}\theta)\left\{2+\cos(\tfrac{1}{2}\theta)\cos(\tfrac{3}{2}\theta)\right\}\right]$$

$$\sigma_{22} = \frac{K_{II}}{\sqrt{2\pi r}}\left[\sin(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta)\cos(\tfrac{3}{2}\theta)\right]$$

$$\sigma_{12} = \frac{K_{II}}{\sqrt{2\pi r}}\left[\cos(\tfrac{1}{2}\theta)\left\{1-\sin(\tfrac{1}{2}\theta)\sin(\tfrac{3}{2}\theta)\right\}\right]$$

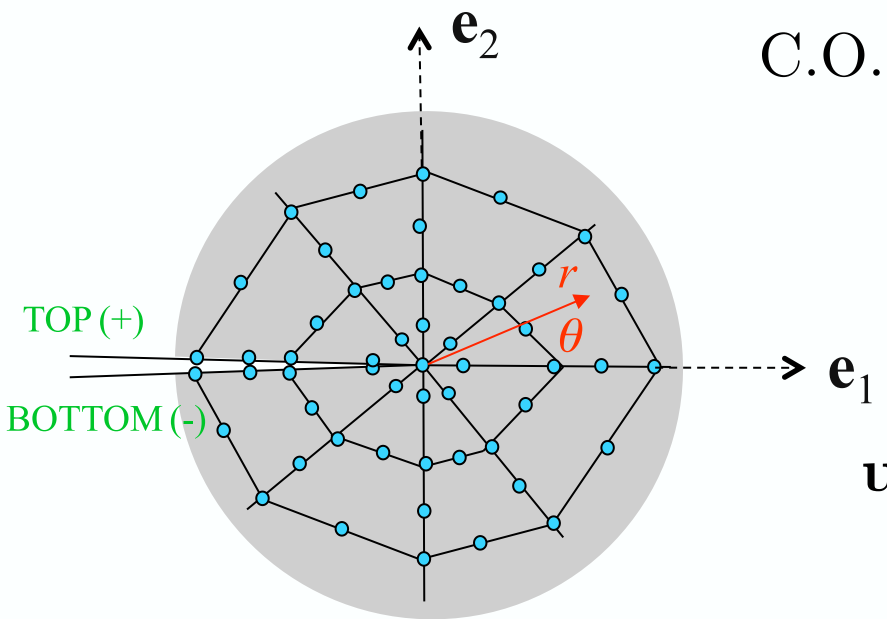
$$u_1 = \frac{K_{II}}{2\mu}\sqrt{\frac{r}{2\pi}}\left[\sin(\tfrac{1}{2}\theta)\left\{\kappa+1+2\cos^2(\tfrac{1}{2}\theta)\right\}\right]$$

$$u_2 = \frac{K_{II}}{2\mu}\sqrt{\frac{r}{2\pi}}\left[-\cos(\tfrac{1}{2}\theta)\left\{\kappa-1-2\sin^2(\tfrac{1}{2}\theta)\right\}\right]$$



NOTE: as $r \to 0$, the displacements $u_i \to 0$ while the stresses $\sigma_{ij} \to \infty$ in such a way that the **energy stored at the entire solid is finite!**

STRESS-INTENSITY FACTORS VIA INTERPOLATION AND SPECIAL ELEMENTS



$$\text{C.O.D. } \mathbf{d}(r) \equiv \mathbf{u}^+(r) - \mathbf{u}^-(r) = \mathbf{u}(r,\pi) - \mathbf{u}(r,-\pi)$$

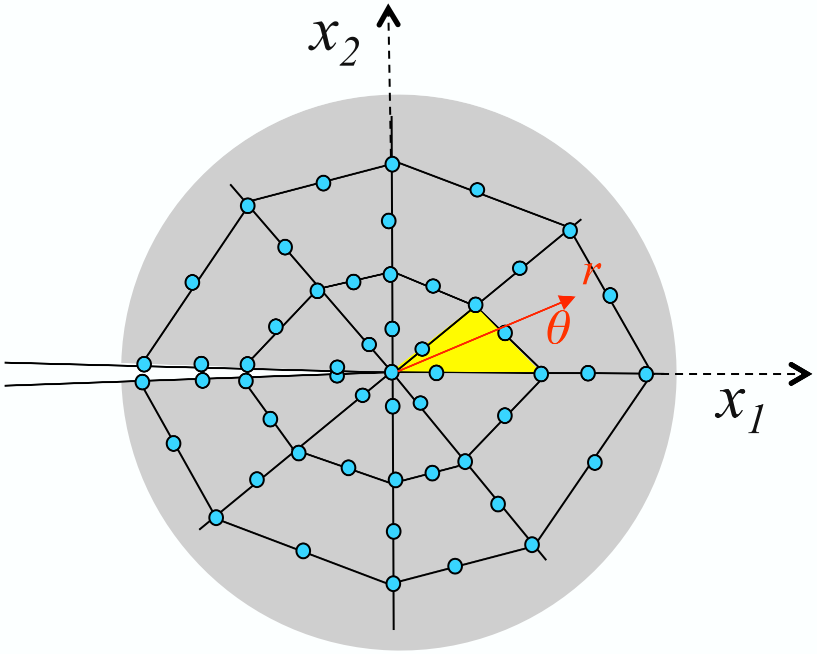
$$\mathbf{u}(r,\pi) = \frac{\kappa+1}{2G}\sqrt{\frac{r}{2\pi}}\left[K_{II}\mathbf{e}_1 + K_I\mathbf{e}_2\right]$$

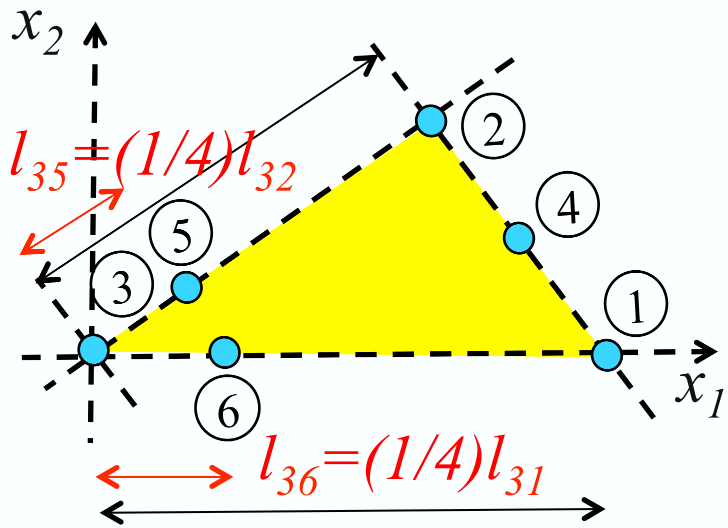
$$\mathbf{u}(r,-\pi) = -\frac{\kappa+1}{2G}\sqrt{\frac{r}{2\pi}}\left[K_{II}\mathbf{e}_1 + K_I\mathbf{e}_2\right]$$

$$\text{fit } \alpha_I,\ \alpha_{II}: \quad \mathbf{d}(r) = \sqrt{r}\left[\alpha_I\mathbf{e}_2 + \alpha_{II}\mathbf{e}_1\right]$$

$$\text{find } K_I,\ K_{II}: \quad K_I = \alpha_I\frac{G\sqrt{2\pi}}{\kappa+1}, \quad K_{II} = \alpha_{II}\frac{G\sqrt{2\pi}}{\kappa+1}$$

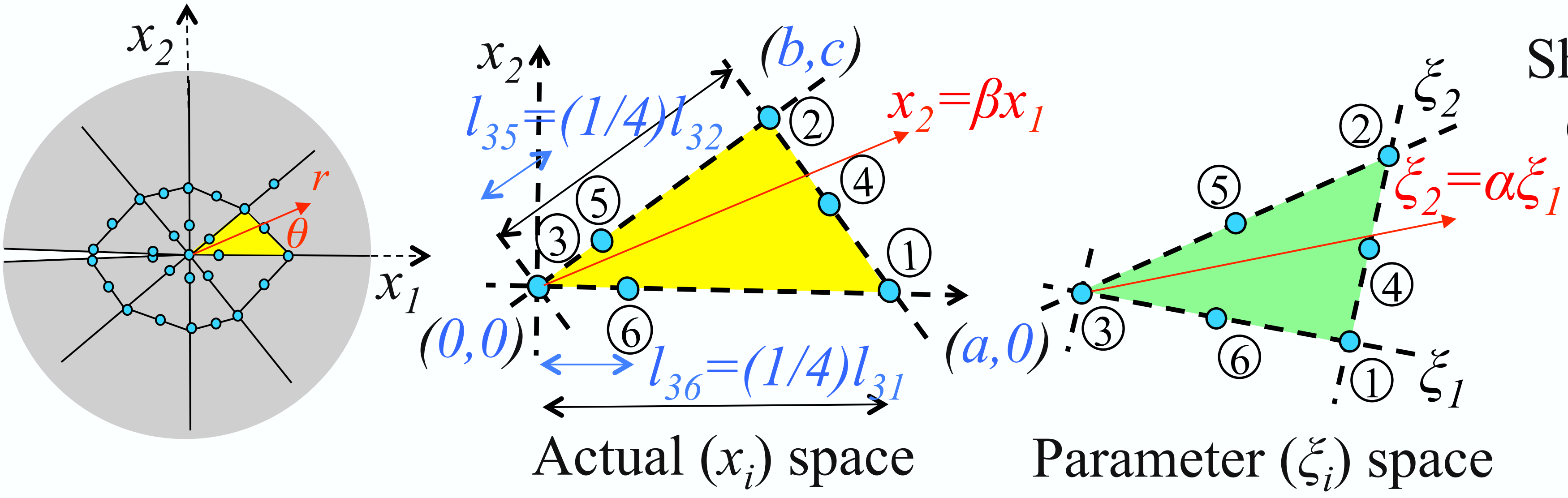
Using an FEM discretization of the solid with acrack, we calculate the crack-opening-displacement (C.O.D.) using the displacement values obtained at its two faces and from the fitted parameters α_I and α_{II} (e.g. using least squares fit) we can evaluate the stress intensity factors K_I and K_{II}





In order to calculate accurately the displacements near the crack tip, one can use special elements that are capable of accounting for $r^{1/2}$ singularity. The simplest possible choice are the 6-node triangles where two of the nodes (5 and 6) are placed at ¼ the distance from the crack tip node (3)

$$\mathbf{x}_6 = (1/4)\mathbf{x}_1\,,\ \mathbf{x}_5 = (1/4)\mathbf{x}_2\,,\ \mathrm{x}_4 = (1/2)(\mathbf{x}_1+\mathbf{x}_2)$$



Shape functions for 6-node triangles

$$N_1 = \xi_1(2\xi_1 - 1)$$

$$N_2 = \xi_2(2\xi_2 - 1)$$

$$N_3 = \xi_3(2\xi_3 - 1)$$

$$N_4 = 4\ \xi_1\xi_2$$

$$N_5 = 4\ \xi_2\xi_3$$

$$N_6 = 4\ \xi_3\xi_1$$

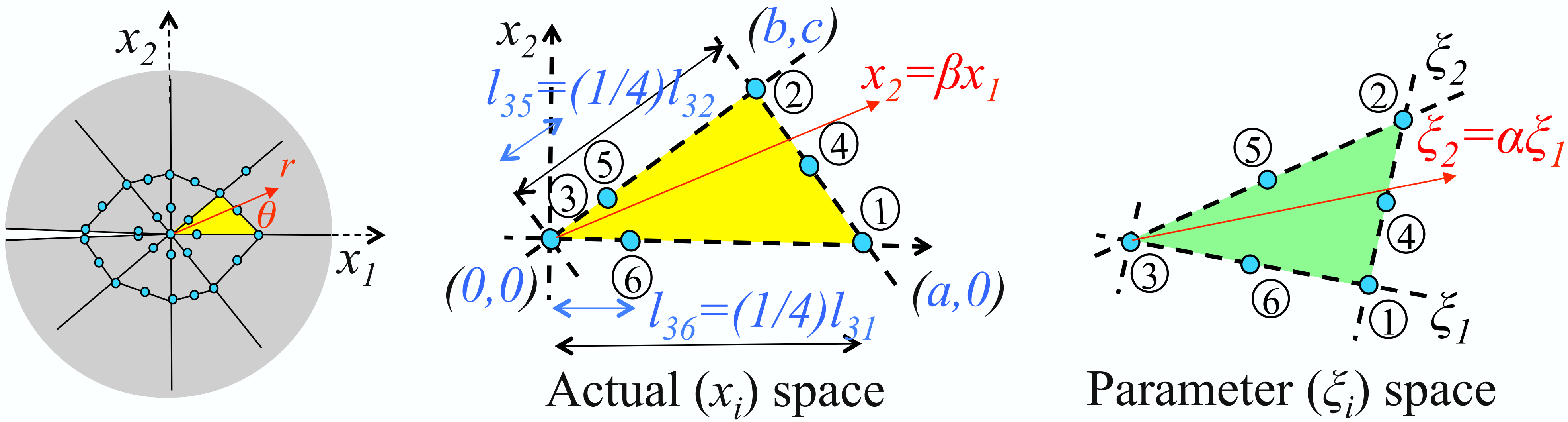
$\mathbf{x}_6 = (1/4)\mathbf{x}_1$, $\mathbf{x}_5 = (1/4)\mathbf{x}_2$, $\mathbf{x}_4 = (1/2)(\mathbf{x}_1+\mathbf{x}_2)$

$\mathbf{x}_1 = a\ \mathbf{e}_1$, $\mathbf{x}_2 = b\ \mathbf{e}_1 + c\ \mathbf{e}_2$

$x_1 = (\xi_1)^2\ [1+\alpha][a+\alpha b]$, $x_2 = (\xi_1)^2\ [1+\alpha][\alpha c]$

$\beta = [\alpha c]/[a+\alpha b]$ depends only on geometry!

Lines through 3 in parameter space map as lines in actual space!



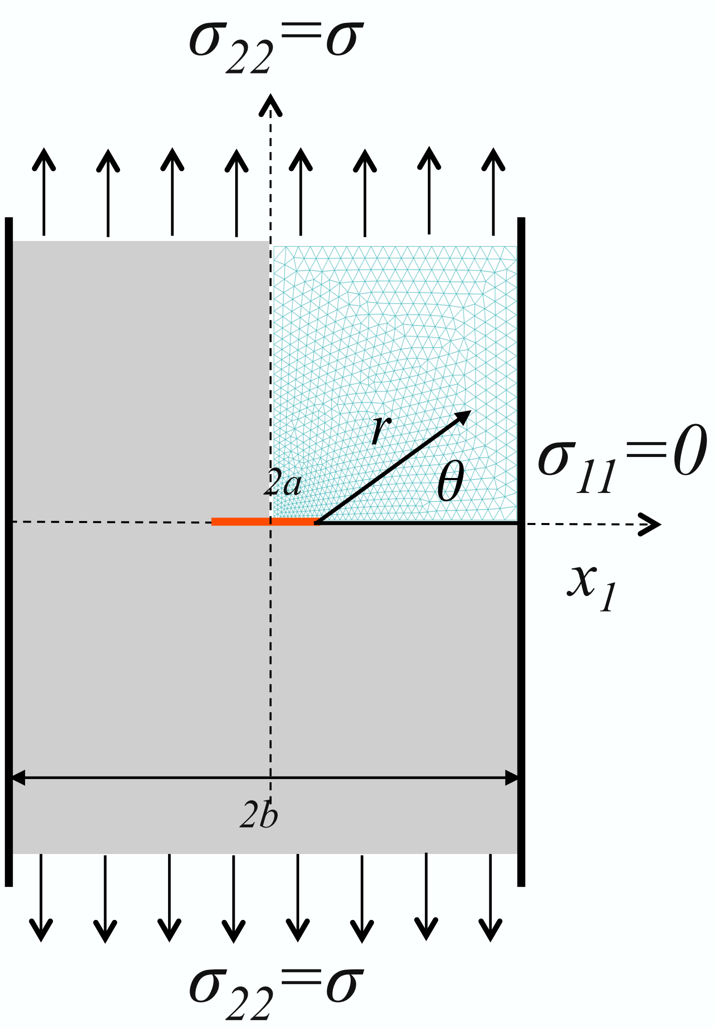
Recall: $x_1 = (\xi_1)^2\,[1+\alpha][a+\alpha b]$, $x_2 = (\xi_1)^2\,[1+\alpha][\alpha c]$

$r = [(x_1)^2 + (x_2)^2]^{1/2} = (\xi_1)^2\,[1+\alpha]\,\gamma$ (geometry-dependent constant) i.e. $\xi_1 = \sqrt{(r/\gamma)}$

$$\mathbf{u}(x_1, x_2) = \Sigma\, N_I(\xi_1, \xi_2)\, \mathbf{x}^I = \Sigma\, N_I(\xi_1, \alpha\xi_1)\, \mathbf{x}^I = \mathbf{u}^3 + \mathbf{f}\,\sqrt{(r/\gamma)} + \mathbf{g}\,(r/\gamma)$$

Notice that the shape functions for the crack tip element are rich enough as to include the correct singularity for the displacement field

FEM CALCULATIONS IN A MODE I GRIFFITH CRACK



An infinite strip of width *2b* contains a crack at the center of length *2a*. The strip is subjected to uniaxial stress $\sigma_{22} = \sigma$.

Only one quarter is analyzed due to symmetry.

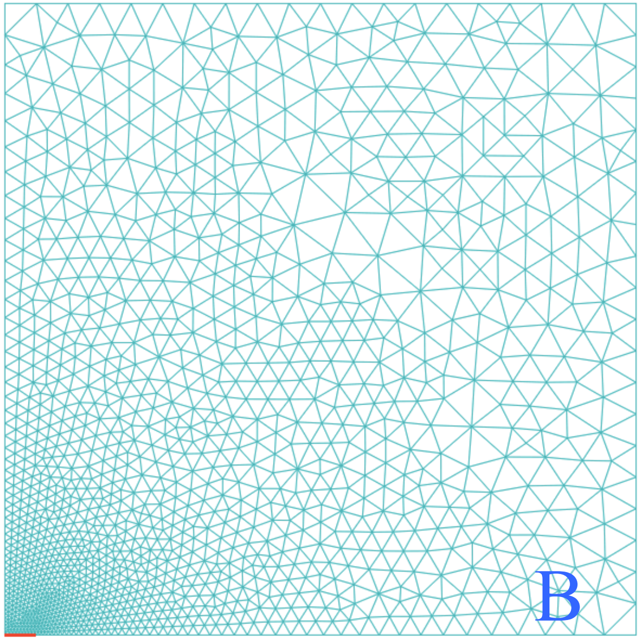
$\sigma_{22}(x_1,0) = \sigma|x_1|/((x_1)^2 - a^2)^{1/2}$ (for $b\rightarrow\infty$)

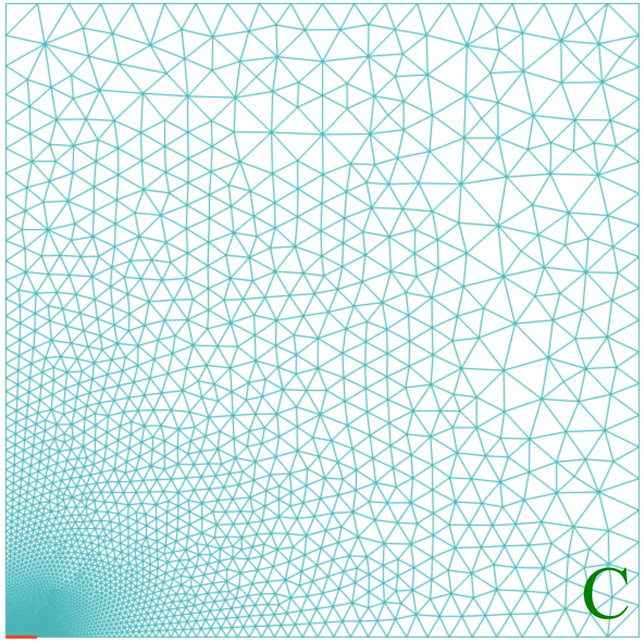
$K_I = \sigma\,(\pi a)^{1/2}$ (for $b\rightarrow\infty$)

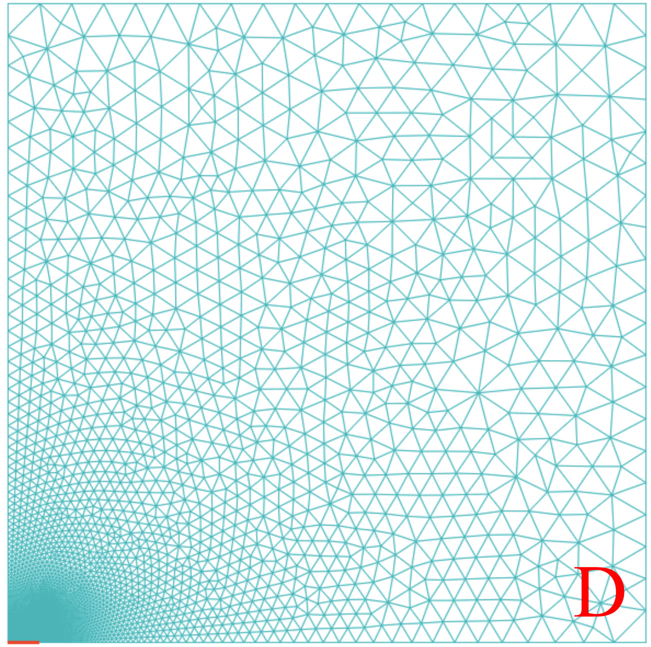
$u_2(x_1,0) = [\sigma(\kappa+1)/4G]\,(a^2 - (x_1)^2)^{1/2}$ (for $b\rightarrow\infty$)

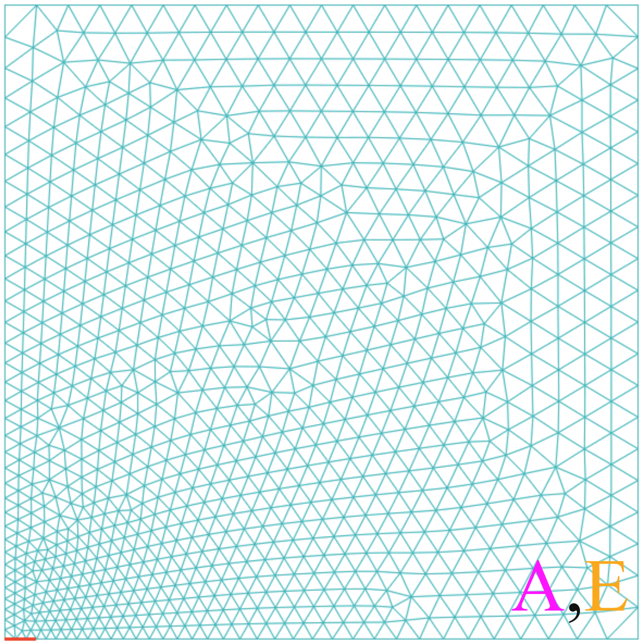
NOTE: $K_I = \sigma\,(\pi a/\cos(\pi a/2b))^{1/2}$ (for finite b)

CRACK OF LENGTH 2a UNDER UNIAXIAL STRESSING (MODE I)







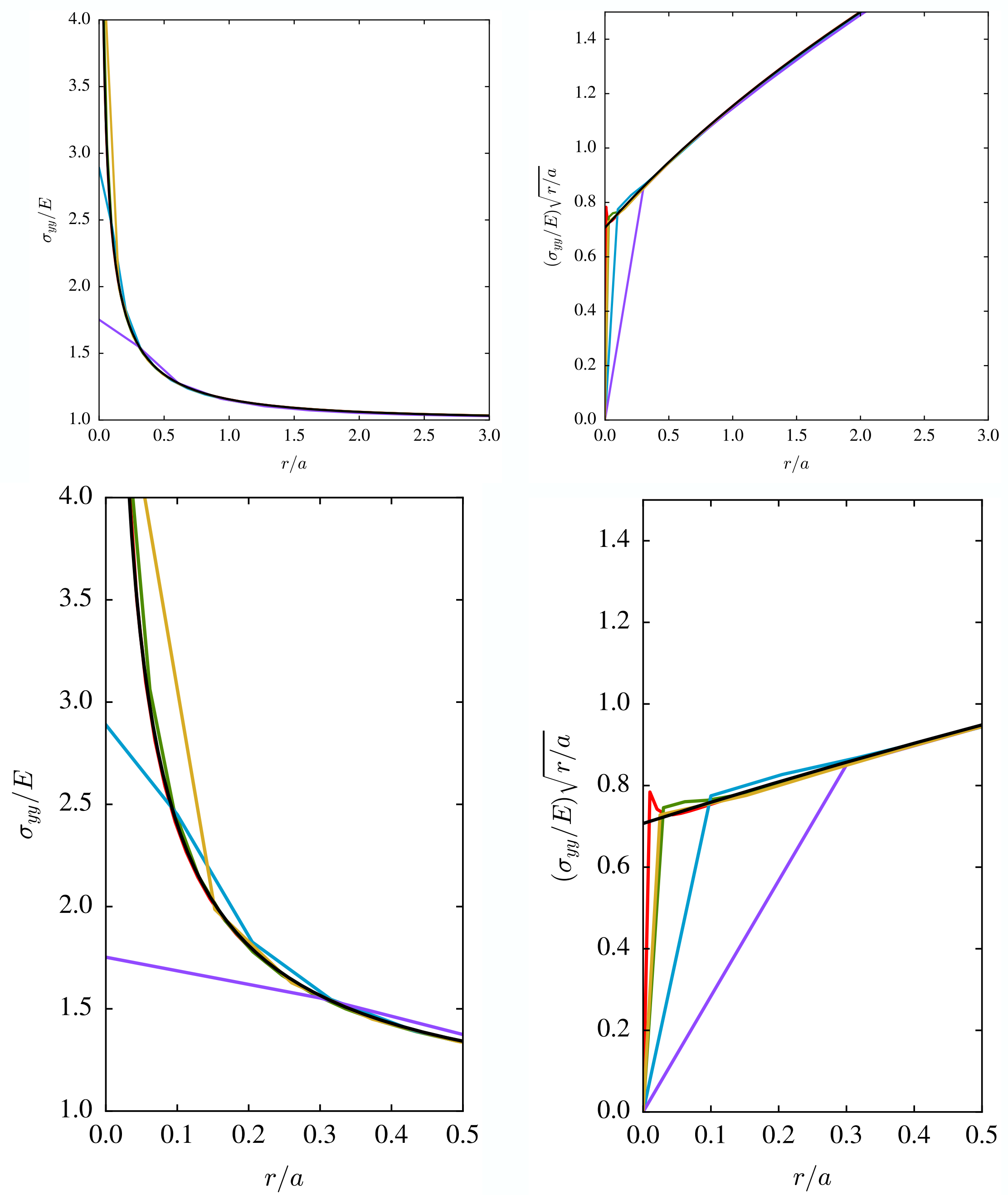


Five different types of meshes were analyzed:

A, B, C, D have simple constant strain triangles

E is made of 6-node triangles, with the ones at the tip having the mid-node at quarter length

CRACK OF LENGTH 2a UNDER UNIAXIAL STRESSING (MODE I)



Calculation of $\sigma_{22}(x_1,0)$

A mesh with *a/3* size CST

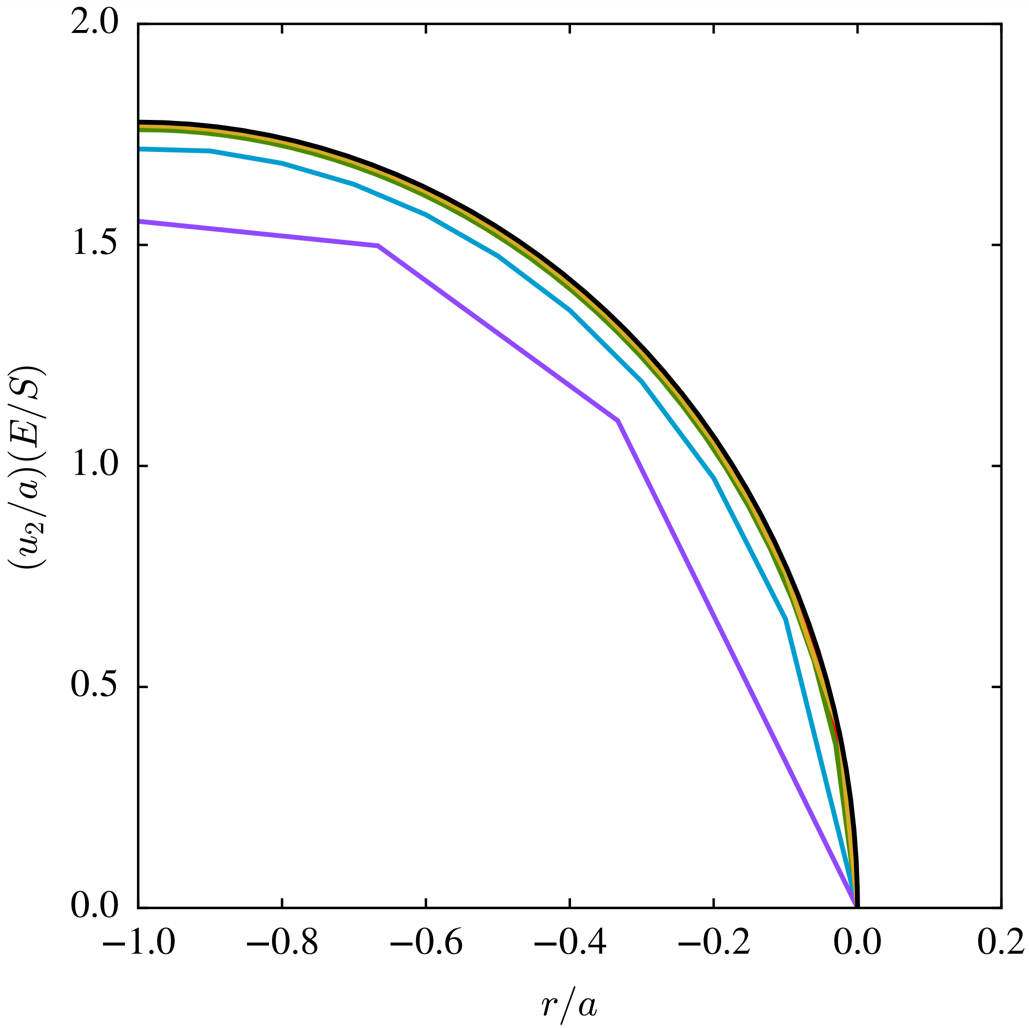
B mesh with *a/10* size CST

C mesh with *a/33* size CST

D mesh with *a/100* size CST

E mesh with *a/3* 6-node triangles (same mesh as A), but with crack-tip elements that account for singular shape functions

Calculation of crack opening displacement $u_2(x_1, 0)$



A mesh with *a/3* size CST

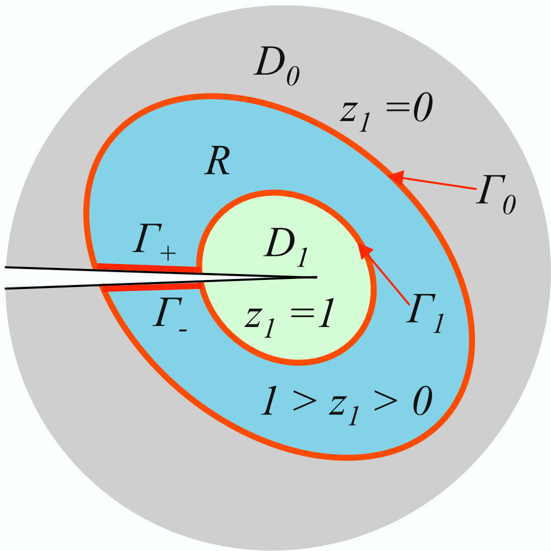
B mesh with *a/10* size CST

C mesh with *a/33* size CST

D mesh with *a/100* size CST

E mesh with *a/3* 6-node triangles (same mesh as A), but with crack-tip elements that account for singular shape functions

STRESS-INTENSITY FACTORS USING ENERGY METHGODS



An advancing crack creates new surface and requires energy/length $\mathcal{J}$, thus reducing the solid's pot. energy U

Need to calculate change of energy due to domain $D(\tau)$ change: $\mathbf{y} = \mathbf{x} + \tau\, \mathbf{z}(\mathbf{x})$ where $\mathbf{z}(\mathbf{x})$: initial velocity of transformation. For crack along x_1, take $\mathbf{z}(\mathbf{x}) = z_1(\mathbf{x})\, \mathbf{e}_1$

Assumptions: $\mathbf{u}(\mathbf{x})$ is the actual equilibrium displacement, there are no body forces

$$\mathcal{J} = -\left[\frac{\partial U}{\partial \tau}\right]_{\tau=0} = -\frac{\partial}{\partial \tau}\left[\int_{D(\tau)} \frac{1}{2} u_{i,j} L_{ijkl} u_{k,l} da\right]_{\tau=0} =$$

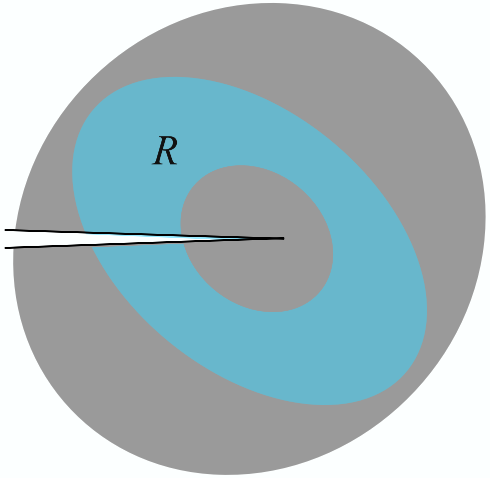
Path-invariant J-integral

$$= \int_R \left[u_{i,j} L_{ijkl} u_{k,p} z_{p,l} - \frac{1}{2} u_{i,j} L_{ijkl} u_{k,l} z_{p,p}\right] da = \int_R \left[u_{i,j} L_{ijpl} u_{k,m} z_m - \frac{1}{2} u_{i,j} L_{ijkl} u_{k,l} z_p\right]_{,p} da$$

$$= \int_{\partial R} \left[u_{i,j} L_{ijpl} u_{k,m} z_m - \frac{1}{2} u_{i,j} L_{ijkl} u_{k,l} z_p\right] n_p ds = \boxed{\int_{\Gamma_1} \left[u_{i,1} \sigma_{ij} n_j - \frac{1}{2} \sigma_{ij} \epsilon_{ij} n_1\right] ds}$$

$$z_1(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in D_0, \quad z_1(\mathbf{x}) = 1 \text{ for } \mathbf{x} \in D_1 \quad \left(\text{e.g. } z_1(\mathbf{x}) = \frac{r - r_0}{r_1 - r_0}, \ \mathbf{x} \in R\right); \quad z_2(\mathbf{x}) = 0$$

ENERGY RELEASE RATE DUE TO PROPAGATING CRACK



Stress intensity factor(s) can be found numerically using the FEM discretized solution (and numerical integration) over a domain R that does not contain the singularity.

$$\mathcal{J} = \int_{\Gamma_1} \left[u_{i,1}\sigma_{ij}n_j - \frac{1}{2}\sigma_{ij}\epsilon_{ij}n_1 \right] ds = \int_0^{2\pi} \left\{ \left[u_{i,1}\sigma_{ij}n_j - \frac{1}{2}\sigma_{ij}\epsilon_{ij}n_1 \right] r \right\}_{r\to 0} d\theta$$

$$\mathcal{J} = \frac{(K_I^2 + K_{II}^2)(\kappa + 1)}{8G} \quad \text{(using asymptotics near crack tip)}$$

$$\mathcal{J} = \int_R \left[u_{i,j}L_{ijkl}u_{k,p}z_{p,l} - \frac{1}{2}u_{i,j}L_{ijkl}u_{k,l}z_{p,p} \right] da$$

$$\mathcal{J} = \sum_e \int_{R_e} \left[u^e_{i,j}L^e_{ijkl}u^e_{k,p}z_{p,l} - \frac{1}{2}u^e_{i,j}L^e_{ijkl}u^e_{k,l}z_{p,p} \right] da \quad \text{(using FEM discretization)}$$