TOPICS COVERED IN THIS LECTURE

1. ISOPARAMETRIC QUADS AND HIGHER ORDER ELEMENTS IN 2D

2. BEAM EXAMPLES USING DIFFERENT 2D ELEMENTS

3. NUMERICAL INTEGRATION IN 1D AND 2D

ISOPARAMETRIC QUADS AND HIGHER ORDER ELEMENTS

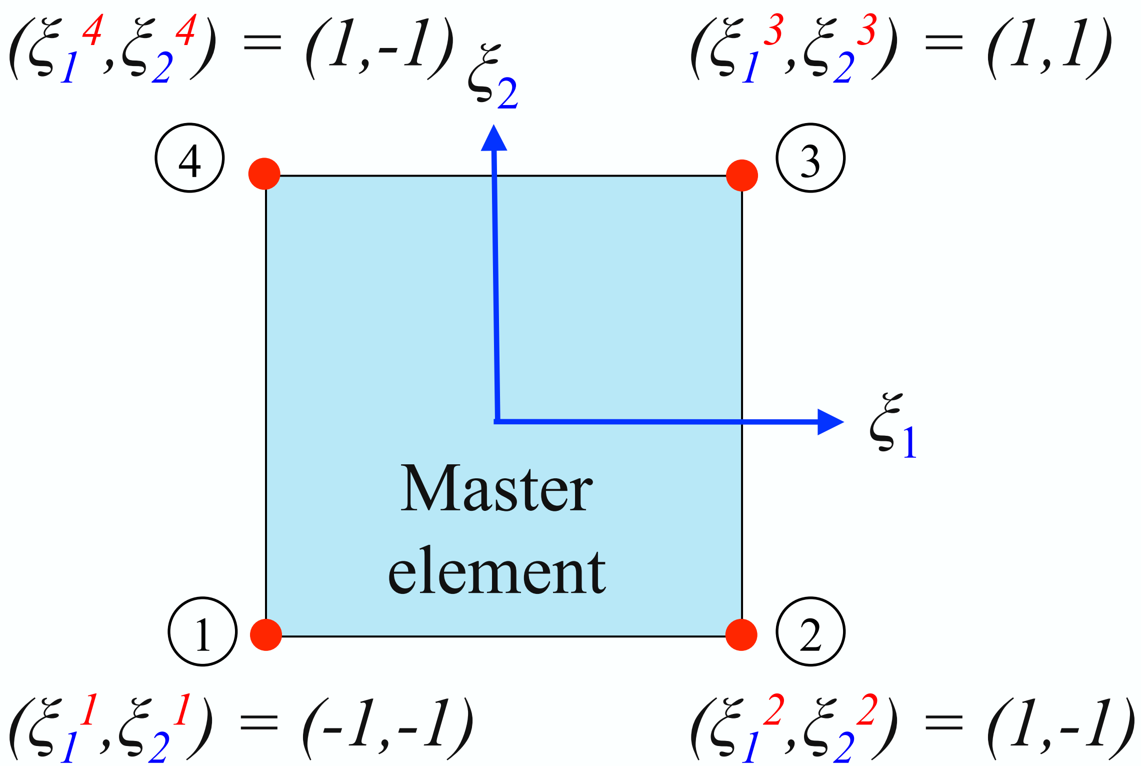
Isoparametric quadrilateral element (quad)

$$u_j\left(\xi_1,\xi_2\right)=\sum_{I=1}^{4} N_I\left(\xi_1,\xi_2\right)U_j^I \qquad N_I(\xi_1^J,\xi_2^J)=\delta_{IJ};\;(I,J=1,\ldots 4)$$

$$u_j\left(\xi_1^I,\xi_2^I\right)=U_j^I$$

$$x_j\left(\xi_1,\xi_2\right)=\sum_{I=1}^{4} N_I\left(\xi_1,\xi_2\right)x_j^I$$

$$x_j\left(\xi_1^I,\xi_2^I\right)=x_j^I$$



$$N_I(\xi_1, \xi_2) = \frac{1}{4}(1 + \xi_1^I \xi_1)(1 + \xi_2^I \xi_2)$$

Shape functions $N_I(\xi_1, \xi_2)$ and coordinate transformation matrix **J** for 4-node isoparametric quads

$$\mathbf{J} \equiv \left[\frac{\partial x_j}{\partial \xi_i}\right] = \left[\sum_{I=1}^{4} \frac{\partial N_I}{\partial \xi_i}(\xi_1, \xi_2) x_j^I\right]$$

$$\mathbf{J} = \begin{bmatrix} \frac{1}{4}\sum_{I=1}^{4} \xi_1^I (1 + \xi_2^I \xi_2) x_1^I & \frac{1}{4}\sum_{I=1}^{4} \xi_1^I (1 + \xi_2^I \xi_2) x_2^I \\ \frac{1}{4}\sum_{I=1}^{4} \xi_2^I (1 + \xi_1^I \xi_1) x_1^I & \frac{1}{4}\sum_{I=1}^{4} \xi_2^I (1 + \xi_1^I \xi_1) x_2^I \end{bmatrix}$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_2} \\ \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_2}{\partial x_1} & 0 & 0 \\ 0 & 0 & \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_2} \\ \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_2} & \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_2}{\partial x_1} \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \dfrac{\partial u_1}{\partial \xi_1} \\ \dfrac{\partial u_1}{\partial \xi_2} \\ \dfrac{\partial u_2}{\partial \xi_1} \\ \dfrac{\partial u_2}{\partial \xi_2} \end{bmatrix}$$

Recall definition of matrix $\mathbf{A} = \dfrac{\mathbf{1}}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$

Recall definition of matrix **G**

$$
\begin{bmatrix} \dfrac{\partial u_1}{\partial \xi_1} \\ \dfrac{\partial u_1}{\partial \xi_2} \\ \dfrac{\partial u_2}{\partial \xi_1} \\ \dfrac{\partial u_2}{\partial \xi_2} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial N_1}{\partial \xi_1} & 0 & \dfrac{\partial N_2}{\partial \xi_1} & 0 & \dfrac{\partial N_3}{\partial \xi_1} & 0 & \dfrac{\partial N_4}{\partial \xi_1} & 0 \\ \dfrac{\partial N_1}{\partial \xi_2} & 0 & \dfrac{\partial N_2}{\partial \xi_2} & 0 & \dfrac{\partial N_3}{\partial \xi_2} & 0 & \dfrac{\partial N_4}{\partial \xi_2} & 0 \\ 0 & \dfrac{\partial N_1}{\partial \xi_1} & 0 & \dfrac{\partial N_2}{\partial \xi_1} & 0 & \dfrac{\partial N_3}{\partial \xi_1} & 0 & \dfrac{\partial N_4}{\partial \xi_1} \\ 0 & \dfrac{\partial N_1}{\partial \xi_2} & 0 & \dfrac{\partial N_2}{\partial \xi_2} & 0 & \dfrac{\partial N_3}{\partial \xi_2} & 0 & \dfrac{\partial N_4}{\partial \xi_2} \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} U_1^1 \\ U_2^1 \\ U_1^2 \\ U_2^2 \\ U_1^3 \\ U_2^3 \\ U_1^4 \\ U_2^4 \end{bmatrix}}_{\mathbf{q}_e}
$$

$$\mathcal{P}^e_{int} = \int_{A_e} \frac{1}{2}[\boldsymbol{\epsilon}^T \boldsymbol{\sigma}(x_1, x_2)]\, dA = \frac{1}{2}\mathbf{q}_e^T \left[\int_{\boldsymbol{\xi}} [\underbrace{\mathbf{G}^T \mathbf{A}^T}_{\boldsymbol{\varepsilon}^T} \mathbf{L}\ \underbrace{\mathbf{A}\ \mathbf{G}]}_{\boldsymbol{\sigma}}\, \underbrace{\det(\mathbf{J})\, d\boldsymbol{\xi}}_{dA} \right] \mathbf{q}_e$$

$$= \frac{1}{2}\mathbf{q}_e^T \mathbf{k}_e \mathbf{q}_e$$

Element stiffness matrix: $\mathbf{k}_\mathrm{e}$

$$-\mathcal{P}^e_{ext} = \int_{A_e} [\mathbf{u}^T(x_1, x_2)\mathbf{b}(x_1, x_2)]\, dA + \int_{\partial A_e} [\mathbf{u}^T(x_1, x_2)\mathbf{t}(x_1, x_2)]\, dl$$

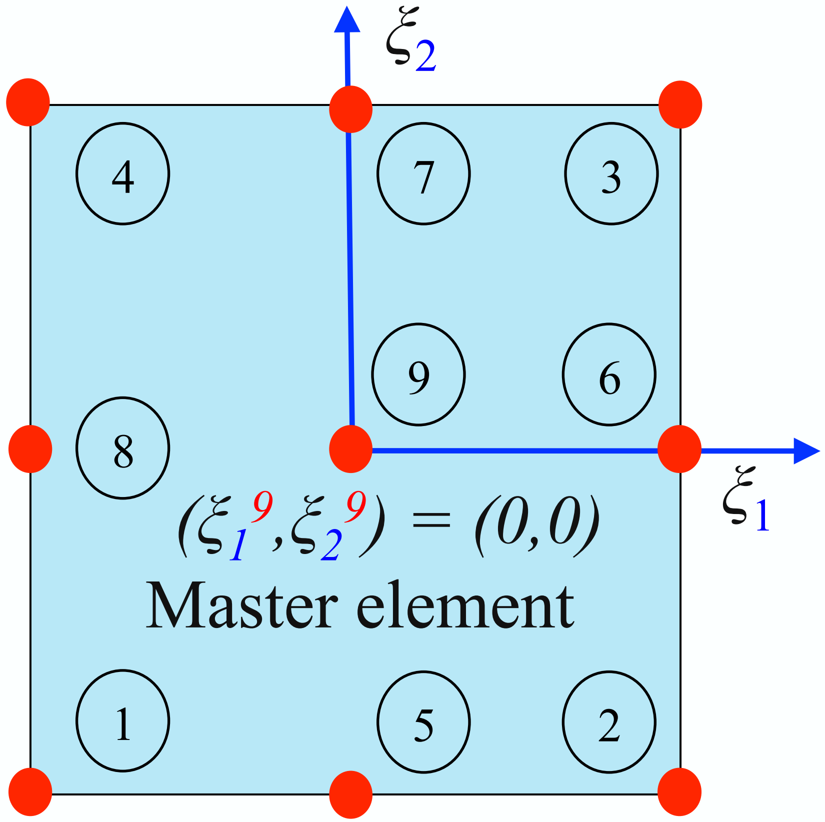
$$= \mathbf{q}_e^T \left[\int_{\boldsymbol{\xi}} [\mathbf{N}^T\ \mathbf{b}] \det(\mathbf{J})\, d\boldsymbol{\xi} + \int_{\partial\boldsymbol{\xi}} [\mathbf{N}^T\ \mathbf{t}]\, dl(\xi) \right]$$

$$= \mathbf{q}_e^T \mathbf{f}_e$$

Element force vector: $\mathbf{f}_\mathrm{e}$

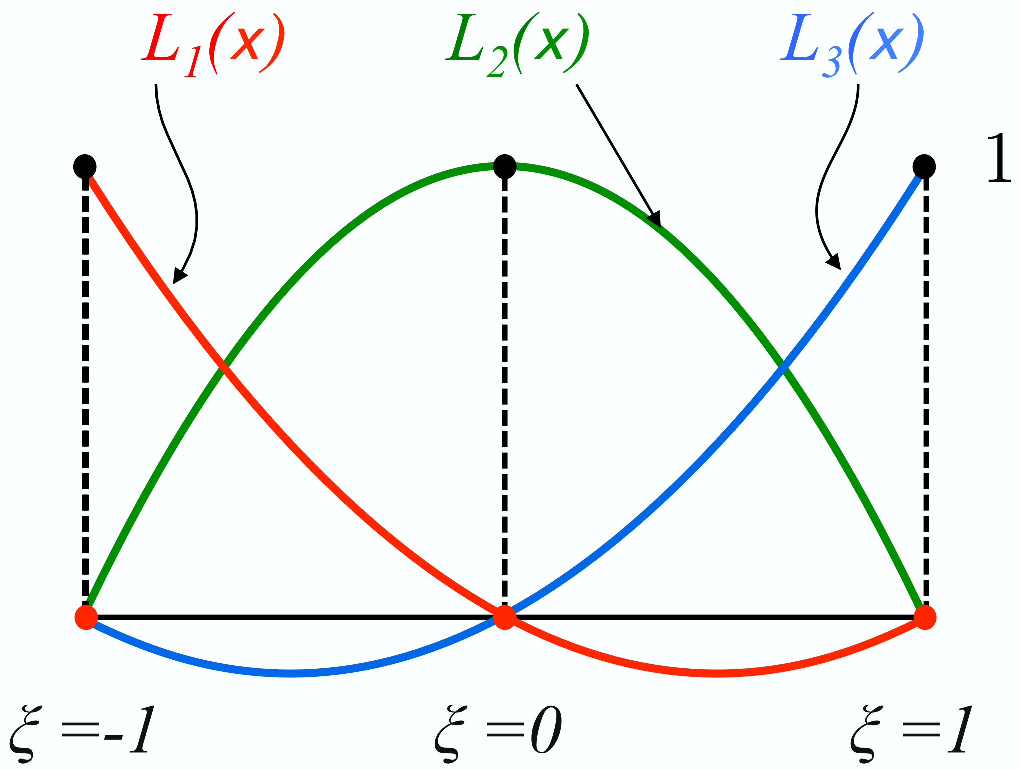
Shape functions must satisfy: $N_I(\xi_1^J, \xi_2^J) = \delta_{IJ}$

$(\xi_1^4, \xi_2^4) = (1,-1)$ $\quad$ $(\xi_1^7, \xi_2^7) = (0,-1)$ $\quad$ $(\xi_1^3, \xi_2^3) = (1,1)$



$(\xi_1^8, \xi_2^8) = (-1,0)$ $\quad$ $(\xi_1^6, \xi_2^6) = (1,0)$

$(\xi_1^1, \xi_2^1) = (-1,-1)$ $\quad$ $(\xi_1^5, \xi_2^5) = (0,-1)$ $\quad$ $(\xi_1^2, \xi_2^2) = (1,-1)$



Recall quadratic Lagrangian functions $L_i(\xi)$ in interval [-1,1]

Shape functions are products:
$N_I(\xi_1, \xi_2)=L_i(\xi_i)\, L_j(\xi_2)$

$$u_j\left(\xi_1,\xi_2\right) = \sum_{I=1}^{9} N_I\left(\xi_1,\xi_2\right) U_j^I$$

$$u_j\left(\xi_1^I,\xi_2^I\right) = U_j^I$$

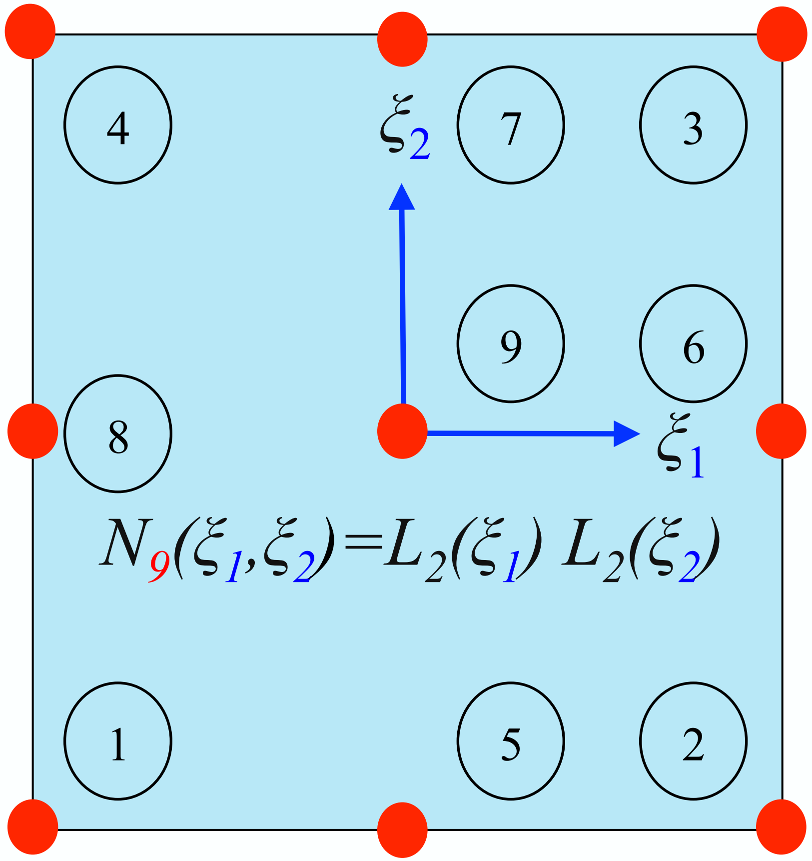
$$x_j\left(\xi_1,\xi_2\right) = \sum_{I=1}^{9} N_I\left(\xi_1,\xi_2\right) x_j^I$$

$$x_j\left(\xi_1^I,\xi_2^I\right) = x_j^I$$

$$N_I(\xi_1^J,\xi_2^J) = \delta_{IJ}; \quad (I,J=1,\dots 9)$$

$$N_4(\xi_1,\xi_2)=L_1(\xi_1)\,L_3(\xi_2) \qquad N_7(\xi_1,\xi_2)=L_2(\xi_1)\,L_3(\xi_2) \qquad N_3(\xi_1,\xi_2)=L_3(\xi_1)\,L_3(\xi_2)$$

$$N_8(\xi_1,\xi_2)=L_1(\xi_1)\,L_2(\xi_2)$$

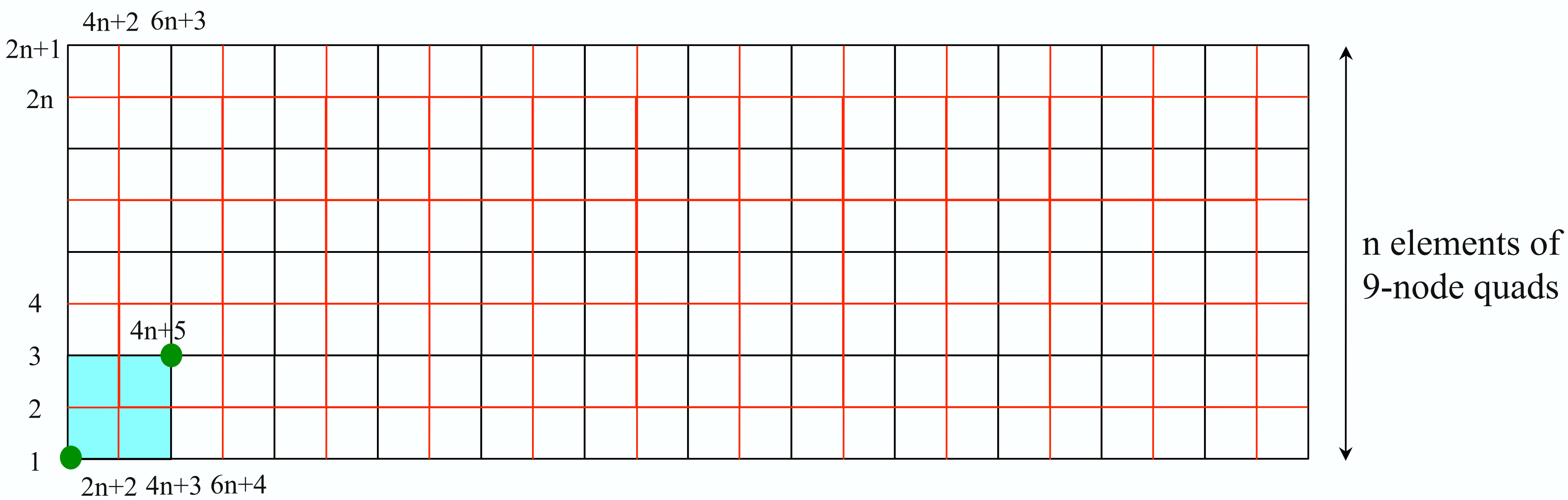


$$N_9(\xi_1,\xi_2)=L_2(\xi_1)\,L_2(\xi_2)$$

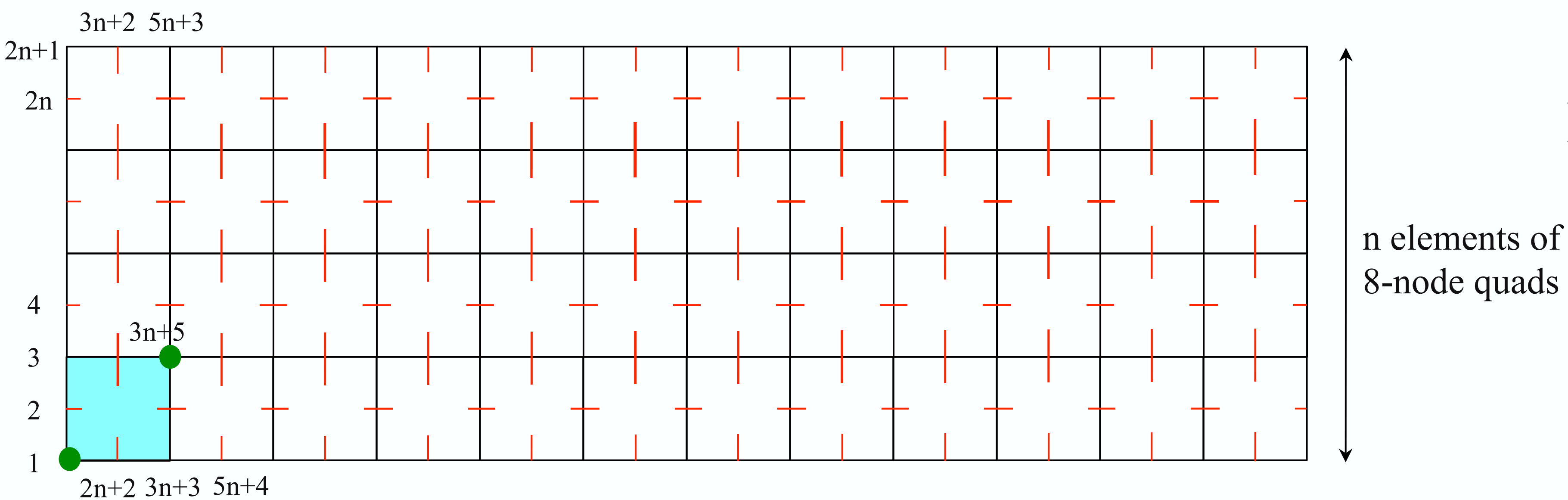
$$N_6(\xi_1,\xi_2)=L_3(\xi_1)\,L_2(\xi_2)$$

$$N_1(\xi_1,\xi_2)=L_1(\xi_1)\,L_1(\xi_2) \qquad N_5(\xi_1,\xi_2)=L_2(\xi_1)\,L_1(\xi_2) \qquad N_2(\xi_1,\xi_2)=L_3(\xi_1)\,L_1(\xi_2)$$

Disadvantage of 9-node quad elements: needless increase of bandwidth

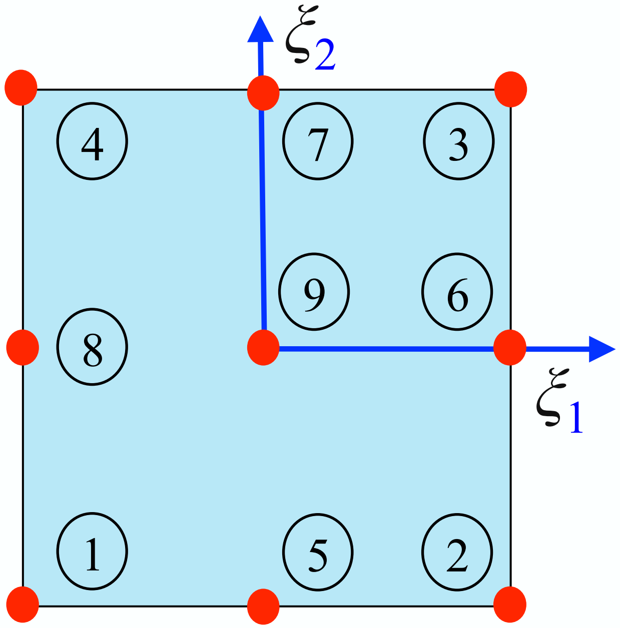


Bandwidth:
$2 \times (4n+5)$
for 9-node quads



Bandwidth:
$2 \times (3n+5)$
for 8-node quads

Way to eliminate internal nodes: static condensation



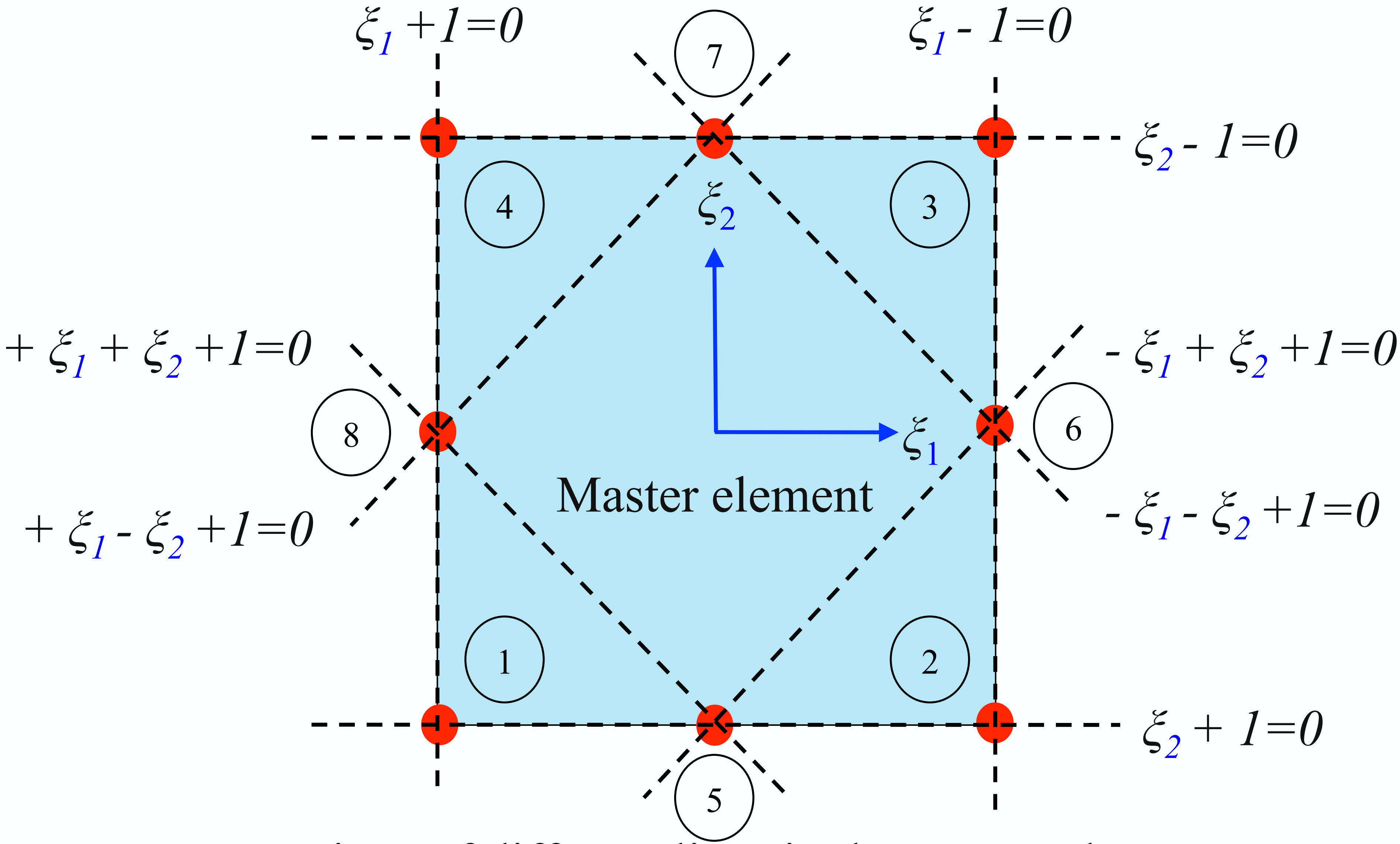
$$
\begin{bmatrix}
\mathbf{K}^G_{11} & \mathbf{K}^G_{12} & \mathbf{K}^G_{13} & \mathbf{K}^G_{14} & \mathbf{K}^G_{15} & \mathbf{K}^G_{16} & \mathbf{K}^G_{17} & \mathbf{K}^G_{18} & \mathbf{k}^e_{19} \\
\mathbf{K}^G_{21} & \mathbf{K}^G_{22} & \mathbf{K}^G_{23} & \mathbf{K}^G_{24} & \mathbf{K}^G_{25} & \mathbf{K}^G_{26} & \mathbf{K}^G_{27} & \mathbf{K}^G_{28} & \mathbf{k}^e_{29} \\
\mathbf{K}^G_{31} & \mathbf{K}^G_{32} & \mathbf{K}^G_{33} & \mathbf{K}^G_{34} & \mathbf{K}^G_{35} & \mathbf{K}^G_{36} & \mathbf{K}^G_{37} & \mathbf{K}^G_{38} & \mathbf{k}^e_{39} \\
\mathbf{K}^G_{41} & \mathbf{K}^G_{42} & \mathbf{K}^G_{43} & \mathbf{K}^G_{44} & \mathbf{K}^G_{45} & \mathbf{K}^G_{46} & \mathbf{K}^G_{47} & \mathbf{K}^G_{48} & \mathbf{k}^e_{39} \\
\mathbf{K}^G_{51} & \mathbf{K}^G_{52} & \mathbf{K}^G_{53} & \mathbf{K}^G_{54} & \mathbf{K}^G_{55} & \mathbf{K}^G_{56} & \mathbf{K}^G_{57} & \mathbf{K}^G_{58} & \mathbf{k}^e_{59} \\
\mathbf{K}^G_{61} & \mathbf{K}^G_{62} & \mathbf{K}^G_{63} & \mathbf{K}^G_{64} & \mathbf{K}^G_{65} & \mathbf{K}^G_{66} & \mathbf{K}^G_{67} & \mathbf{K}^G_{68} & \mathbf{k}^e_{69} \\
\mathbf{K}^G_{71} & \mathbf{K}^G_{72} & \mathbf{K}^G_{73} & \mathbf{K}^G_{74} & \mathbf{K}^G_{75} & \mathbf{K}^G_{76} & \mathbf{K}^G_{77} & \mathbf{K}^G_{78} & \mathbf{k}^e_{79} \\
\mathbf{K}^G_{81} & \mathbf{K}^G_{82} & \mathbf{K}^G_{83} & \mathbf{K}^G_{84} & \mathbf{K}^G_{85} & \mathbf{K}^G_{86} & \mathbf{K}^G_{87} & \mathbf{K}^G_{88} & \mathbf{k}^e_{89} \\
\mathbf{k}^e_{91} & \mathbf{k}^e_{92} & \mathbf{k}^e_{93} & \mathbf{k}^e_{94} & \mathbf{k}^e_{95} & \mathbf{k}^e_{96} & \mathbf{k}^e_{97} & \mathbf{k}^e_{98} & \mathbf{k}^e_{99}
\end{bmatrix}
\begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \\ \mathbf{Q}_4 \\ \mathbf{Q}_5 \\ \mathbf{Q}_6 \\ \mathbf{Q}_7 \\ \mathbf{Q}_8 \\ \mathbf{Q}_9 \end{bmatrix}
=
\begin{bmatrix} \mathbf{F}^G_1 \\ \mathbf{F}^G_2 \\ \mathbf{F}^G_3 \\ \mathbf{F}^G_4 \\ \mathbf{F}^G_5 \\ \mathbf{F}^G_6 \\ \mathbf{F}^G_7 \\ \mathbf{F}^G_8 \\ \mathbf{f}^e_9 \end{bmatrix}
$$

Change element stiffness & force of boundary nodes by:

$$\mathbf{k}^e_{ij} \rightarrow \mathbf{k}^e_{ij} - \mathbf{k}^e_{i9}[\mathbf{k}^e_{99}]^{-1}\mathbf{k}^e_{9j}$$

$$\mathbf{f}^e_i \rightarrow \mathbf{f}^e_i - \mathbf{k}^e_{i9}[\mathbf{k}^e_{99}]^{-1}\mathbf{f}^e_9$$

Equilibrium equation for node 9 solved immediately



Equations of different lines in the master element

Shape functions of node I are products of line equations with remaining nodes

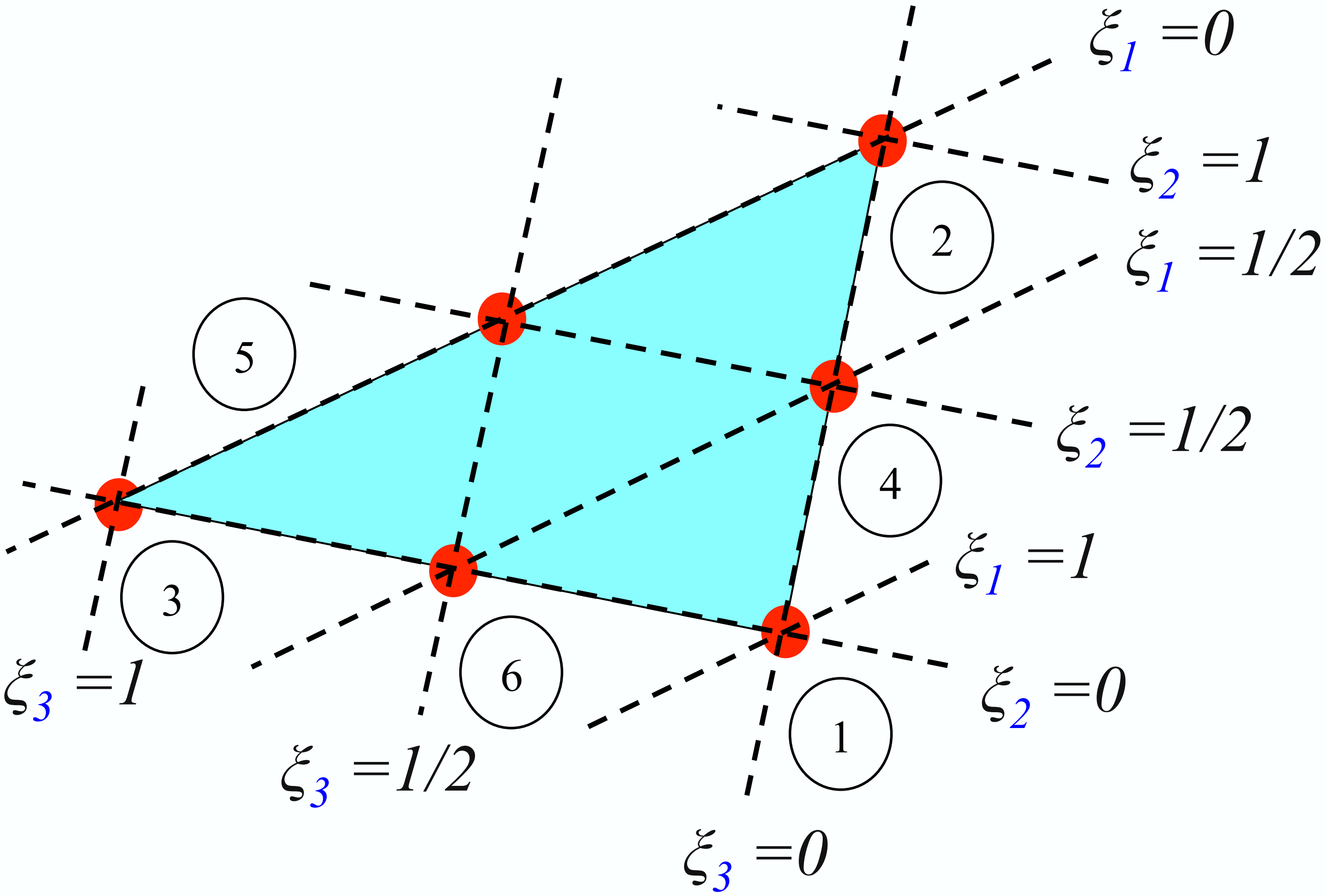
$$N_1(\xi_1,\xi_2) = -\frac{1}{4}(1-\xi_1)(1-\xi_2)(1+\xi_1+\xi_2),\quad N_5(\xi_1,\xi_2) = \frac{1}{2}(1-\xi_1^2)(1-\xi_2)$$

$$N_2(\xi_1,\xi_2) = -\frac{1}{4}(1+\xi_1)(1-\xi_2)(1-\xi_1+\xi_2),\quad N_6(\xi_1,\xi_2) = \frac{1}{2}(1+\xi_1)(1-\xi_2^2)$$

$$N_3(\xi_1,\xi_2) = -\frac{1}{4}(1+\xi_1)(1+\xi_2)(1-\xi_1-\xi_2),\quad N_7(\xi_1,\xi_2) = \frac{1}{2}(1-\xi_1^2)(1+\xi_2)$$

$$N_4(\xi_1,\xi_2) = -\frac{1}{4}(1-\xi_1)(1+\xi_2)(1+\xi_1-\xi_2),\quad N_8(\xi_1,\xi_2) = \frac{1}{2}(1-\xi_1)(1-\xi_2^2)$$

Shape functions of node I are products of equations avoiding that node



$$N_1 = \xi_1(2\xi_1 - 1)$$

$$N_2 = \xi_2(2\xi_2 - 1)$$

$$N_3 = \xi_3(2\xi_3 - 1)$$

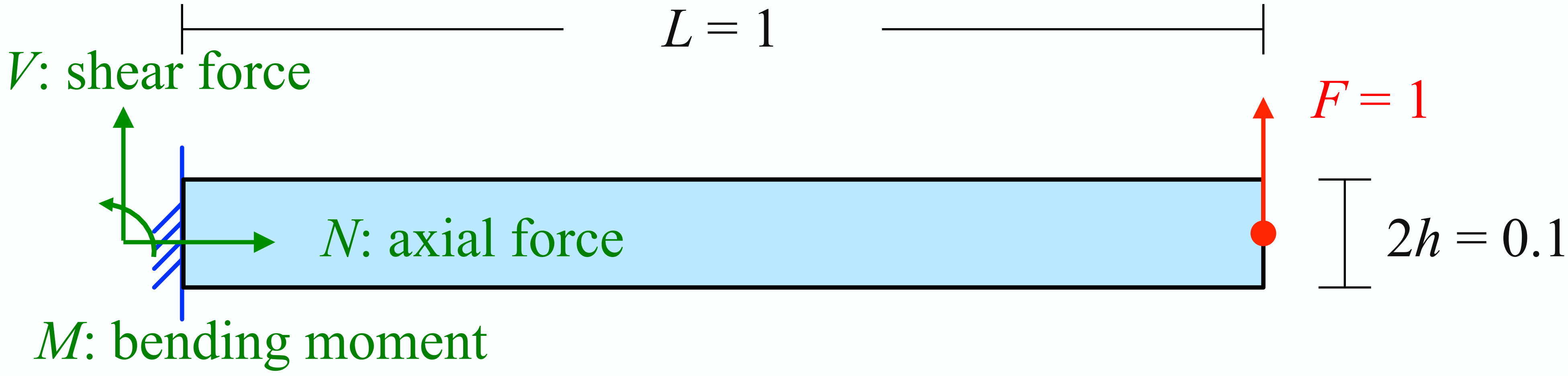
$$N_4 = 4\,\xi_1\xi_2$$

$$N_5 = 4\,\xi_2\xi_3$$

$$N_6 = 4\,\xi_3\xi_1$$

Triangular coordinates satisfy: $\xi_1+\xi_2+\xi_3 = 1$

BEAM EXAMPLES USING DIFFERENT 2D ELEMENTS

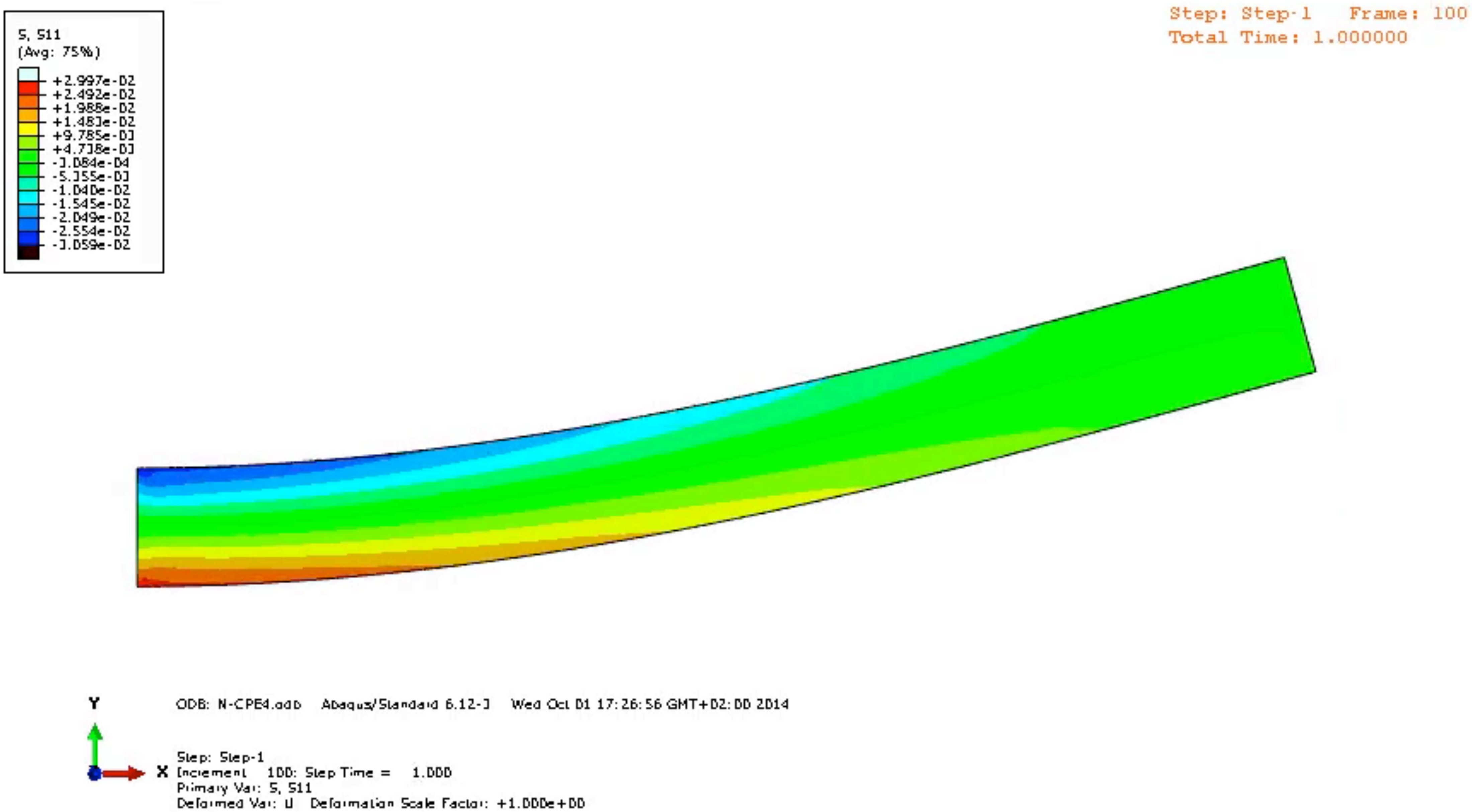


Problem will be solved using different elements in 2D

Analytical solution found with the help of Airy's stress function satisfies average equilibrium conditions at each end, while FEM solution considers correct boundary conditions (all nodes fixed at left, have forces applied at right)

Analytical solution correct away from boundaries (boundary layers)

FEM solution of cantilever beam showing end effects



Airy stress function has correct axial, shear forces and moment at ends

$$\phi(x_1,x_2) = \frac{h^2}{4}\frac{F}{L}\frac{x_2}{h}\left(\left(\frac{x_1}{L}-1\right) - 3\frac{x_1}{L}\left(\frac{x_2}{h}\right)^2\right)\left(\frac{L}{h}\right)^2$$

Left: *(x_1=0)*

N = 0

V = 1

M = L

$$\sigma_{11} = \frac{\partial^2\phi}{\partial x_2 \partial x_2} = \frac{3}{2}\frac{x_2}{h}\frac{F}{L}\left(\frac{x_1}{L}-1\right)\left(\frac{L}{h}\right)^2$$

$$\sigma_{22} = \frac{\partial^2\phi}{\partial x_1 \partial x_1} = 0$$

$$\sigma_{12} = -\frac{\partial^2\phi}{\partial x_1 \partial x_2} = -\frac{3}{4}\frac{F}{L}\left(\left(\frac{x_2}{h}\right)^2-1\right)\frac{L}{h}$$

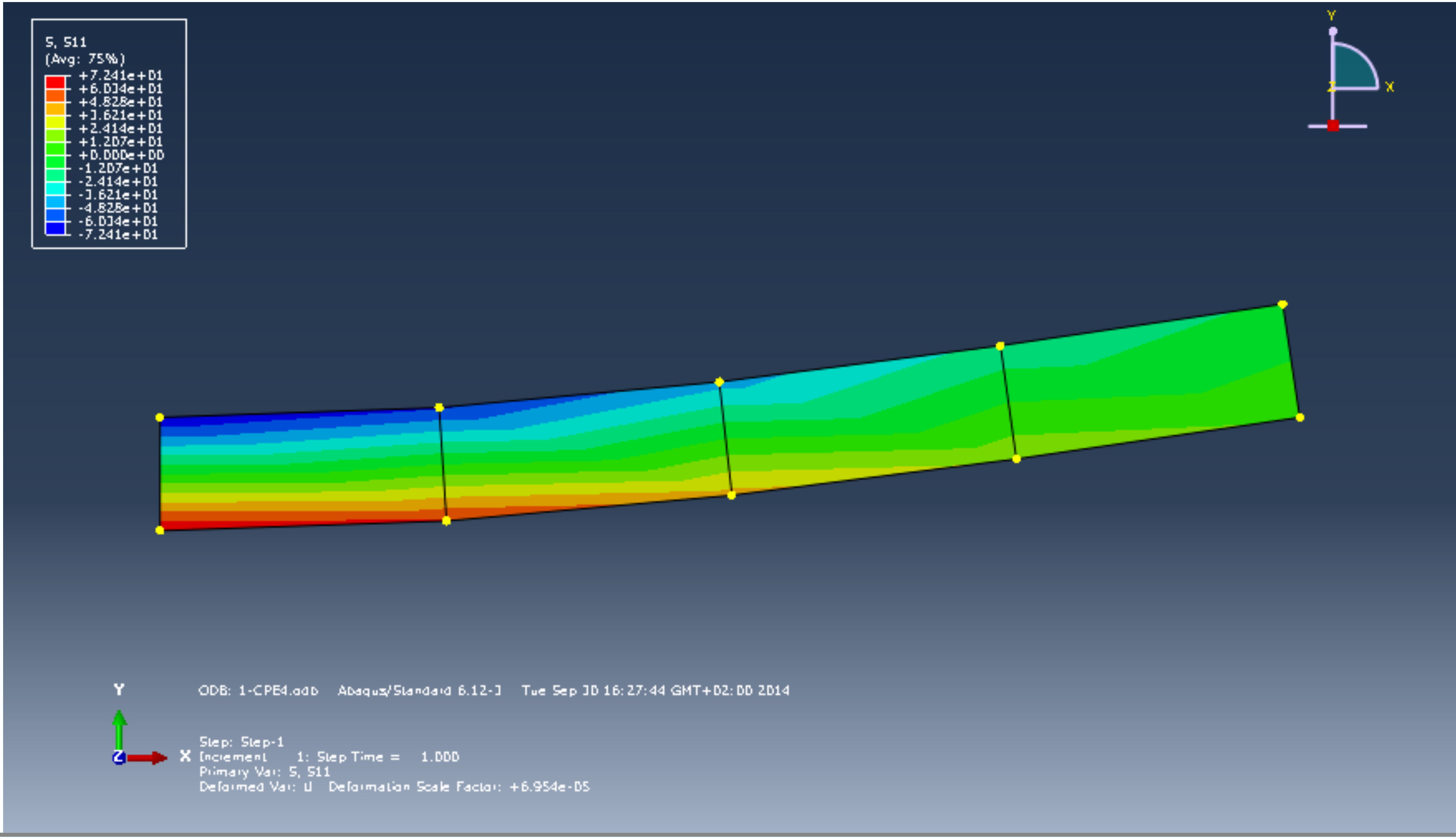
Right: *(x_1=L)*

N = 0

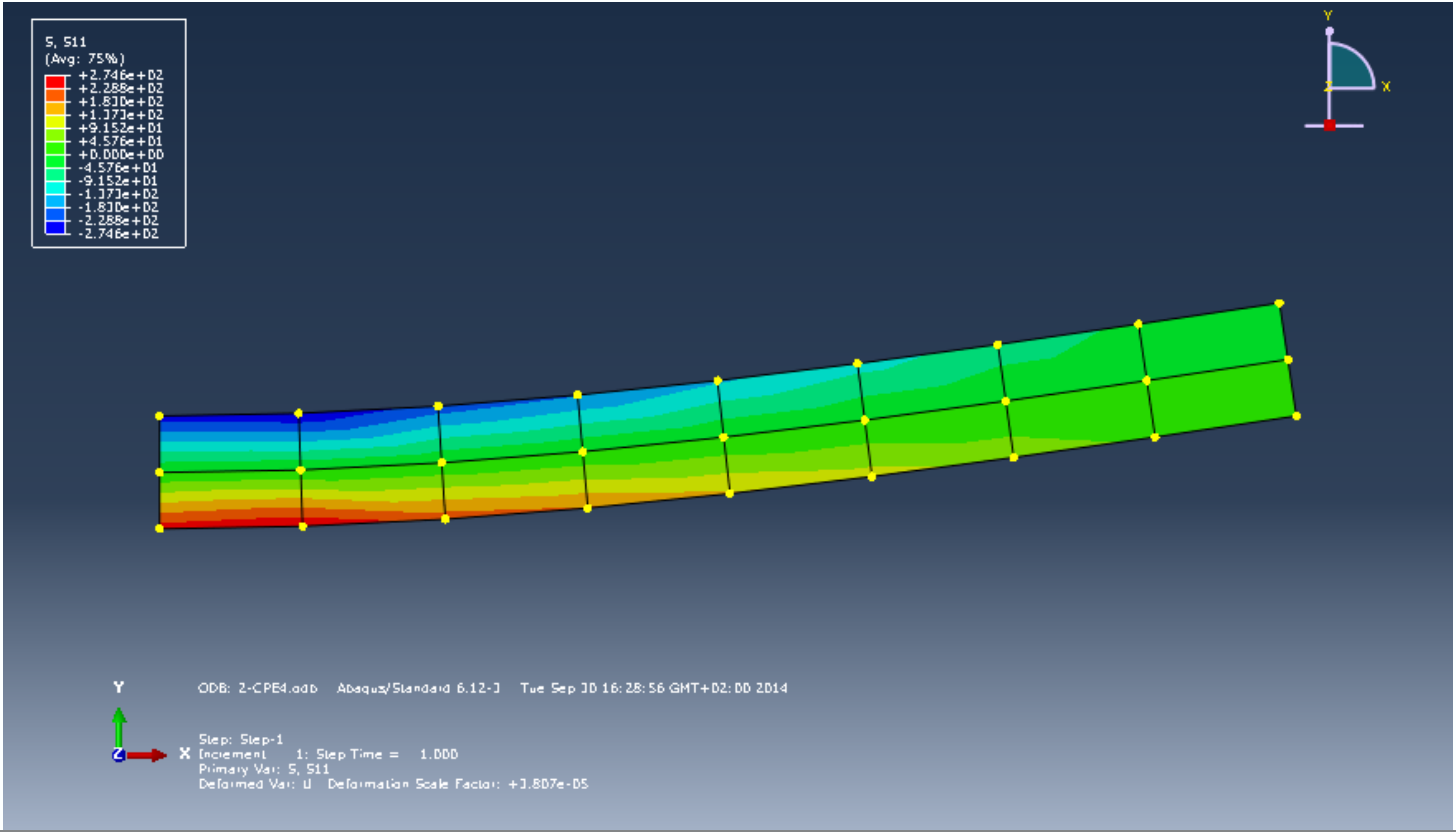
V = 1

M = 0

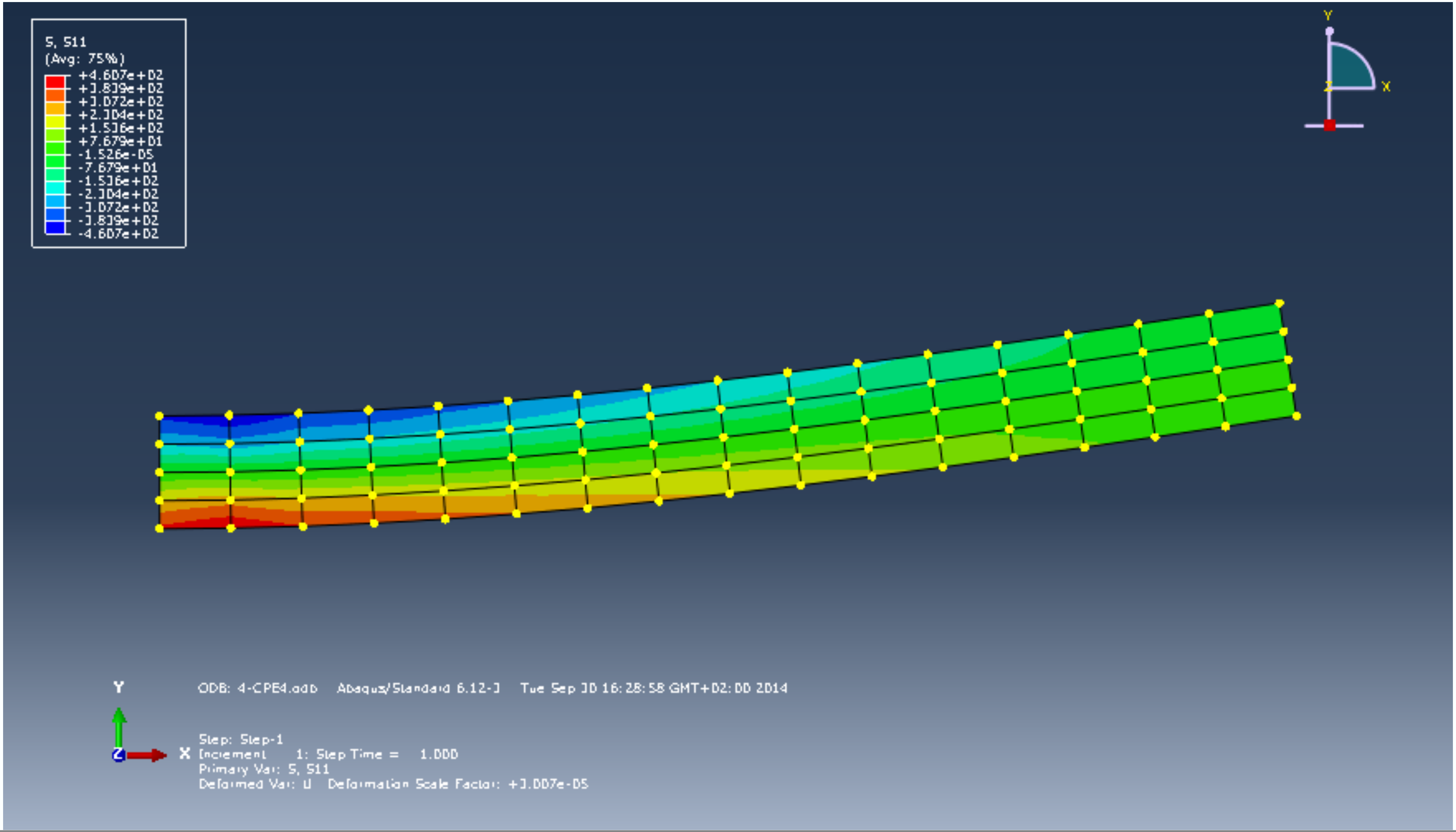
FEM solution using 4 4-node quads



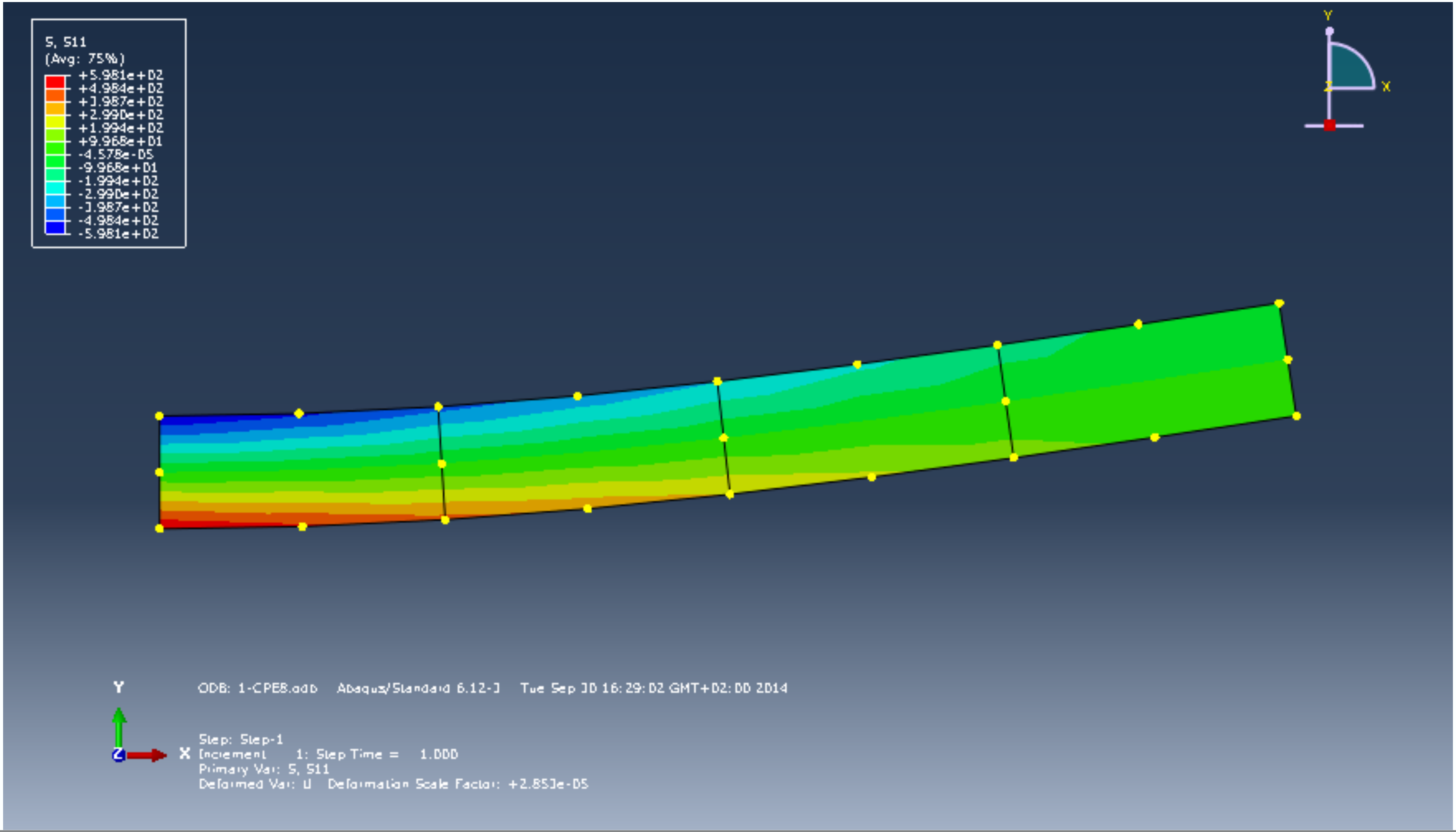
FEM solution using 16 4-node quads



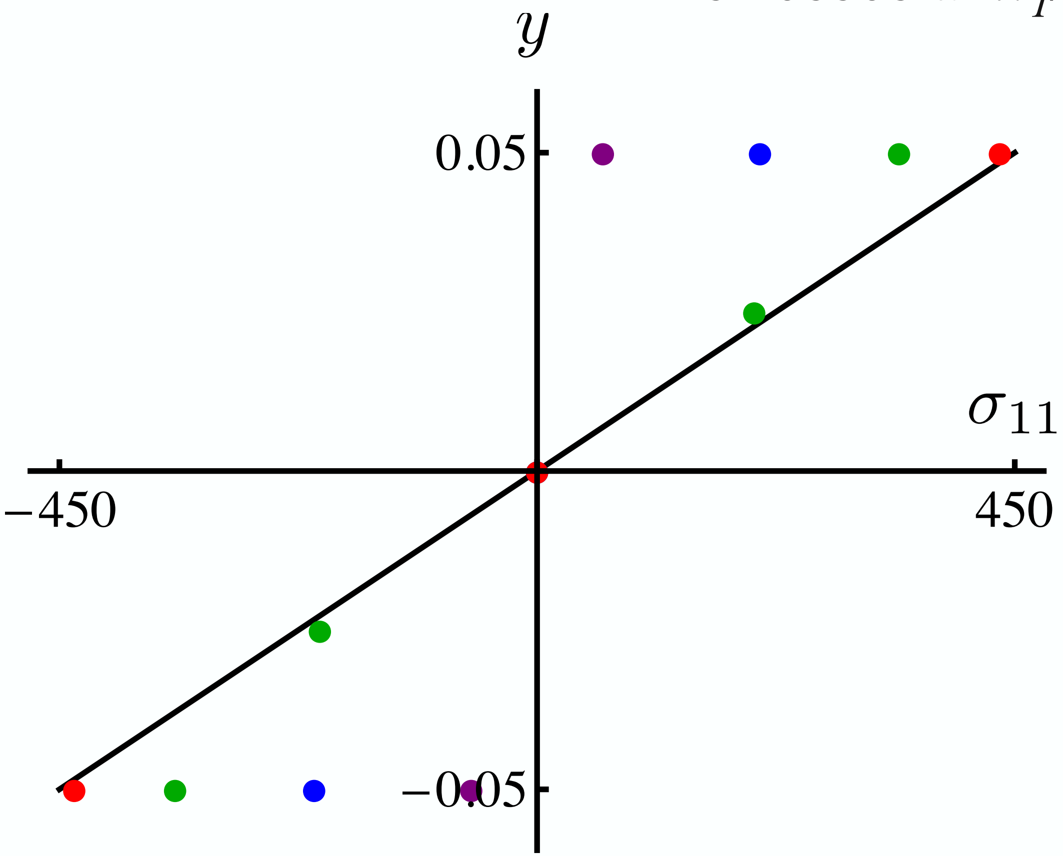
FEM solution using 64 4-node quads

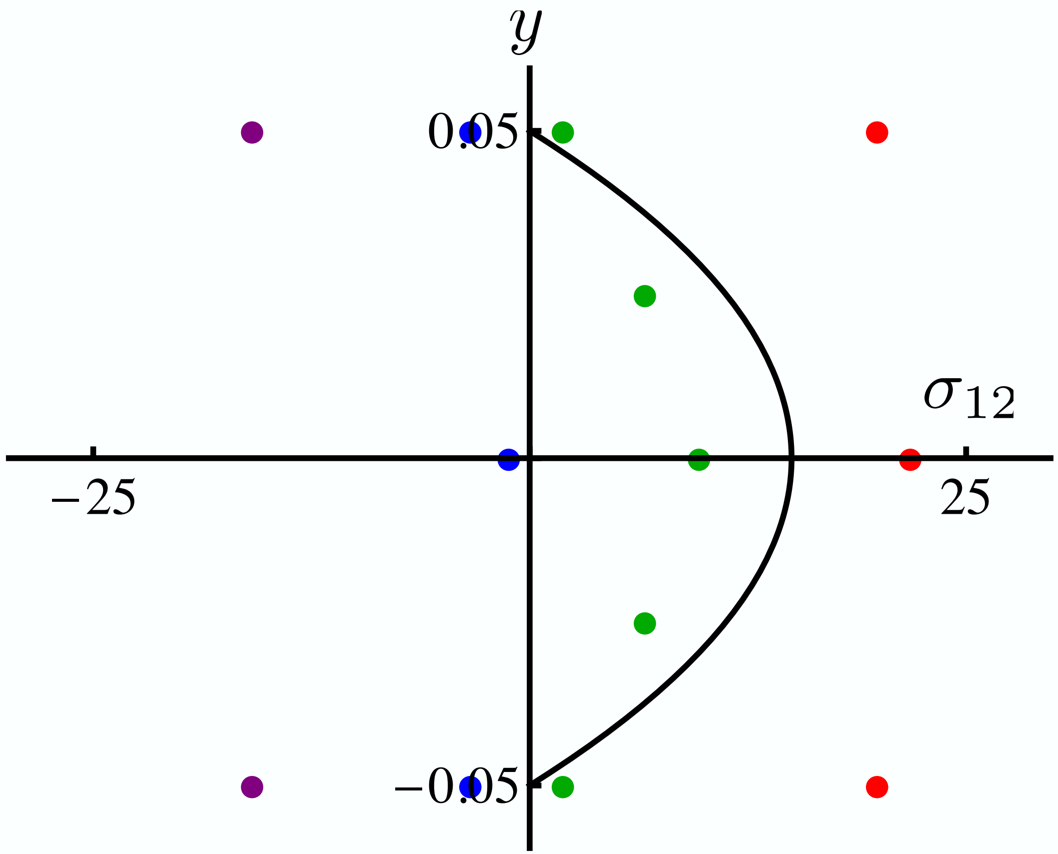


FEM solution using 4 8-node quads



stresses at $x_1/L=l=1/4$





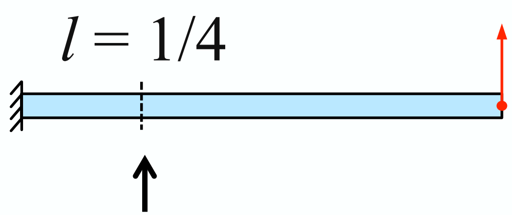
Analytical solution $\sigma_{11} = \dfrac{3F(l-L)x_2}{2h^3}$, $\sigma_{12} = \dfrac{3F\left(h^2 - x_2^2\right)}{4h^3}$

linear element: 2 layers

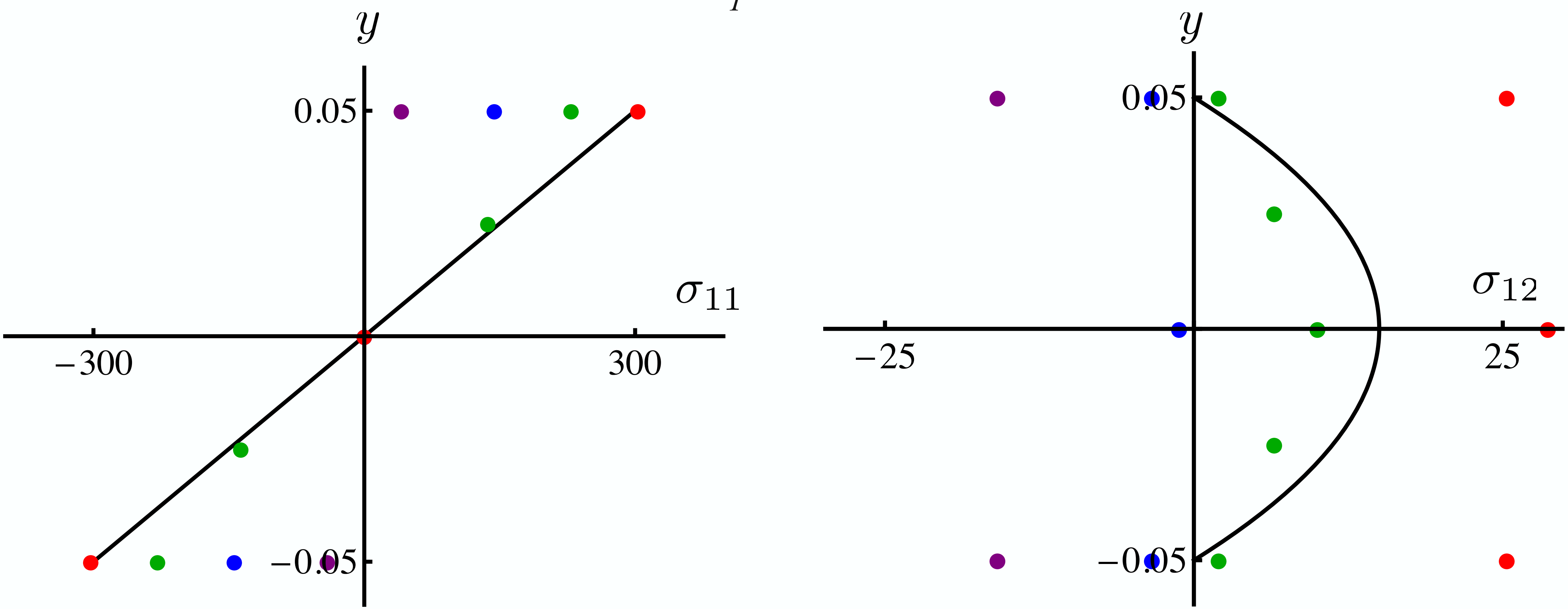
quadratic element: 1 layer

linear element: 1 layers

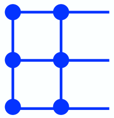
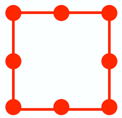
linear element: 4 layers

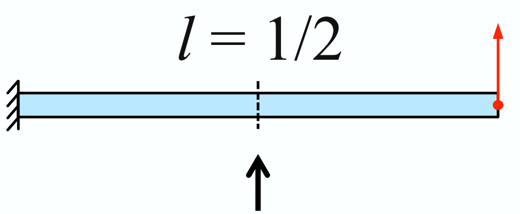


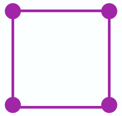
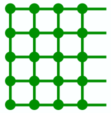
stresses at $x_1/L=l=1/2$



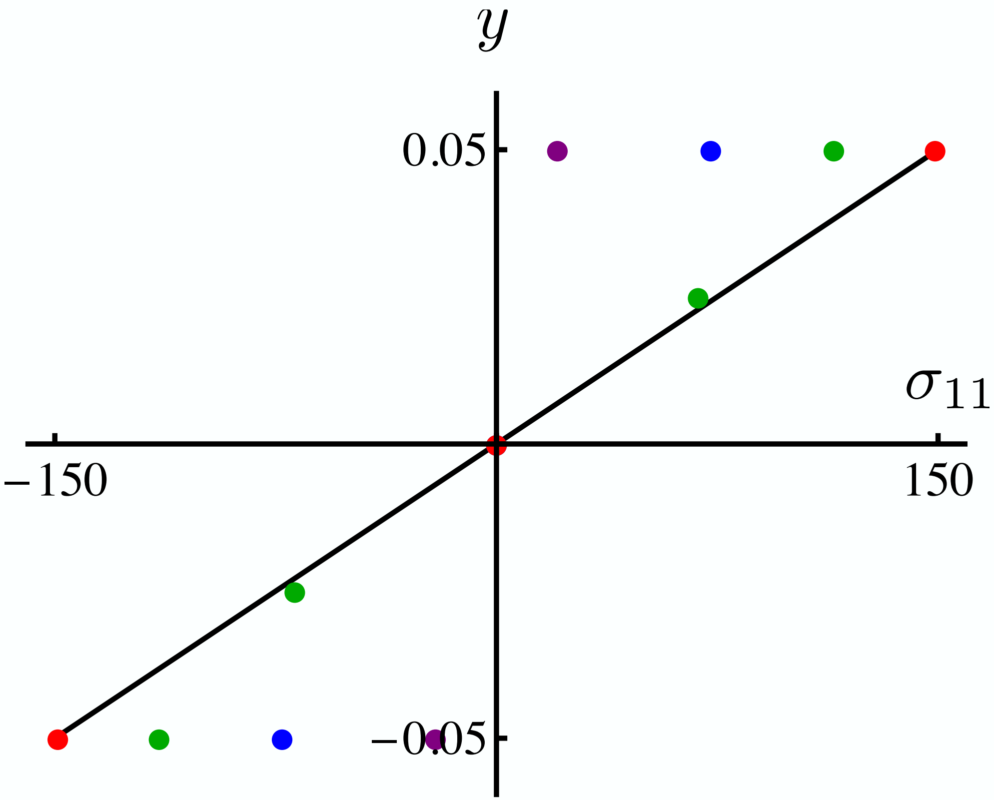
Analytical solution $\sigma_{11} = \dfrac{3F(l-L)x_2}{2h^3}, \quad \sigma_{12} = \dfrac{3F\left(h^2 - x_2^2\right)}{4h^3}$

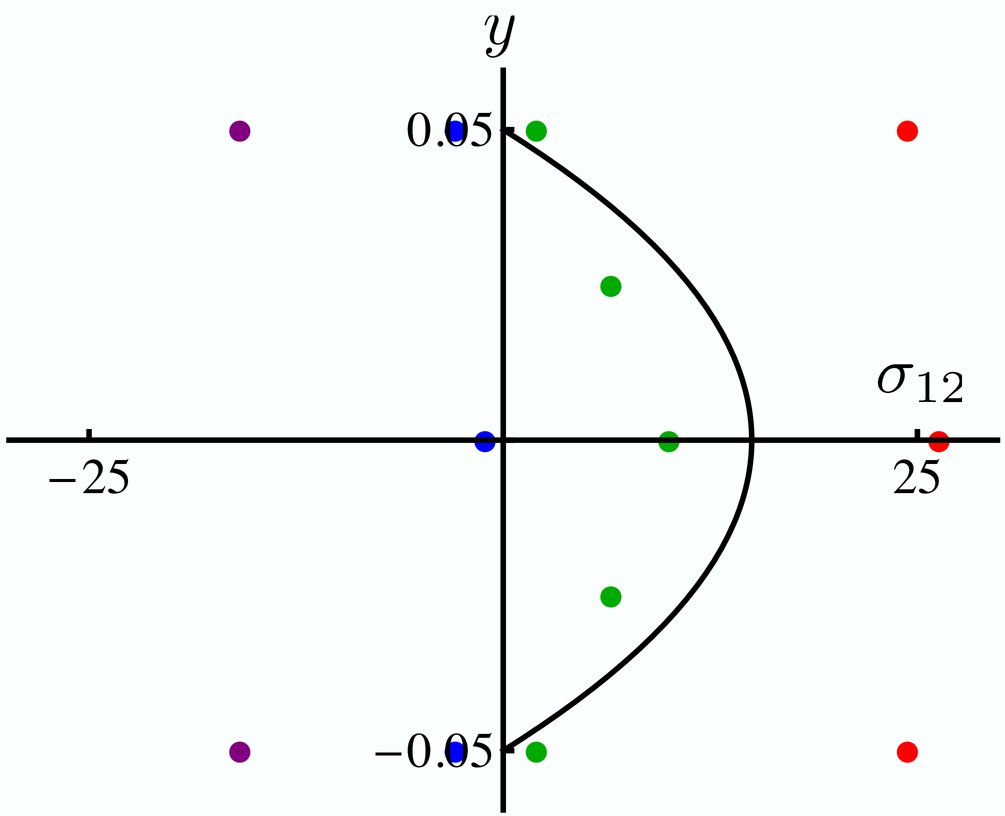
linear element: 2 layers  quadratic element: 1 layer 



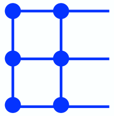
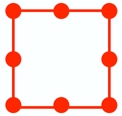
linear element: 1 layers  linear element: 4 layers 

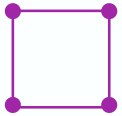
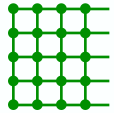
stresses at $x_1/L=l=3/4$

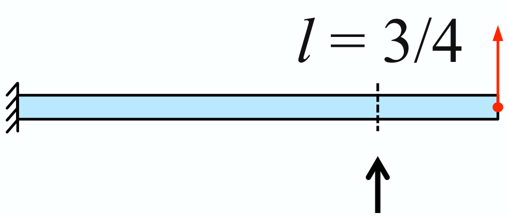


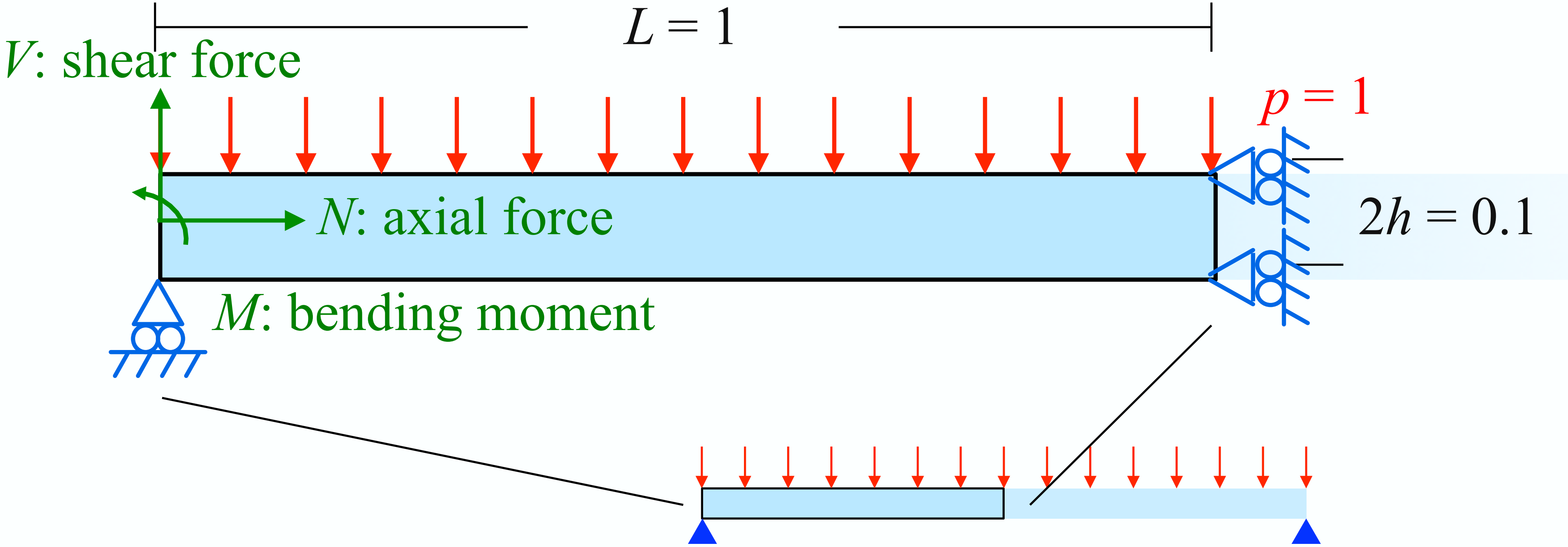


Analytical solution $\sigma_{11} = \dfrac{3F(l-L)x_2}{2h^3}, \quad \sigma_{12} = \dfrac{3F\left(h^2 - x_2^2\right)}{4h^3}$

linear element: 2 layers  quadratic element: 1 layer 

linear element: 1 layers  linear element: 4 layers 

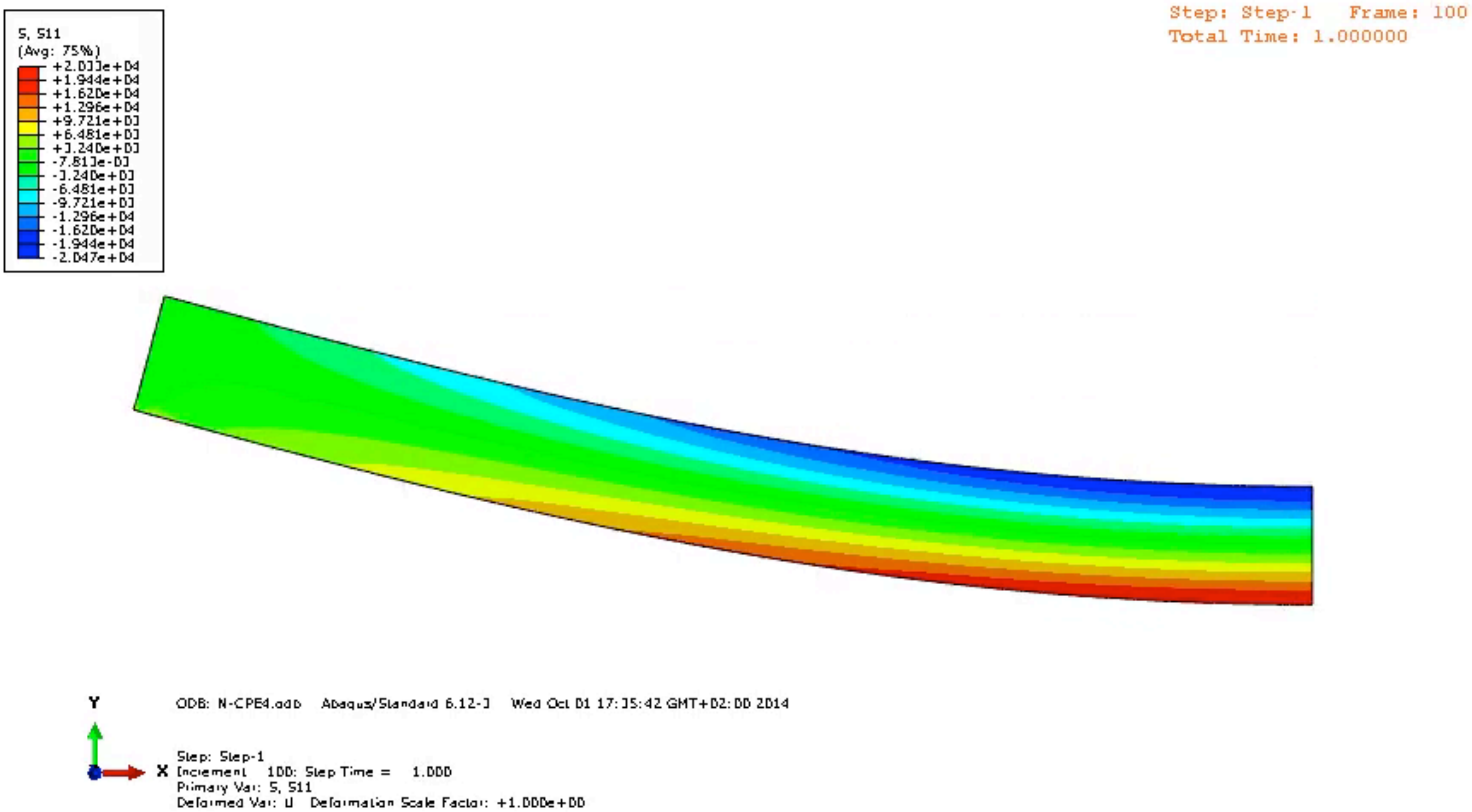




Beam is simply supported at each end.
Due to symmetry only half needs to be modeled.

Analytical solution found with the help of Airy's stress function satisfies average equilibrium conditions at each end, while FEM solution considers correct boundary conditions

FEM solution of simply supported beam showing end effects



$$\phi(x_1, x_2) = \frac{ph^2}{40}\left\{ \left(\frac{x_2}{h}\right)^3 \left(2 - \left(\frac{x_2}{h}\right)^2\right) + \right.$$

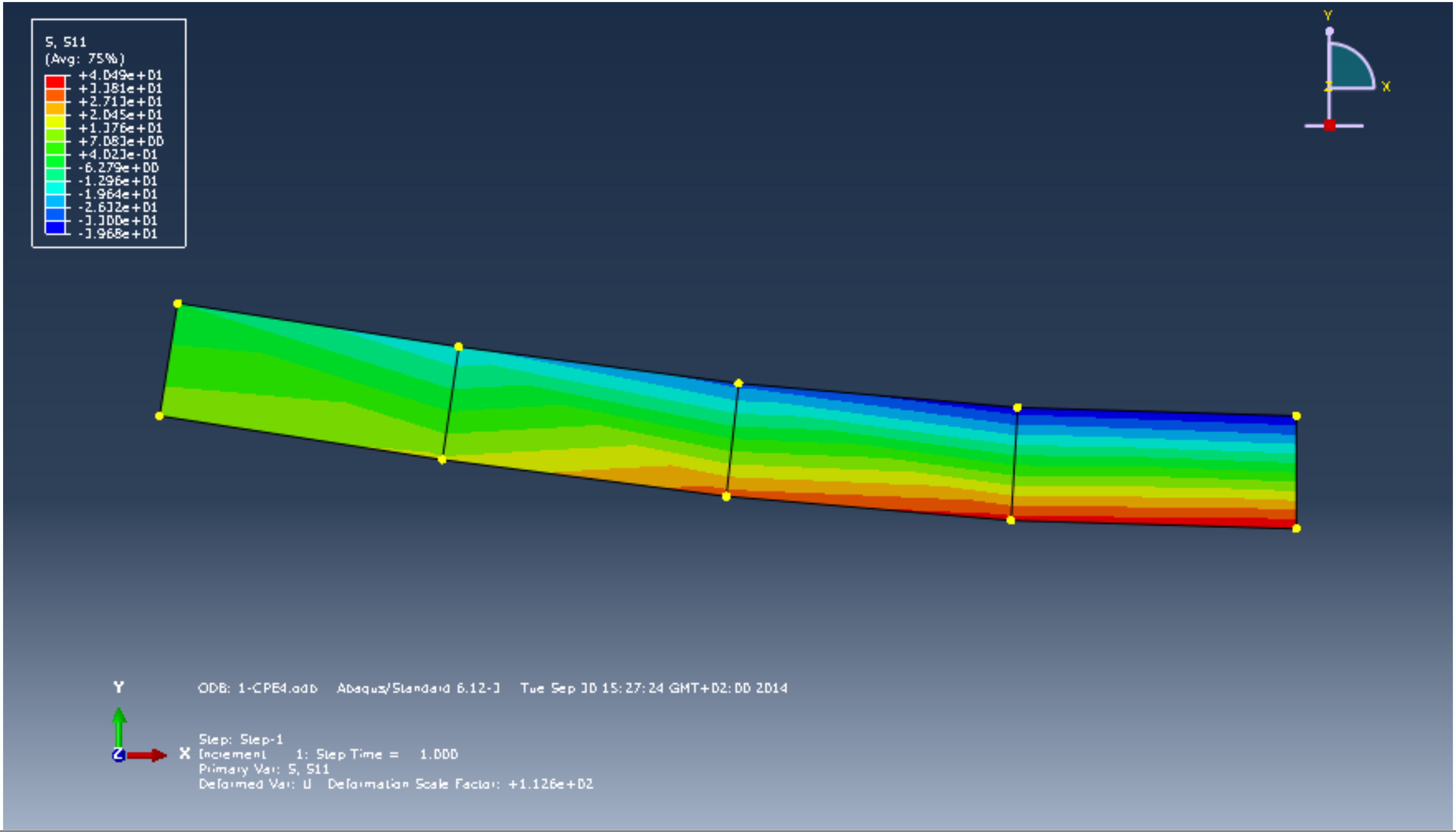
$$\left. + 5\left(\frac{L}{h}\right)^2 \left[\left(\frac{x_1}{L}\right)^2 \left(\left(\frac{x_2}{h}\right)^3 - 3\left(\frac{x_2}{h}\right) - 2\right) - \left(\frac{x_2}{h}\right)^3\right]\right\}$$

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2 \partial x_2} = \frac{3p}{4}\left(\frac{L}{h}\right)^2 \frac{x_2}{h}\left[\left(\frac{x_1}{L}\right)^2 - 1\right] + \frac{p}{10}\frac{x_2}{h}\left(3 - 5\left(\frac{x_2}{h}\right)^2\right)$$

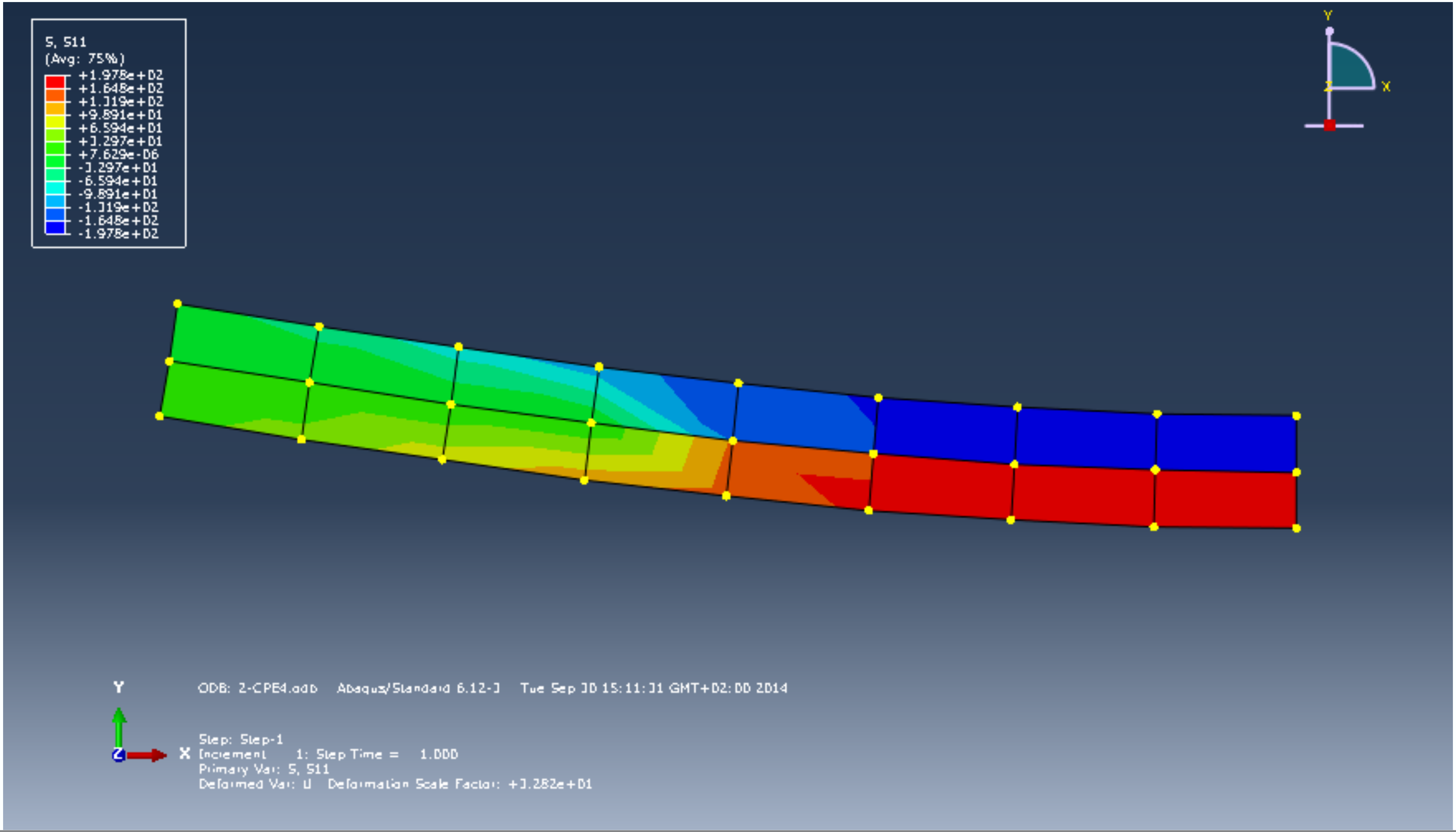
$$\sigma_{22} = \frac{\partial^2 \phi}{\partial x_1 \partial x_1} = \frac{p}{4}\left[\left(\frac{x_2}{h}\right)^3 - 3\left(\frac{x_2}{h}\right) - 2\right]$$

$$\sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = \frac{3p}{4}\left(\frac{L}{h}\right)\left(\frac{x_1}{L}\right)\left[1 - \left(\frac{x_2}{h}\right)^2\right]$$

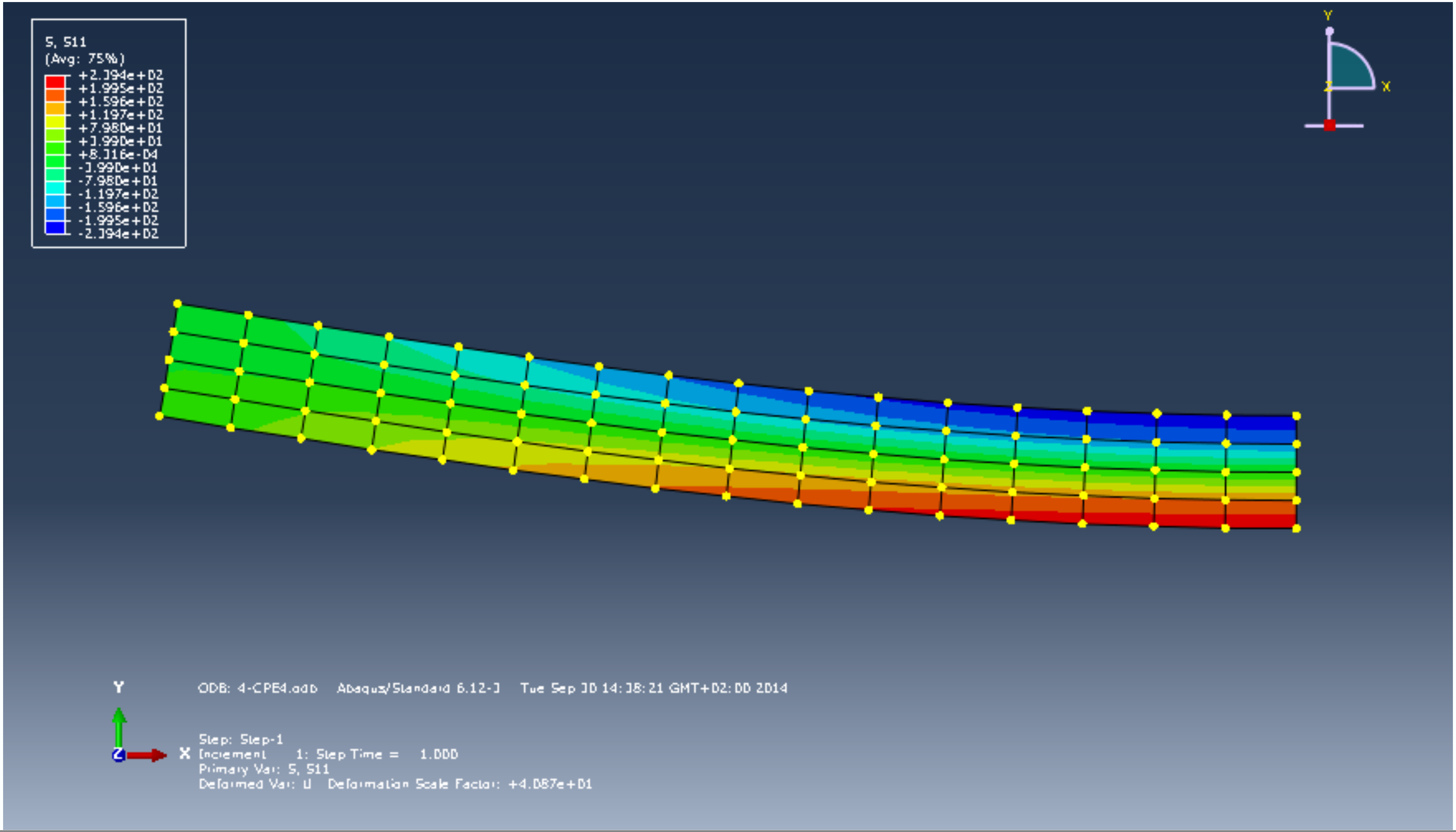
FEM solution using 4 4-node quads



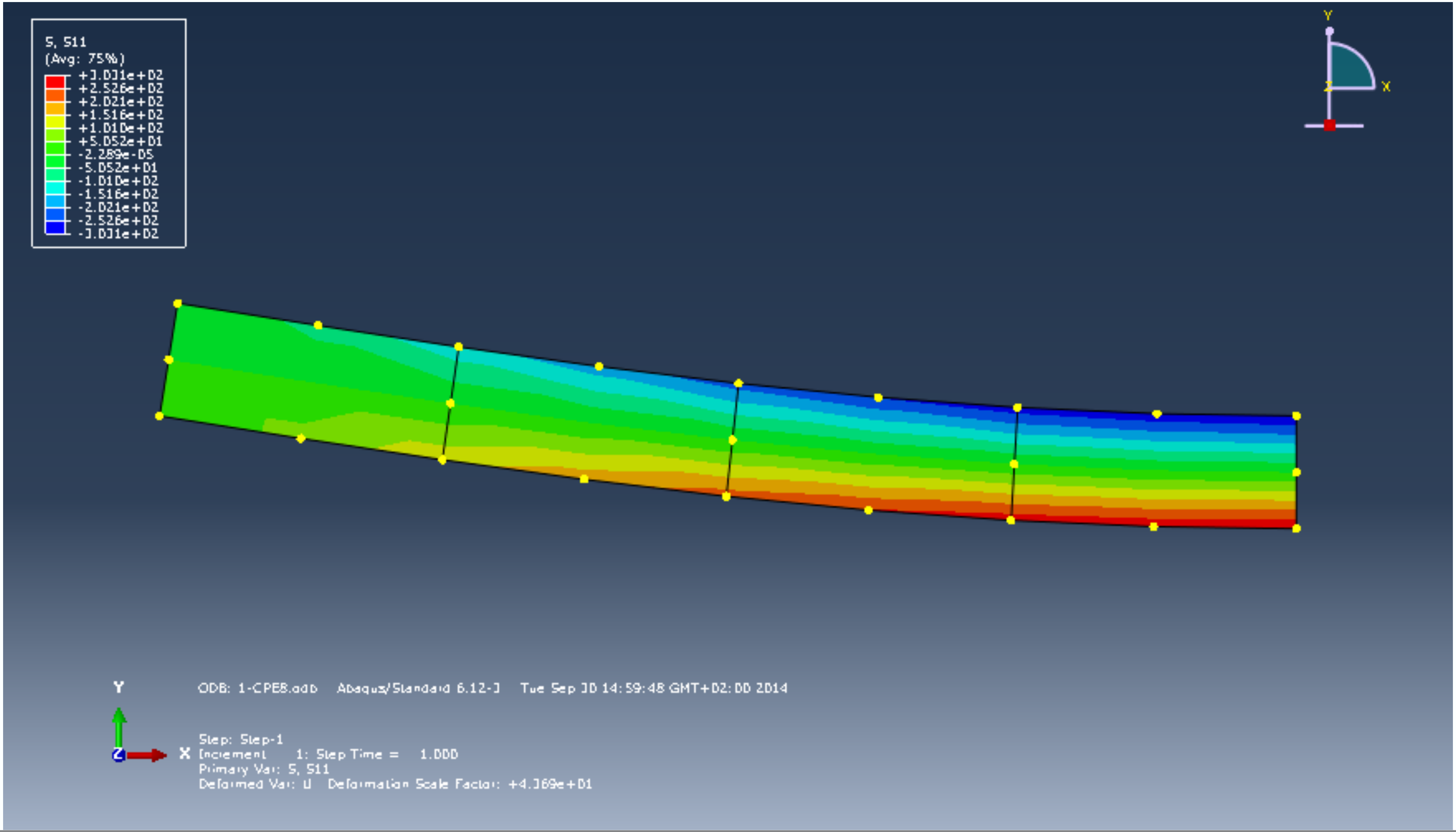
FEM solution using 16 4-node quads



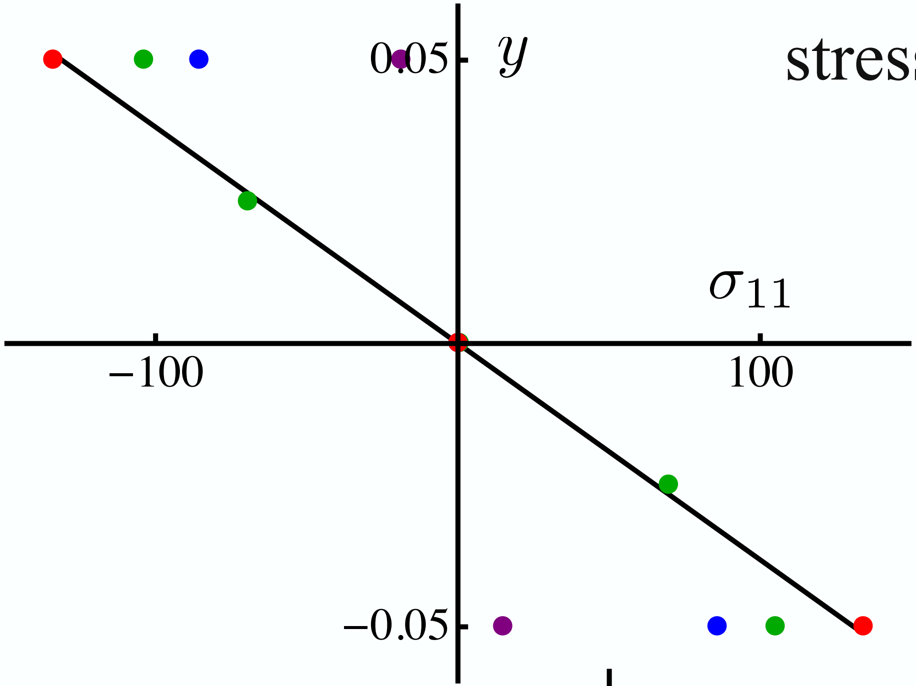
FEM solution using 64 4-node quads

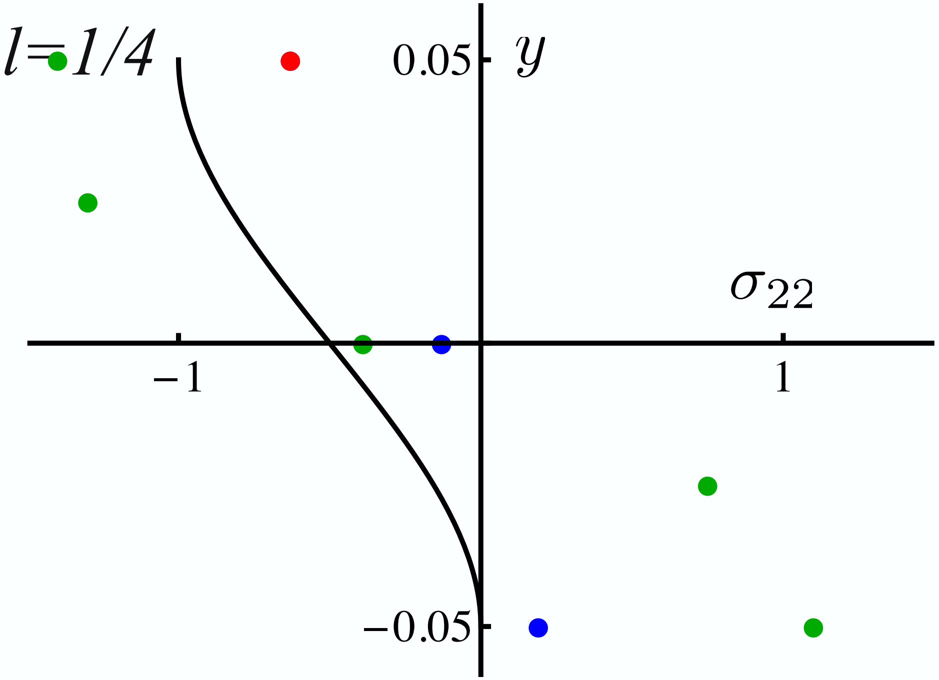


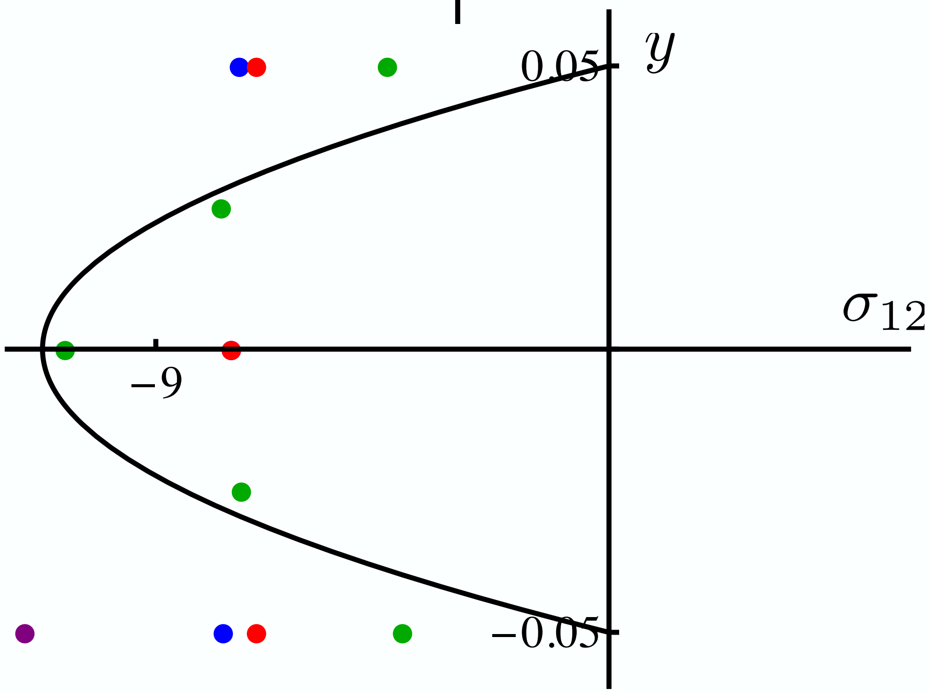
FEM solution using 4 8-node quads



stresses at $x_1/L=l=1/4$



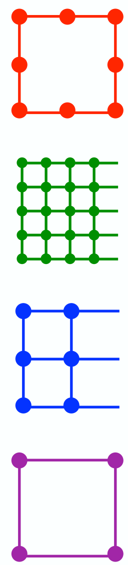




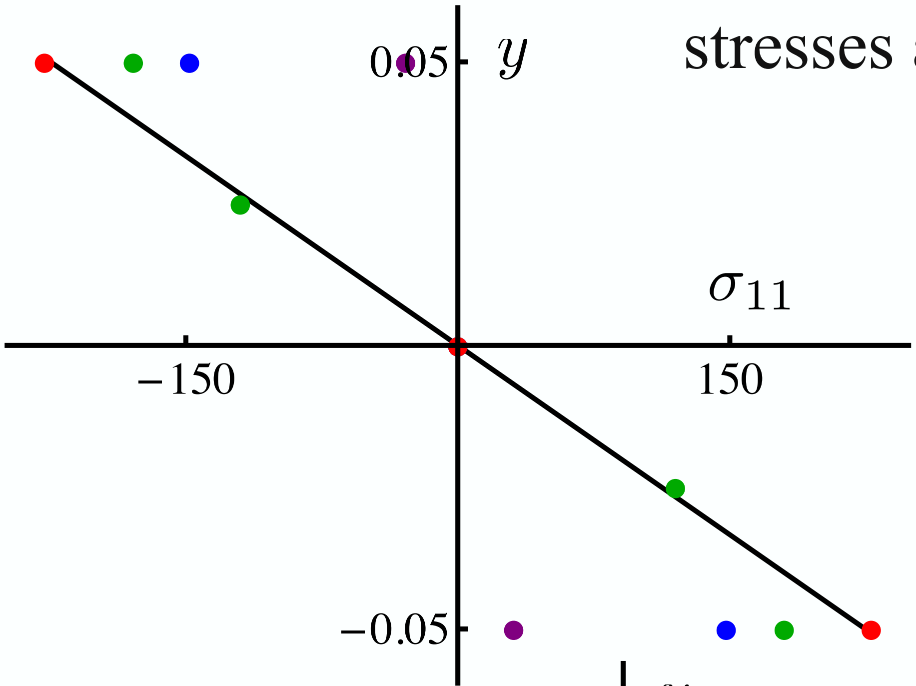
$$\sigma_{11} = \frac{3p}{4}\left(\frac{L}{h}\right)^2 \frac{x_2}{h}\left[\left(\frac{x_1}{L}\right)^2 - 1\right] + O(1)$$

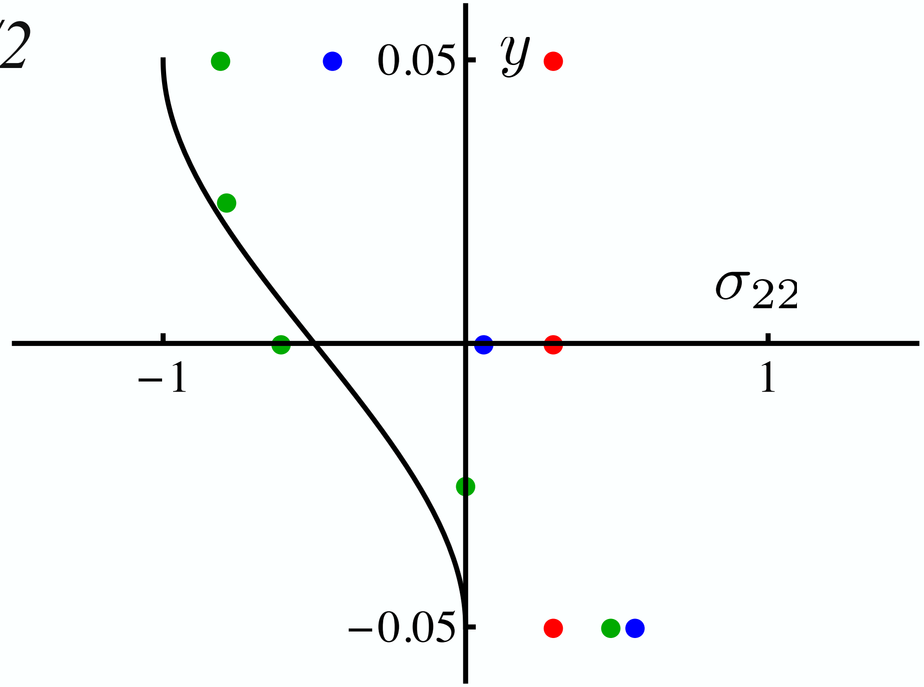
$$\sigma_{12} = \frac{3p}{4}\left(\frac{L}{h}\right)\left(\frac{x_1}{L}\right)\left[1 - \left(\frac{x_2}{h}\right)^2\right]$$

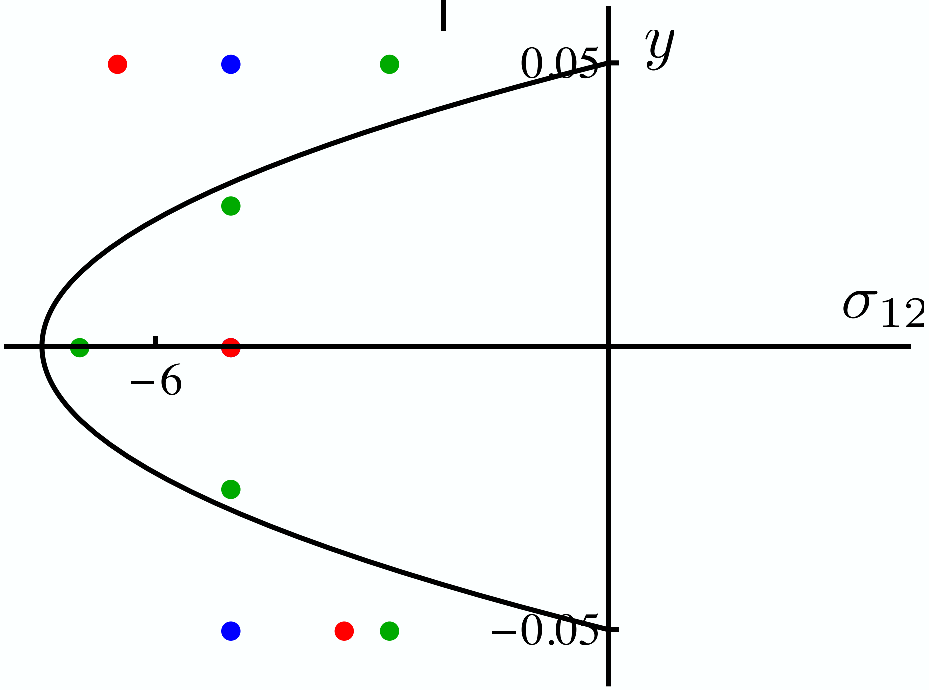
$$\sigma_{22} = \frac{p}{4}\left[\left(\frac{x_2}{h}\right)^3 - 3\left(\frac{x_2}{h}\right) - 2\right]$$



stresses at $x_1/L=l=1/2$



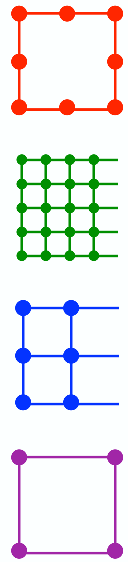




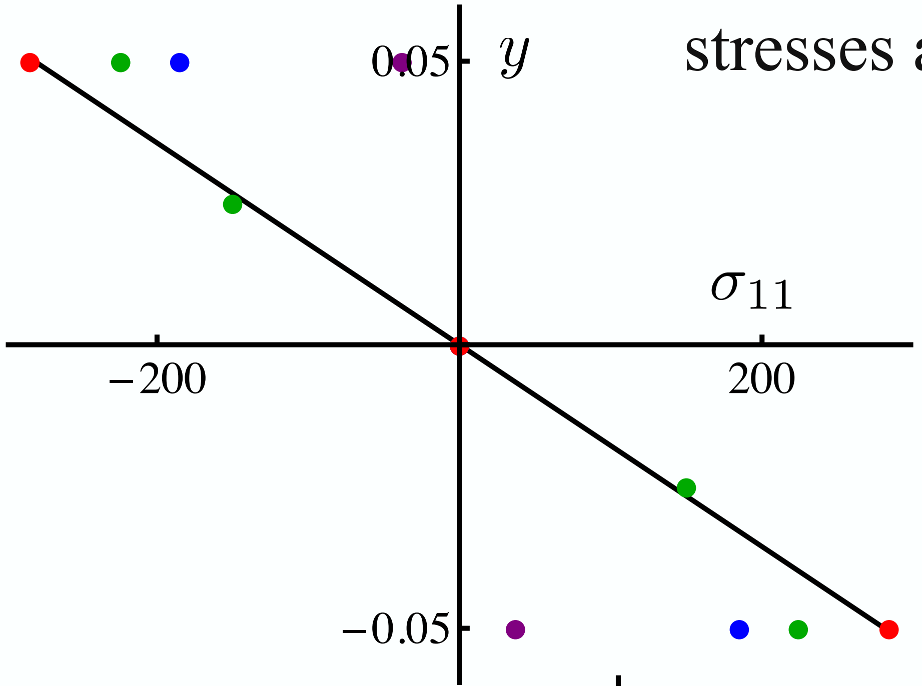
$$\sigma_{11} = \frac{3p}{4}\left(\frac{L}{h}\right)^2 \frac{x_2}{h}\left[\left(\frac{x_1}{L}\right)^2 - 1\right] + O(1)$$

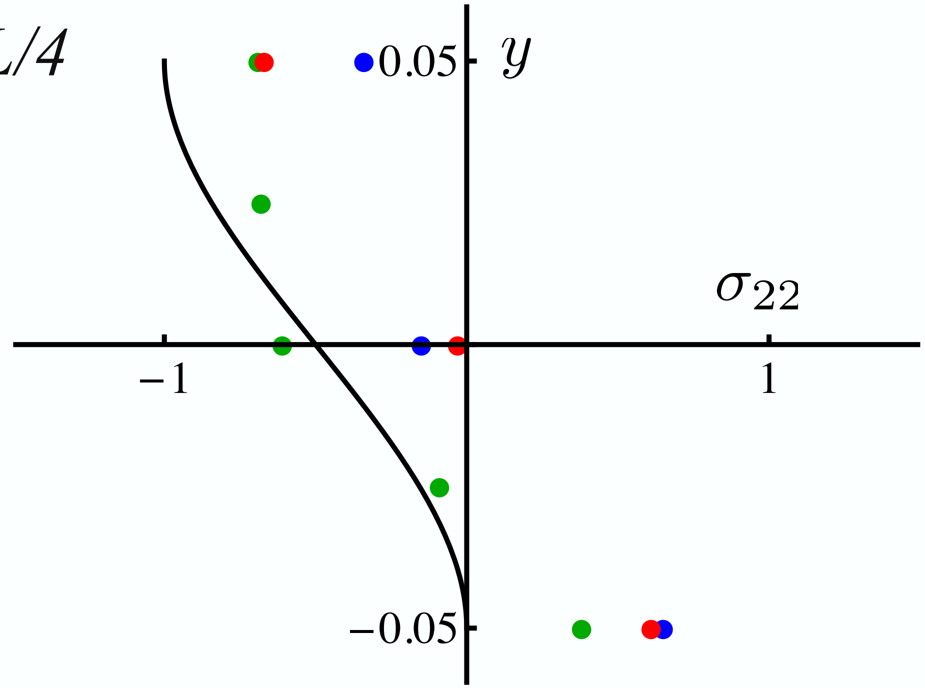
$$\sigma_{12} = \frac{3p}{4}\left(\frac{L}{h}\right)\left(\frac{x_1}{L}\right)\left[1 - \left(\frac{x_2}{h}\right)^2\right]$$

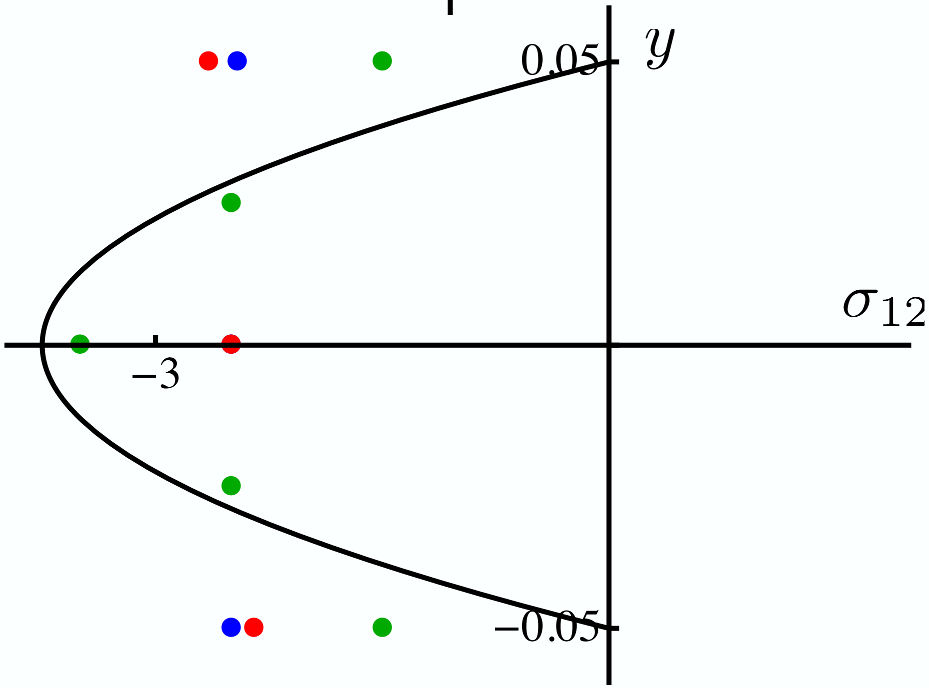
$$\sigma_{22} = \frac{p}{4}\left[\left(\frac{x_2}{h}\right)^3 - 3\left(\frac{x_2}{h}\right) - 2\right]$$



stresses at $x_1/L = l = 3L/4$



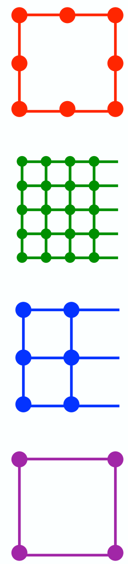




$$\sigma_{11} = \frac{3p}{4}\left(\frac{L}{h}\right)^2 \frac{x_2}{h}\left[\left(\frac{x_1}{L}\right)^2 - 1\right] + O(1)$$

$$\sigma_{12} = \frac{3p}{4}\left(\frac{L}{h}\right)\left(\frac{x_1}{L}\right)\left[1 - \left(\frac{x_2}{h}\right)^2\right]$$

$$\sigma_{22} = \frac{p}{4}\left[\left(\frac{x_2}{h}\right)^3 - 3\left(\frac{x_2}{h}\right) - 2\right]$$



NUMERICAL INTEGRATION IN 1D AND 2D

$$\text{quadrature}: \boxed{\int_{-1}^{+1} f(\xi)\, d\xi = \sum_{j=1}^{n_I} w_I f(\xi^I) + R \approx \sum_{j=1}^{n_I} w_I f(\xi^I)}$$

Weights **quadrature points** **remainder**

$$\text{trapezoidal}: n_I = 2; \quad \xi^1 = -1,\ \xi^2 = +1$$

accuracy order

$$w_1 = w_2 = 1; \quad R = -\frac{2}{3}\frac{d^2 f}{d\xi^2}(\xi^*); \quad \xi^* \in (-1, 1)$$

$$\text{Simpson}: n_I = 3; \quad \xi^1 = -1,\ \xi^2 = 0,\ \xi^3 = +1$$

$$w_1 = w_3 = \frac{1}{3},\ w_2 = \frac{4}{3}; \quad R = -\frac{1}{90}\frac{d^4 f}{d\xi^4}(\xi^*)$$

Gaussian quadrature of *1, 2 and 3 points*

$$1 \text{ point Gauss}: n_I = 1; \quad \xi^1 = 0$$

$$w_1 = 2; \quad R = \frac{1}{3}\frac{d^2 f}{d\xi^2}(\xi^*); \quad \xi^* \in (-1, 1)$$

$$2 \text{ point Gauss}: n_I = 2; \quad \xi^1 = -\frac{1}{\sqrt{3}}, \ \xi^2 = \frac{1}{\sqrt{3}}$$

$$w_1 = w_2 = 1; \quad R = \frac{1}{135}\frac{d^4 f}{d\xi^4}(\xi^*)$$

$$3 \text{ point Gauss}: n_I = 3; \quad \xi^1 = -\frac{\sqrt{3}}{\sqrt{5}}, \ \xi_2 = 0, \ \xi^3 = \frac{\sqrt{3}}{\sqrt{5}}$$

$$w_1 = w_3 = \frac{5}{9}, \ w_2 = \frac{8}{9}; \quad R = \frac{1}{15750}\frac{d^6 f}{d\xi^6}(\xi^*)$$

General Gaussian quadrature of n_I points is $2n_I$ accurate

$$\text{Gauss points : } \xi^I \text{ are roots of Legendre polynomial } P_{n_I}(\xi^I) = 0$$

$$\text{weight : } w_I = \frac{2}{[(1-(\xi^I)^2]\left[\frac{dP_{n_I}}{d\xi}(\xi^I)\right]^2}$$

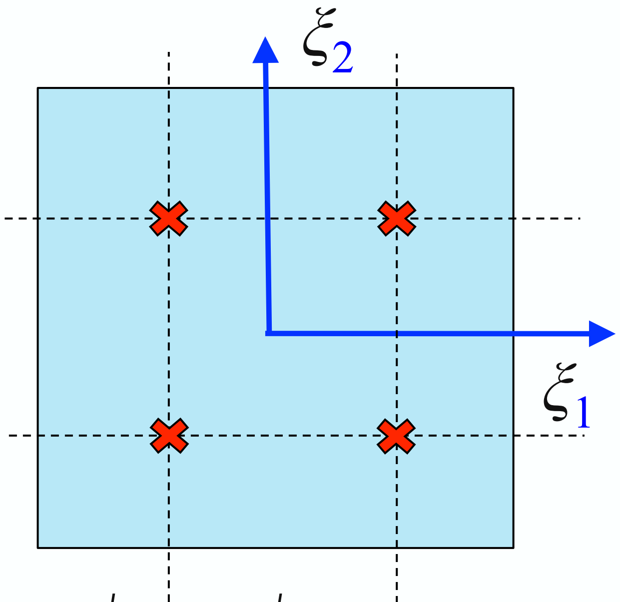
$$\text{remainder : } R = \frac{2^{(2n_I)}(n_I!)^4}{(2n_I+1)[(2n_I)!]^3}\frac{d^{(2n_I)}f}{d\xi^{(2n_I)}}(\xi^*)$$

$$n_I \text{ order Legendre : } P_{n_I}(\xi) \equiv \frac{1}{2^{n_I}(n_I)!}\frac{d^{(n_I)}}{d\xi^{(n_I)}}(\xi^2-1)^{n_I}$$

General Gaussian quadrature in 2D uses master element

$$\int_{-1}^{+1}\int_{-1}^{+1} f(\xi_1,\xi_2)\, d\xi_1 d\xi_2 = \sum_{I=1}^{I=n_I}\sum_{J=1}^{J=n_I} w_I w_J f(\xi_1^I,\xi_2^J)$$

$(\xi_1^2,\xi_2^1) = (1/\sqrt{3},-1/\sqrt{3})$ $(\xi_1^2,\xi_2^2) = (1/\sqrt{3},1/\sqrt{3})$



A 2×2 Gauss integration uses grid points with coordinates and weights taken from 1D

$(\xi_1^1,\xi_2^1) = (-1/\sqrt{3},-1/\sqrt{3})$ $(\xi_1^1,\xi_2^2) = (-1/\sqrt{3},1/\sqrt{3})$