TOPICS COVERED IN THIS LECTURE

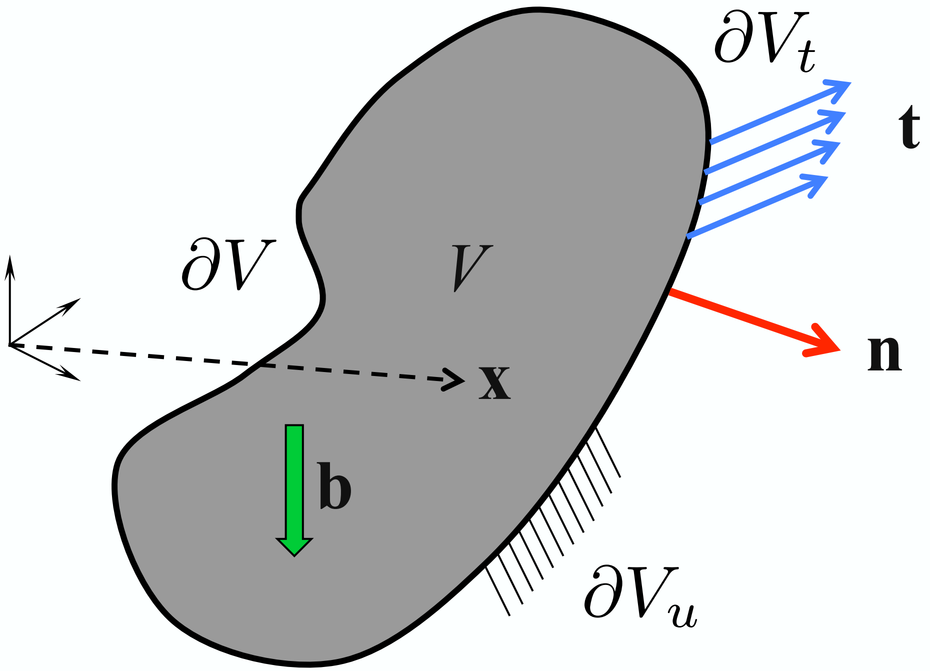
1. **REVIEW OF SMALL STRAIN LINEAR ELASTICITY (2D & 3D)**

2. **2D EXAMPLE: FLAMANT PROBLEM (POINT LOAD IN HALFSPACE)**

3. **CONSTANT STRAIN TRIANGLES – ELEMENT STIFFNESS & FORCE DERIVATIONS**

4. **ISOPARAMETRIC CONSIDERATIONS FOR C.S.T.**

REVIEW OF SMALL STRAIN LINEAR ELASTICITY (2D & 3D)



Solid occupies domain: V

Domain boundary: ∂V

Body forces: $\mathbf{b}$

Surface traction: $\mathbf{t}$

Surface normal (outward): $\mathbf{n}$

Traction prescribed on: ∂V_t

Displacement prescribed on: ∂V_u

Position vector: $\mathbf{x}$

Energy density: $W(\boldsymbol{\varepsilon})$

Stress-strain: $\sigma_{ij} = \dfrac{\partial W}{\partial \epsilon_{ij}}$

(general nonlinear elastic material)

Potential : $\mathcal{P} = \mathcal{P}_{int} + \mathcal{P}_{ext}$

Internal : $\mathcal{P}_{int} = \int_V W(\epsilon_{ij})\, dV \; ; \quad \sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$

External : $\mathcal{P}_{ext} = -\int_V b_i u_i \, dV - \int_{\partial V_t} t_i u_i \, dS$

$$\mathcal{P}(\mathbf{u} + \epsilon \delta \mathbf{u}) \geq \mathcal{P}(\mathbf{u}) \; ; \qquad \delta \mathbf{u}(\mathbf{x}) = 0 \; \forall \; \mathbf{x} \in \partial V_u \; , \; \epsilon \in \mathbb{R}$$

$$\frac{d}{d\epsilon} \left[\mathcal{P}(\mathbf{u} + \epsilon \delta \mathbf{u}) \right]_{\epsilon=0} = 0 \; ; \qquad \text{extremum (1)}$$

$$\frac{d^2}{d\epsilon^2} \left[\mathcal{P}(\mathbf{u} + \epsilon \delta \mathbf{u}) \right]_{\epsilon=0} > 0 \; ; \qquad \text{minimum (2)}$$

Linearized kinematics : $\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

Boundary traction : $t_j = n_i \, \sigma_{ij}$; (Cauchy tetrahedron)

Linear elasticity : $\sigma_{ij} = L_{ijkl} \, \epsilon_{kl}$

Major symmetry : $L_{ijkl} = L_{klij}$; (due to energy existence)

Minor symmetries : $L_{ijkl} = L_{jikl} = L_{ijlk}$; $(\sigma_{ij} = \sigma_{ji} \;,\; \epsilon_{ij} = \epsilon_{ji})$

Energy density : $W = \int_0^{\epsilon} \sigma_{ij} \, \epsilon_{ij} \, d\epsilon = \frac{1}{2} L_{ijkl} \, \epsilon_{ij} \, \epsilon_{kl}$

Linearized strain: ε_{ij} Cauchy stress: σ_{ij} Elastic moduli tensor: L_{ijkl}

$$(1) \implies \int_V \left[\frac{\partial W}{\partial \epsilon_{ij}} \, \delta\epsilon_{ij} \right] dV - \int_V [b_i \, \delta u_i] \, dV - \int_{\partial V} [t_i \, \delta u_i] \, dS = 0$$

$$\int_V \left[\frac{\partial W}{\partial \epsilon_{ij}} \, \delta\epsilon_{ij} \right] dV = \int_V \left[\sigma_{ij} \frac{\partial \delta u_i}{\partial x_j} \right] dV$$

$$\int_V \left[\sigma_{ij} \frac{\partial \delta u_j}{\partial x_i} \right] dV = \int_V \left[\frac{\partial}{\partial x_i} (\sigma_{ij} \delta u_j) - \frac{\partial \sigma_{ij}}{\partial x_i} \delta u_j \right] dV$$

$$\int_V \left[\frac{\partial}{\partial x_i} (\sigma_{ij} \delta u_j) \right] dV = \int_{\partial V} [n_i \sigma_{ij} \delta u_j] \, dS = \int_{\partial V_t} [t_j \delta u_j] \, dS$$

$$(1) \implies \int_V \left[\left(\frac{\partial \sigma_{ij}}{\partial x_i} + b_j \right) \delta u_j \right] dV - \int_{V_t} [(n_i \sigma_{ij} - t_j) \delta u_j] \, dS = 0$$

Equilibrium **Traction (natural) boundary condition**

elastic moduli : L_{ijkl} have at most 21 constants Here: *i = 1, 2, 3*

energy invariance : $W(\epsilon_{pq}) = W(R_{pi}\ \epsilon_{ij} R_{qj}) \quad \forall\ \mathbf{R} \in \mathcal{G}$

moduli invariance : $L_{pqrs} = R_{pi} R_{qj} R_{rk} R_{sl} L_{ijkl}$

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for isotropy : $\mathcal{G} = SO(3)$ (all rigid body rotations)

isotropic moduli : $$L_{ijkl} = \frac{E}{1+\nu}\left[\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \frac{\nu}{1-2\nu}\left(\delta_{ij}\delta_{kl}\right)\right]$$

two constants : E : Young modulus, ν : Poisson ratio

$$(2) \Longrightarrow \int_V \left[L_{ijkl}\delta\epsilon_{ij}\delta\epsilon_{kl}\right]\, dV > 0 \ (E > 0\ , -1 < \nu < 0.5)$$

equilibrium : $$\frac{\partial \sigma_{ij}}{\partial x_i} + b_j = 0 \,, \ \mathbf{x} \in V$$

Here: $i = 1, 2$

boundary : $$t_j = n_i \sigma_{ij}, \ \mathbf{x} \in \partial V_t \ ; \quad u_i = \text{given}, \ \mathbf{x} \in \partial V_u$$

kinematics : $$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

constitutive : $$\sigma_{ij} = L_{ijkl}\epsilon_{kl}$$

plane strain : $$L_{ijkl} = \frac{E}{1+\nu}\left[\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \frac{\nu}{1-2\nu}\left(\delta_{ij}\delta_{kl}\right)\right]$$

$u_i(x_1,x_2)$, $u_3=0$

plane stress : $$L_{ijkl} = \frac{E}{1-\nu^2}\left[\frac{1-\nu}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \nu\left(\delta_{ij}\delta_{kl}\right)\right]$$

$\sigma_{i3}=\sigma_{33}=0$

For 2D linear elasticity problems in infinite domains, an analytical solution is easier to obtain by introducing the Airy stress function ϕ

$$\phi \text{ Airy function} \quad : \quad \sigma_{11} = \frac{\partial^2 \phi}{\partial x_2 \partial x_2} \ , \ \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1 \partial x_1} \ , \ \sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$$

$$\text{Compatibility} \quad : \quad \frac{\partial^2 \epsilon_{11}}{\partial x_2 \partial x_2} - 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1 \partial x_1} = 0$$

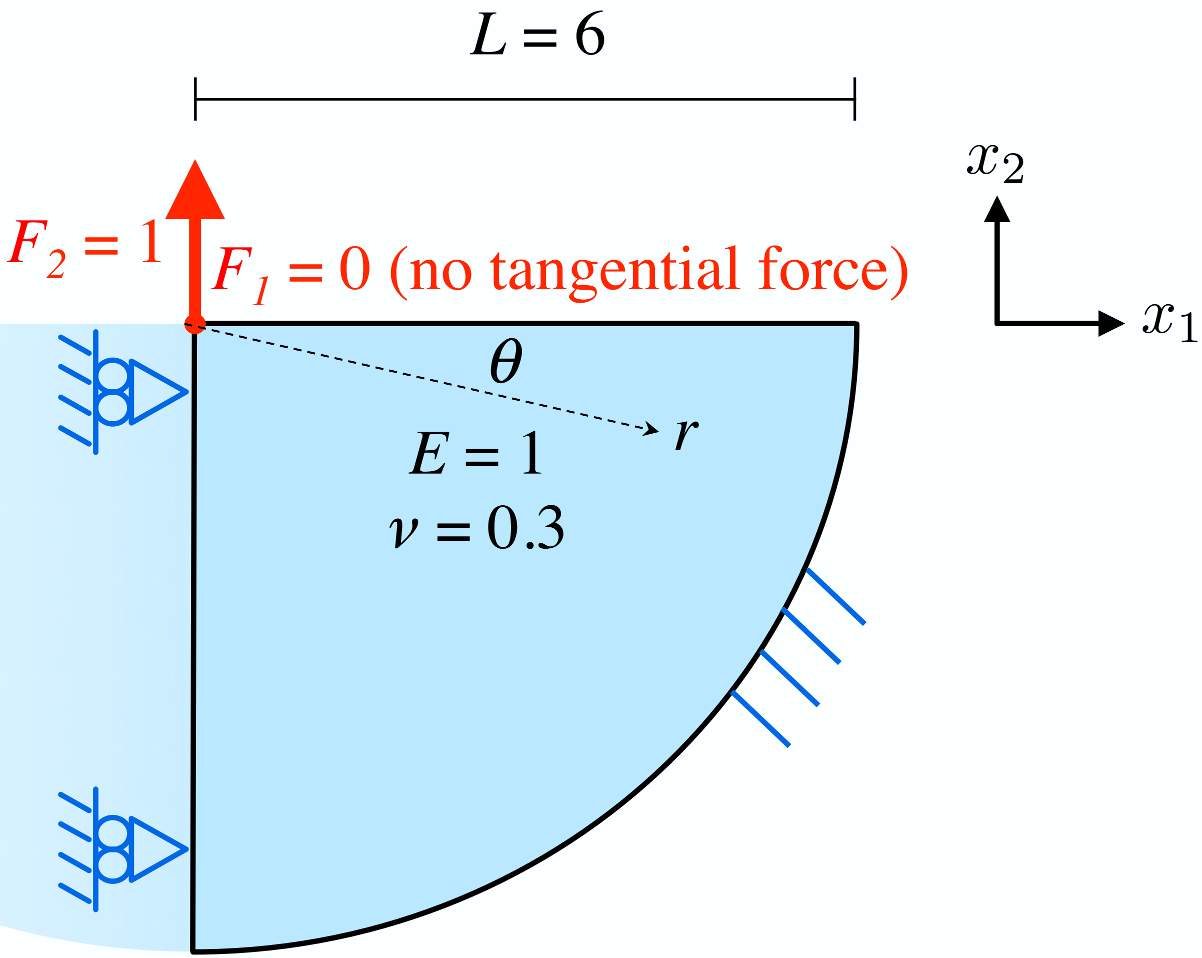
$$\text{for } \mathbf{b} = \mathbf{0} \quad : \quad \boxed{\nabla^4 \phi = 0}; \qquad \phi(x_1, x_2) \text{ is a biharmonic function}$$

By expressing the strain as a function of stress and substituting into the strain compatibility equation we obtain that the Airy stress function satisfies the bi-harmonic equation for either plane stress or plane strain.

2D LINEAR ELASTICITY EXAMPLE: FLAMANT PROBLEM

Flamant problem: point load on a linearly isotropic, elastic half-space

Here only normal point load is considered ($F_1 = 0$) and hence symmetry is used to solve the problem (rollers along the x_2 axis)



$$f(r, \theta) = (r\theta/\pi)(-F_1 \sin\theta + F_2 \cos\theta)$$

In the absence of a natural length, we model a half disc of radius L and remove translation d.o.f. by requiring that $u_2(0, -L) = 0$

Flamant problem: analytical solution

$$u_1 = -\frac{F_1(\kappa+1)\ln|x|}{4\pi\mu} + \frac{F_2(\kappa-1)\mathrm{sgn}(x)}{8\mu}$$

Eliminate translation:
change x to x/L

$$u_2 = -\frac{F_1(\kappa-1)\mathrm{sgn}(x)}{8\mu} - \frac{F_2(\kappa+1)\ln|x|}{4\pi\mu}$$

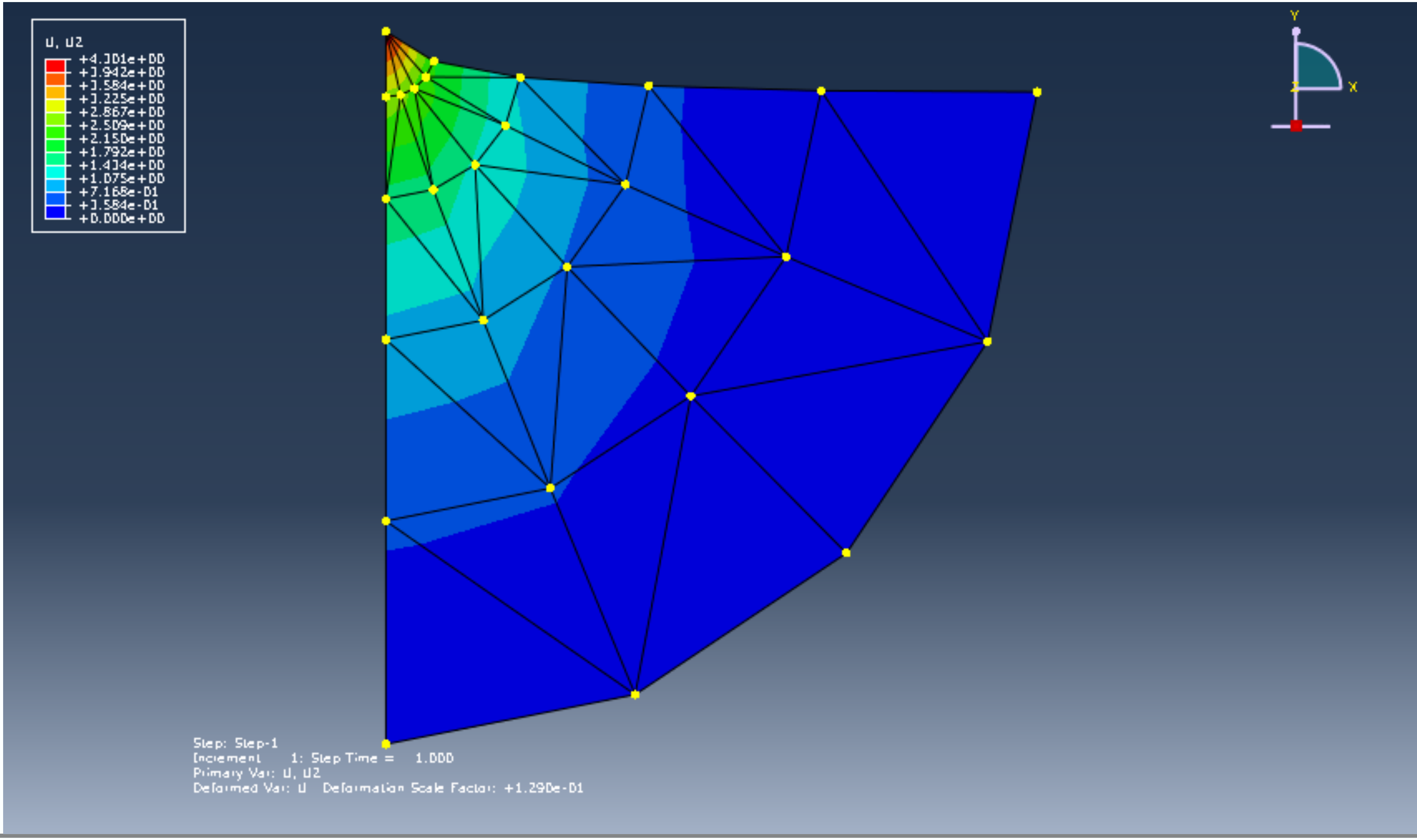
$$\sigma_{rr} = -\frac{2F_1}{\pi r}\cos\theta + \frac{2F_2}{\pi r}\sin\theta$$

$$\sigma_{r\theta} = 0$$

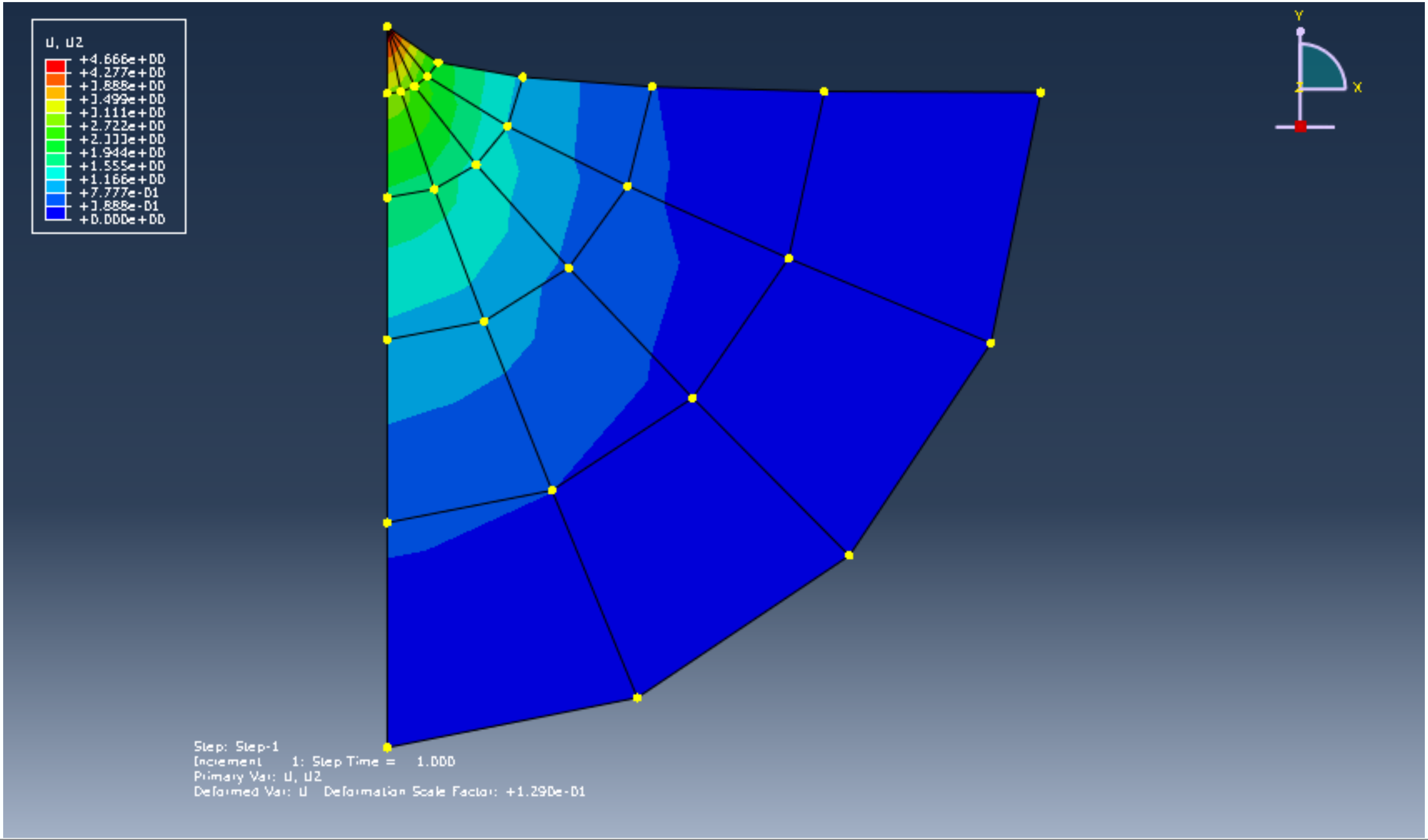
$$\sigma_{\theta\theta} = 0$$

$$\kappa = \begin{cases} 3-4\nu & \text{plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

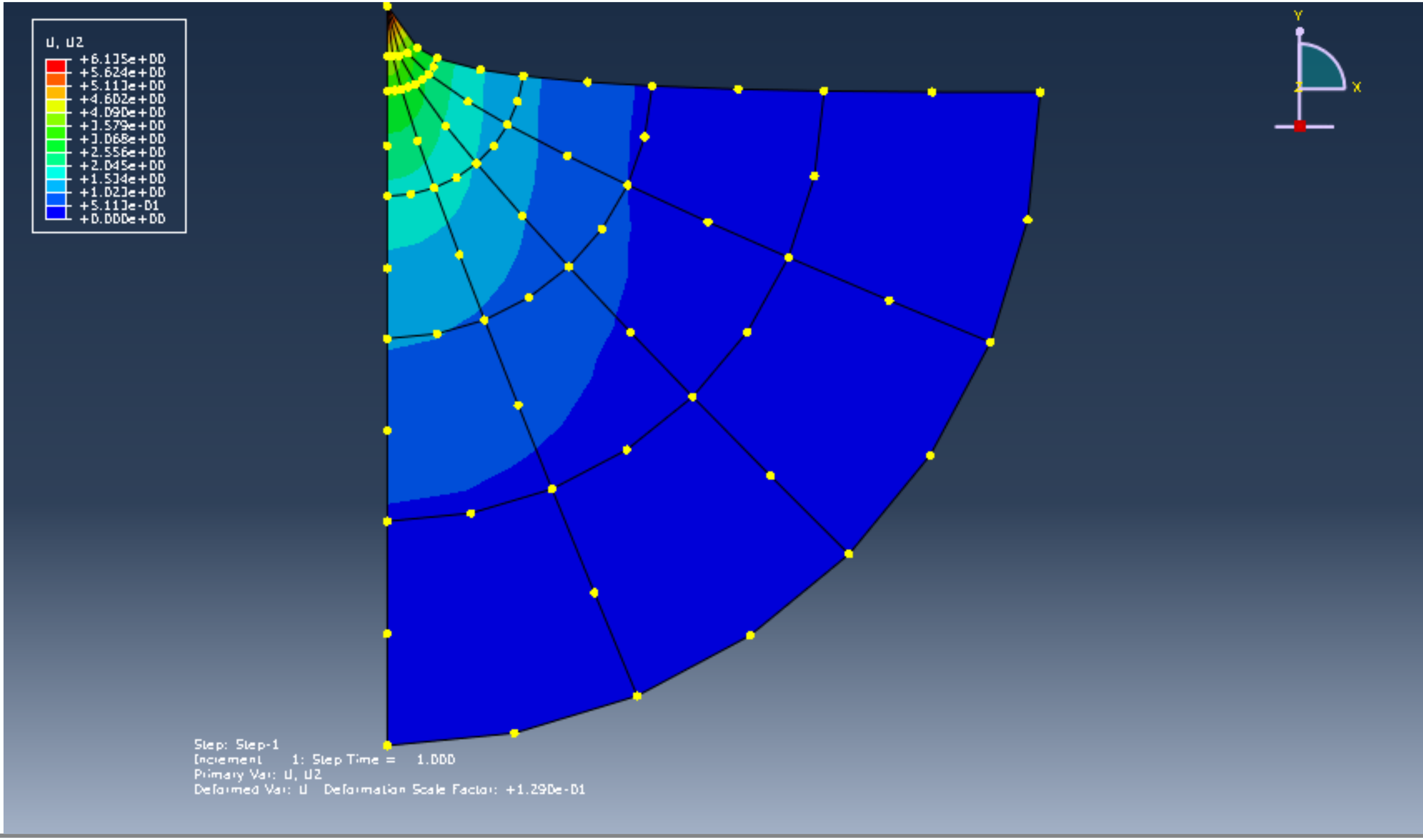
Flamant problem: constant strain triangular elements



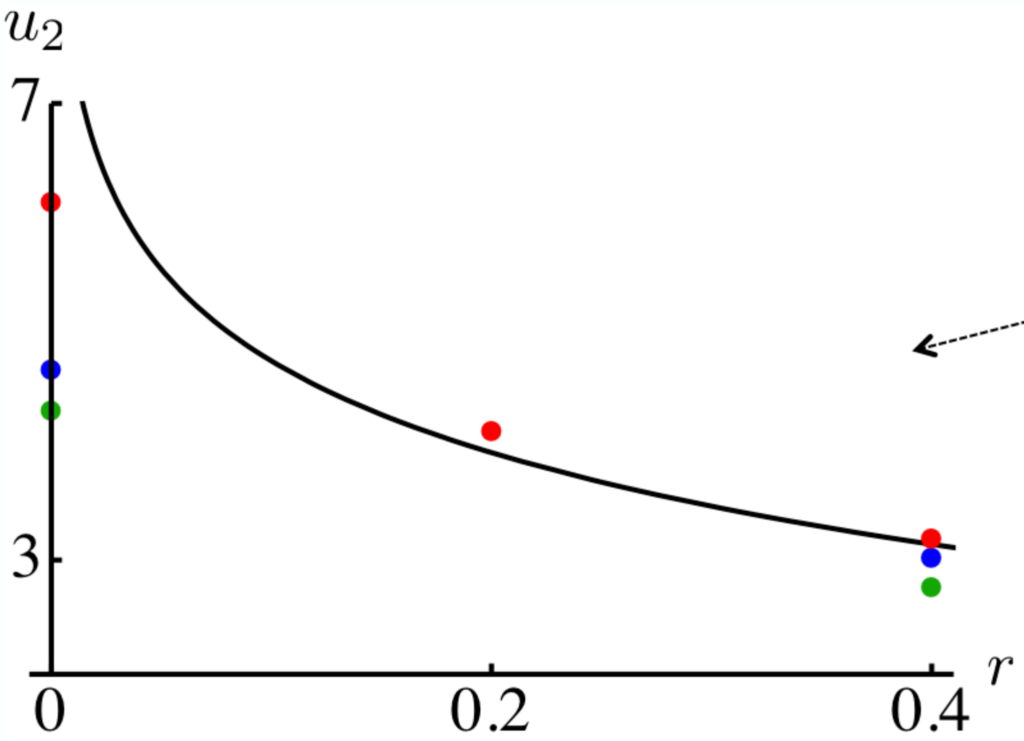
Flamant problem: triangular & quad elements (bilinear)

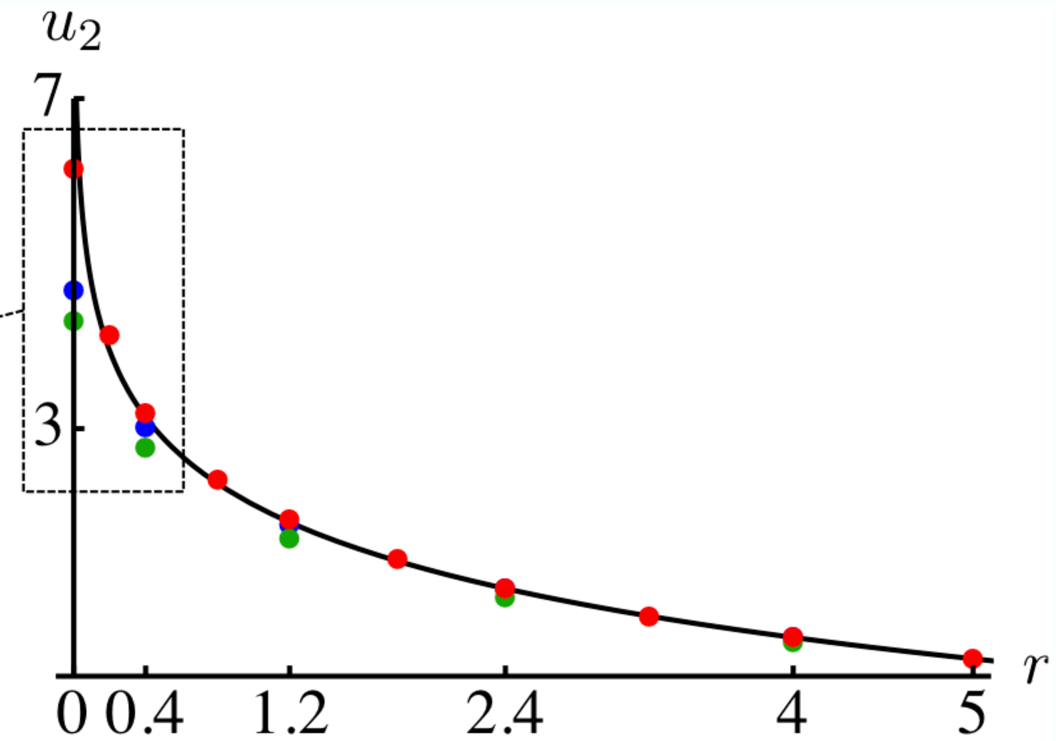


Flamant problem: 6-node triangular & 8-node quad elements (quadratic)



Flamant problem: comparing different elements for accuracy of $u_2(0,x_2)$



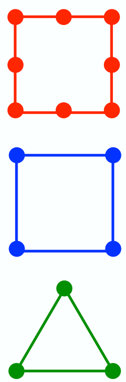


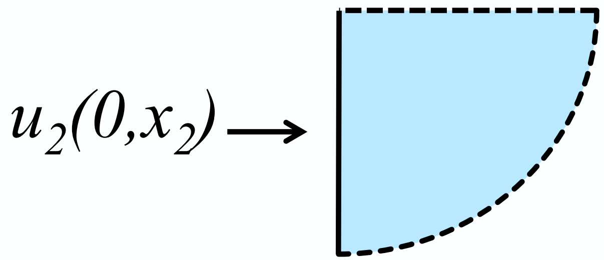
Analytical sol. $u_2 = \frac{4(1-\nu^2)}{\pi} \ln(x_2/L)$

FEM result with quadratic elements

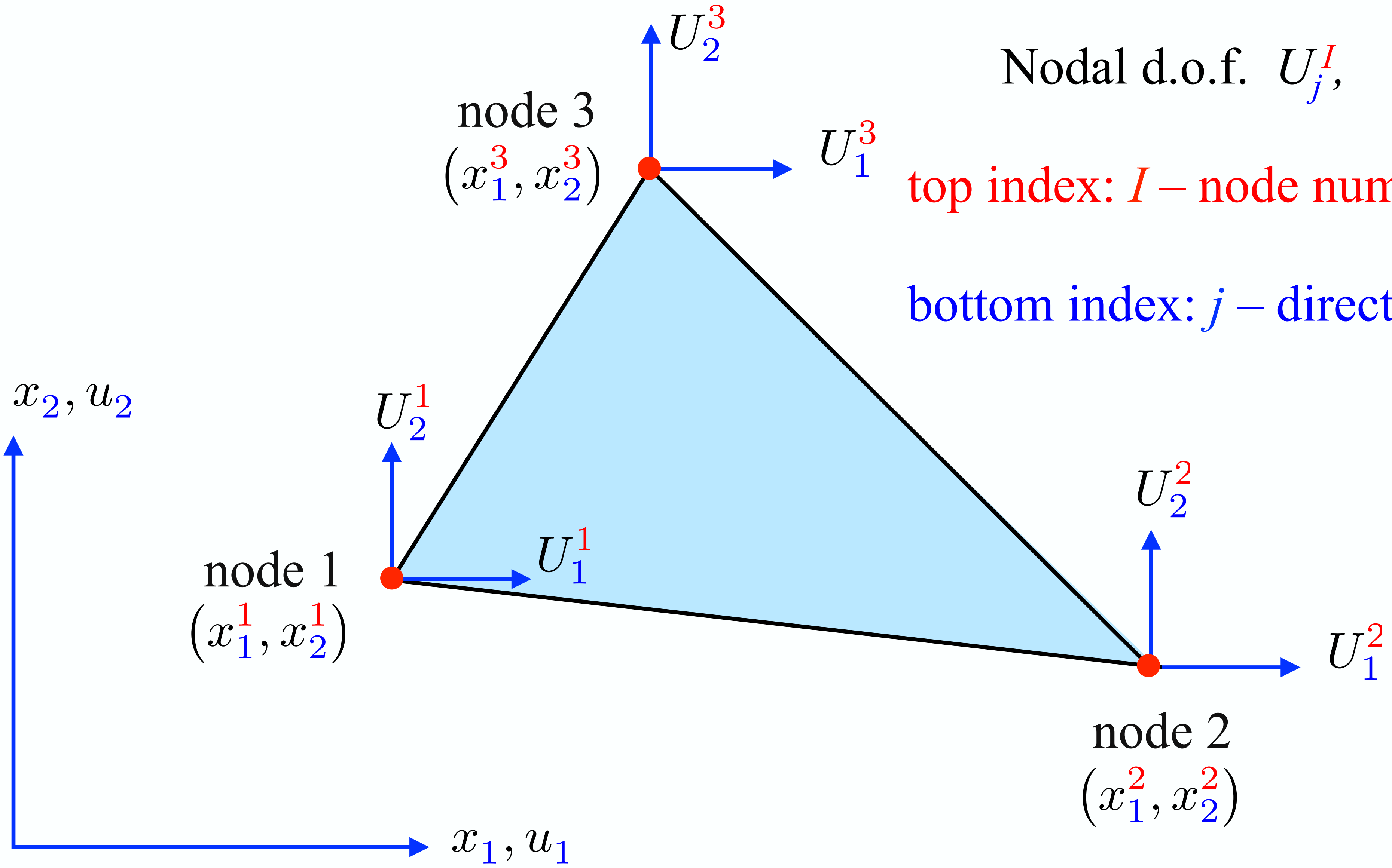
FEM result with bilinear elements

FEM result with triangular elements





SIMPLEST 2D ELEMENT: CONSTANT STRAIN TRIANGLE



Nodal d.o.f. U_j^I,

top index: I – node number

bottom index: j – direction

Element d.o.f. $\mathbf{q}_e^T = [U_1^1, U_2^1, U_1^2, U_2^2, U_1^3, U_2^3]$

$$u_i(x_1, x_2) = N_1(x_1, x_2) U_i^1 + N_2(x_1, x_2) U_i^2 + N_3(x_1, x_2) U_i^3$$

$$u_i(x_1, x_2) = \sum_{I=1}^{3} N_I(x_1, x_2) U_i^I$$

displacement interpolation

$$u_i\left(x_1^I, x_2^I\right) = U_i^I \ , \quad \text{nodal value requirement}$$

$$N_1\left(x_1^1, x_2^1\right) = 1 \ , \quad N_1\left(x_1^2, x_2^2\right) = 0 \ , \quad N_1\left(x_1^3, x_2^3\right) = 0$$

$$N_2\left(x_1^1, x_2^1\right) = 0 \ , \quad N_2\left(x_1^2, x_2^2\right) = 1 \ , \quad N_2\left(x_1^3, x_2^3\right) = 0$$

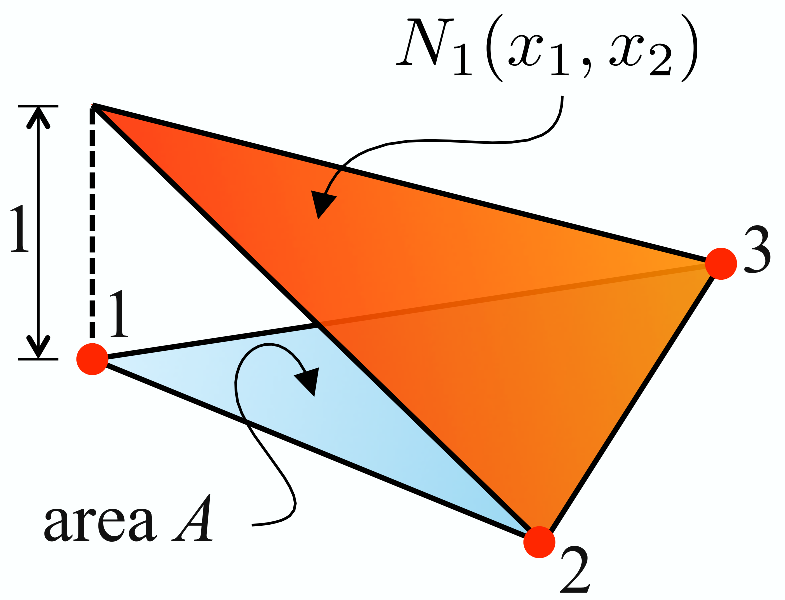
$$N_3\left(x_1^1, x_2^1\right) = 0 \ , \quad N_3\left(x_1^2, x_2^2\right) = 0 \ , \quad N_3\left(x_1^3, x_2^3\right) = 1$$

Shape functions $N_I(x_1, x_2)$ are bilinear in terms of coordinates

(3 constants found from the 3 nodal conditions – example N_1

$$N_1\left(x_1, x_2\right) = a x_1 + b x_2 + c$$

$$\begin{cases} 1 = a x_1^1 + b x_2^1 + c \\ 0 = a x_1^2 + b x_2^2 + c \\ 0 = a x_1^3 + b x_2^3 + c \end{cases}$$



Solution of the 3 × 3 linear system for the coefficients of $N_1(x_1, x_2)$

$$a = \frac{1}{2A} \det \begin{pmatrix} 1 & x_2^1 & 1 \\ 0 & x_2^2 & 1 \\ 0 & x_2^3 & 1 \end{pmatrix}, \quad b = \frac{1}{2A} \det \begin{pmatrix} x_1^1 & 1 & 1 \\ x_1^2 & 0 & 1 \\ x_1^3 & 0 & 1 \end{pmatrix}$$

$$c = \frac{1}{2A} \det \begin{pmatrix} x_1^1 & x_2^1 & 1 \\ x_1^2 & x_2^2 & 0 \\ x_1^3 & x_2^3 & 0 \end{pmatrix}, \quad 2A = \det \begin{pmatrix} x_1^1 & x_2^1 & 1 \\ x_1^2 & x_2^2 & 1 \\ x_1^3 & x_2^3 & 1 \end{pmatrix}$$

The three bilinear shape functions $N_I(x_1, x_2)$; $(I=1, 2, 3)$

$$N_1(x_1, x_2) = \frac{1}{2A}\left[x_1^2 x_2^3 - x_1^3 x_2^2 + \left(x_2^2 - x_2^3\right) x_1 - \left(x_1^2 - x_1^3\right) x_2\right]$$

$$N_2(x_1, x_2) = \frac{1}{2A}\left[x_1^3 x_2^1 - x_1^1 x_2^3 + \left(x_2^3 - x_2^1\right) x_1 - \left(x_1^3 - x_1^1\right) x_2\right]$$

$$N_3(x_1, x_2) = \frac{1}{2A}\left[x_1^1 x_2^2 - x_1^2 x_2^1 + \left(x_2^1 - x_2^2\right) x_1 - \left(x_1^1 - x_1^2\right) x_2\right]$$

Displacement discretization is conveniently written in matrix form: $\mathbf{u} = \mathbf{N}\mathbf{q}_e$

$$\underbrace{\begin{bmatrix} u_1(x_1,x_2) \\ u_2(x_1,x_2) \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} N_1(x_1,x_2) & 0 & N_2(x_1,x_2) & 0 & N_3(x_1,x_2) & 0 \\ 0 & N_1(x_1,x_2) & 0 & N_2(x_1,x_2) & 0 & N_3(x_1,x_2) \end{bmatrix}}_{\mathbf{N}} \underbrace{\begin{bmatrix} U_1^1 \\ U_2^1 \\ U_1^2 \\ U_2^2 \\ U_1^3 \\ U_2^3 \end{bmatrix}}_{\mathbf{q}_e}$$

Kinematic discretization is also written in matrix form: $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{q}_e$

Kinematics discretization is conveniently written in matrix form: $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{q}_e$

$$\mathbf{B} \equiv \frac{1}{2A}\begin{bmatrix} x_2^2 - x_2^3 & 0 & x_2^3 - x_2^1 & 0 & x_2^1 - x_2^2 & 0 \\ 0 & -\left(x_1^2 - x_1^3\right) & 0 & -\left(x_1^3 - x_1^1\right) & 0 & -\left(x_1^1 - x_1^2\right) \\ -\left(x_1^2 - x_1^3\right) & x_2^2 - x_2^3 & -\left(x_1^3 - x_1^1\right) & x_2^3 - x_2^1 & -\left(x_1^1 - x_1^2\right) & x_2^1 - x_2^2 \end{bmatrix}$$

NOTE: **B** matrix is constant (constant strain triangle!)

$\varepsilon^T = [\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}]$; where $\gamma_{12} = 2\,\varepsilon_{12}$

Reacall: $\mathbf{q}_e^T = [U_1^1, U_2^1, U_1^2, U_2^2, U_1^3, U_2^3]$

Constitutive equation also written in matrix form: $\boldsymbol{\sigma} = \mathbf{L}\,\boldsymbol{\varepsilon}$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{bmatrix} \qquad 2\epsilon_{12}$$

$$\mathbf{L} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}, \quad \mathbf{L} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

plane strain — plane stress

$$\mathcal{P}^e_{int} = \int_A \frac{1}{2}[\boldsymbol{\epsilon}^T \boldsymbol{\sigma}(x_1, x_2)]\, dA = \frac{1}{2}\mathbf{q}_e^T \left[\int_A [\mathbf{B}^T \mathbf{L}(x_1, x_2)\mathbf{B}]\, dA\right] \mathbf{q}_e$$

$$= \frac{1}{2}\mathbf{q}_e^T \mathbf{k}_e \mathbf{q}_e$$

Element stiffness matrix: $\mathbf{k}_\mathrm{e}$

$$-\mathcal{P}^e_{ext} = \int_A [\mathbf{u}^T(x_1, x_2)\mathbf{b}(x_1, x_2)]\, dA + \int_{\partial A} [\mathbf{u}^T(x_1, x_2)\mathbf{t}(x_1, x_2)]\, dl$$

$$= \mathbf{q}_e^T \left[\int_A [\mathbf{N}^T(x_1, x_2)\, \mathbf{b}(x_1, x_2)]\, dA + \int_{\partial A} [\mathbf{N}^T(x_1, x_2)\, \mathbf{t}(x_1, x_2)]\, dl\right]$$

$$= \mathbf{q}_e^T \mathbf{f}_e$$

Element force vector: $\mathbf{f}_\mathrm{e}$

Element stiffness matrix: $\mathbf{k}_e$ for constant moduli $\mathbf{L}$

$$\mathbf{k}_e = \int_A \left[\mathbf{B}^T \mathbf{L} \mathbf{B}\right] dA$$

$$\mathbf{k}_e = A\, \mathbf{B}^T \mathbf{L} \mathbf{B}$$

Element force vector: $\mathbf{f}_e$ for constant body forces $\mathbf{b}$ & traction $\mathbf{t}$

$$\mathbf{f}_e = \int_A \left[\mathbf{N}^T(x_1, x_2)\, \mathbf{b}\right] dA + \int_{\partial A} \left[\mathbf{N}^T(x_1(l), x_2(l))\, \mathbf{t}\right] dl$$

$$\mathbf{f}_e^T = \left[\frac{b_1}{3} + \frac{t_1}{2}, \frac{b_2}{3} + \frac{t_2}{2}, \frac{b_1}{3} + \frac{t_1}{2}, \frac{b_2}{3} + \frac{t_2}{2}, \frac{b_1}{3}, \frac{b_2}{3}\right]$$

NOTE: element has traction applied on the side defined by nodes 1 & 2

ISOPARAMETRIC CONSIDERATIONS FOR C.S.T.

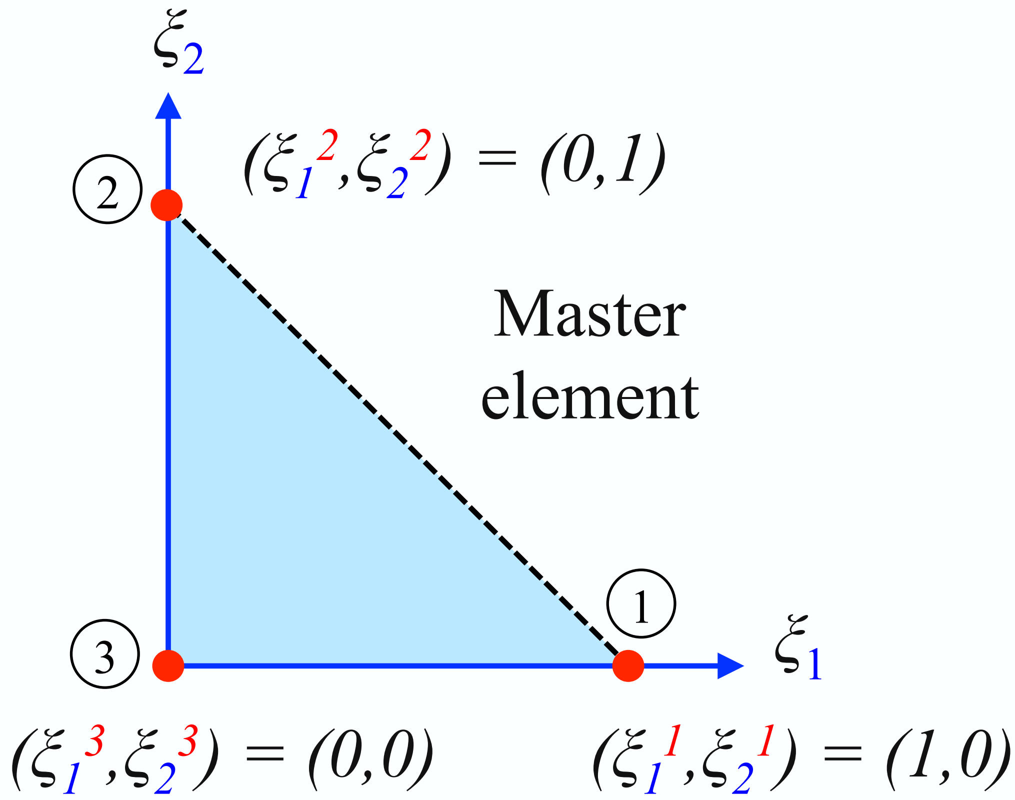
$$u_j\left(\xi_1,\xi_2\right)=\sum_{I=1}^{3}N_I\left(\xi_1,\xi_2\right)U_j^I$$

Parameters: ξ_1, ξ_2

$$u_j\left(\xi_1^I,\xi_2^I\right)=U_j^I$$

$$x_j\left(\xi_1,\xi_2\right)=\sum_{I=1}^{3}N_I\left(\xi_1,\xi_2\right)x_j^I$$

$$x_j\left(\xi_1^I,\xi_2^I\right)=x_j^I$$



Master element shape functions $N_I(\xi_1, \xi_2)$ are found to be:

$$N_1\left(\xi_1^1, \xi_2^1\right) = 1, \quad N_1\left(\xi_1^2, \xi_2^2\right) = 0, \quad N_1\left(\xi_1^3, \xi_2^3\right) = 0$$

$$\Longrightarrow \quad N_1\left(\xi_1, \xi_2\right) = \xi_1$$

$$N_2\left(\xi_1^1, \xi_2^1\right) = 0, \quad N_2\left(\xi_1^2, \xi_2^2\right) = 1, \quad N_2\left(\xi_1^3, \xi_2^3\right) = 0$$

$$\Longrightarrow \quad N_2\left(\xi_1, \xi_2\right) = \xi_2$$

$$N_3\left(\xi_1^1, \xi_2^1\right) = 0, \quad N_3\left(\xi_1^2, \xi_2^2\right) = 0, \quad N_3\left(\xi_1^3, \xi_2^3\right) = 1$$

$$\Longrightarrow \quad N_3\left(\xi_1, \xi_2\right) = 1 - \xi_1 - \xi_2$$

For strains we need the transformation (Hessian) matrix **J** and its inverse **J**$^{-1}$

$$\begin{bmatrix} \dfrac{\partial u_i}{\partial \xi_1} \\ \\ \dfrac{\partial u_i}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x_1}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_1} \\ \\ \dfrac{\partial x_1}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_2} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u_i}{\partial x_1} \\ \\ \dfrac{\partial u_i}{\partial x_2} \end{bmatrix} , \quad \mathbf{J} = \begin{bmatrix} x_1^1 - x_1^3 & x_2^1 - x_2^3 \\ \\ x_1^2 - x_1^3 & x_2^2 - x_2^3 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial u_i}{\partial x_1} \\ \\ \dfrac{\partial u_i}{\partial x_2} \end{bmatrix} = \mathbf{J}^{-\mathbf{1}} \begin{bmatrix} \dfrac{\partial u_i}{\partial \xi_1} \\ \\ \dfrac{\partial u_i}{\partial \xi_2} \end{bmatrix} , \quad \text{recall} : x_j(\xi_1, \xi_2) = \sum_{I=1}^{3} N_I(\xi_1, \xi_2) x_j^I$$

Definition of matrix A

$$\boldsymbol{\epsilon} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_2} \\ \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} \end{bmatrix} = \overbrace{\begin{bmatrix} \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_2}{\partial x_1} & 0 & 0 \\ 0 & 0 & \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_2} \\ \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_2} & \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_2}{\partial x_1} \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} \dfrac{\partial u_1}{\partial \xi_1} \\ \dfrac{\partial u_1}{\partial \xi_2} \\ \dfrac{\partial u_2}{\partial \xi_1} \\ \dfrac{\partial u_2}{\partial \xi_2} \end{bmatrix}$$

Definition of matrix **G**

$$\begin{bmatrix} \dfrac{\partial u_1}{\partial \xi_1} \\ \dfrac{\partial u_1}{\partial \xi_2} \\ \dfrac{\partial u_2}{\partial \xi_1} \\ \dfrac{\partial u_2}{\partial \xi_2} \end{bmatrix} = \overbrace{\begin{bmatrix} \dfrac{\partial N_1}{\partial \xi_1} & 0 & \dfrac{\partial N_2}{\partial \xi_1} & 0 & \dfrac{\partial N_3}{\partial \xi_1} & 0 \\ \dfrac{\partial N_1}{\partial \xi_2} & 0 & \dfrac{\partial N_2}{\partial \xi_2} & 0 & \dfrac{\partial N_3}{\partial \xi_2} & 0 \\ 0 & \dfrac{\partial N_1}{\partial \xi_1} & 0 & \dfrac{\partial N_2}{\partial \xi_1} & 0 & \dfrac{\partial N_3}{\partial \xi_1} \\ 0 & \dfrac{\partial N_1}{\partial \xi_2} & 0 & \dfrac{\partial N_2}{\partial \xi_2} & 0 & \dfrac{\partial N_3}{\partial \xi_2} \end{bmatrix}}^{\mathbf{G}} \begin{bmatrix} U_1^1 \\ U_2^1 \\ U_1^2 \\ U_2^2 \\ U_1^3 \\ U_2^3 \end{bmatrix}$$

Finding element stiffness using master element

$$\mathcal{P}^e_{int} = \int_A \frac{1}{2}[\boldsymbol{\epsilon}^T \boldsymbol{\sigma}(x_1, x_2)]\, dA$$

$$= \frac{1}{2}\mathbf{q}_e^T \left. \int_{\boldsymbol{\xi}} [\mathbf{G}^T \mathbf{A}^T \mathbf{L}(\xi_1, \xi_2) \mathbf{A}\mathbf{G}(\det \mathbf{J})]\, d\boldsymbol{\xi} \right] \mathbf{q}_e$$

$$= \frac{1}{2}\mathbf{q}_e^T \mathbf{k}_e \mathbf{q}_e$$