TOPICS COVERED IN THIS LECTURE

1. BAR MODEL (1D) – LINEAR (REVIEW) AND HIGHER ORDER ELEMENTS
2. BAR MODEL (1D) – MASTER ELEMENT & ISOPARAMETRIC INTERPOLATION
3. PLANAR TRUSSES (2D)
4. SPACE FRAMES (3D)
5. THERMAL LOADING OF TRUSSES

Potential : $\mathcal{P} = \mathcal{P}_{int} + \mathcal{P}_{ext}$

Internal : $\mathcal{P}_{int} = \int_V \frac{1}{2}\sigma(x)\,\epsilon(x)\,dV = \frac{1}{2}\int_0^L E\,A(x)\left(\frac{du}{dx}\right)^2 dx$

External : $\mathcal{P}_{ext} = -\int_V \rho\,g\,u(x)\,dV = -\int_0^L \rho\,g\,A(x)\,u(x)\,dx$

Numerical solution of equilibrium problem – Raleigh-Ritz method:
Instead of minimizing energy in an infinite dimensional space, we do minimize in a finite dimensional space, in which case we end up with an algebraic problem. Since energy in quadratic in *u(x)*, system is linear!

$$\mathcal{P}(u^{app}(x)) = \mathcal{P}(\mathbf{Q}); \quad u^{app} = \sum_{i=1}^{i=n} Q_i N_i(x), \quad \mathbf{Q} \equiv [Q_1, Q_2, \cdots Q_n]$$

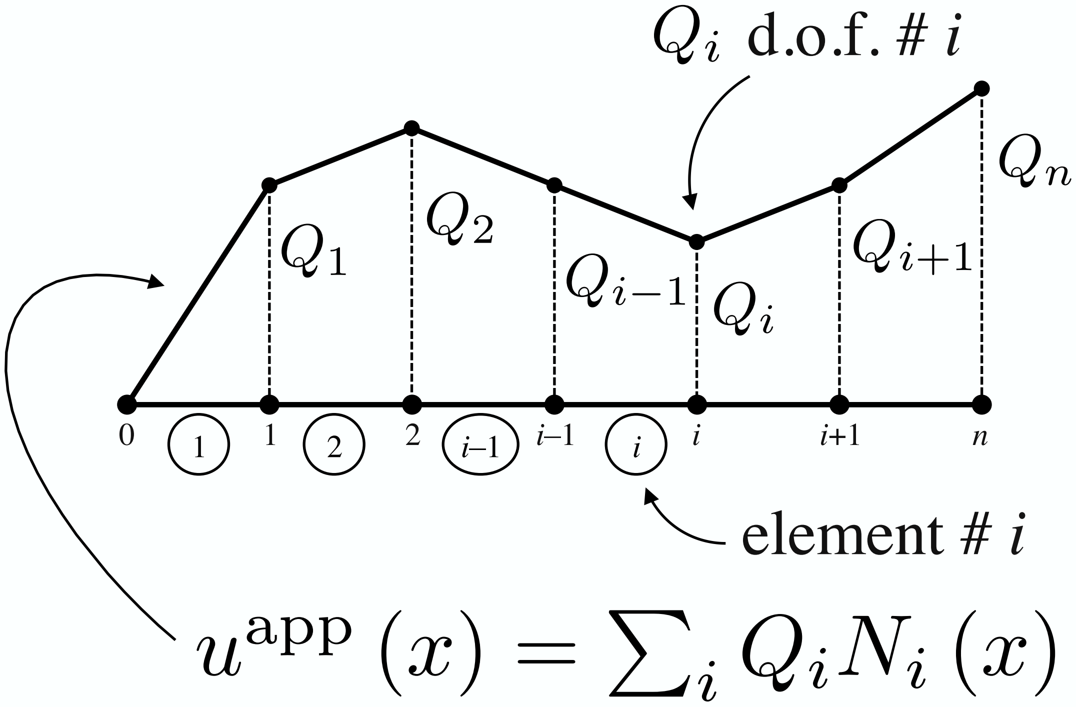
$$\frac{\partial \mathcal{P}(\mathbf{Q})}{\partial Q_i} = \int_0^L \left[EA \frac{du^{app}}{dx} \frac{\partial}{\partial Q_i} \left(\frac{du^{app}}{dx} \right) - \rho g A \frac{\partial u^{app}}{\partial Q_i} \right] dx = 0$$

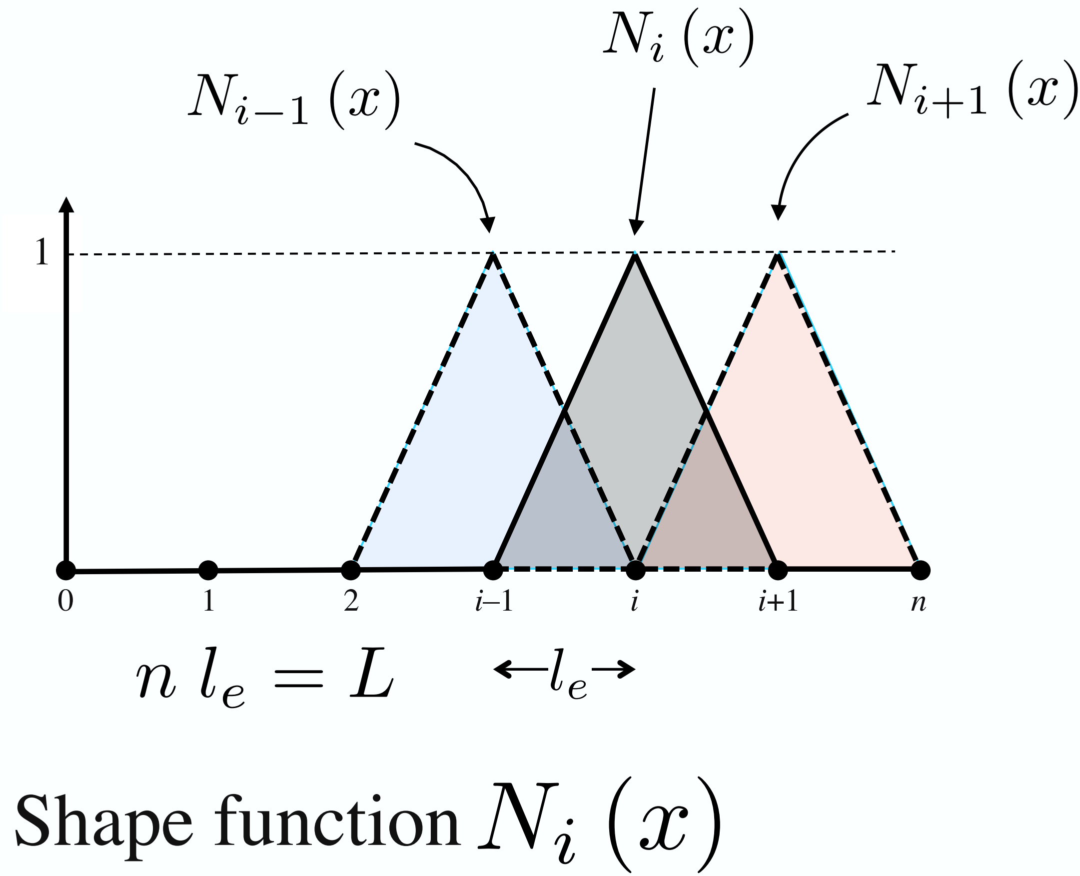
$$\sum_{j=1}^{j=n} \left\{ \int_0^L \left[EA \frac{dN_i}{dx} \frac{dN_j}{dx} \right] dx \right\} Q_j - \int_0^L \left[\rho g A N_i \right] dx = 0$$

$$\sum_{j=1}^{j=n} K_{ij} Q_j - F_i = 0 \; ; \quad (\text{in compact form} : \mathbf{K}\mathbf{Q} = \mathbf{F})$$

$$K_{ij} \equiv \int_0^L \left[EA \frac{dN_i}{dx} \frac{dN_j}{dx} \right] dx \; , F_i \equiv \int_0^L \left[\rho g A N_i \right] dx$$

Stiffness matrix: **K**, Force vector: **F**, Degrees of Freedom: **Q**





Easy physical interpretation of d.o.f. (degree of freedom) Q_i at node x_i: due to its construction, $u^{app}(x_i) = Q_i$

Shape functions $N_i(x)$ have compact support: $N_i(x_i) = 1$, $N_i(x_{i-1}) = N_i(x_{i+1}) = 0$.
Compactness of support of shape function great advantage of FEM

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & 0 \\ 0 & K_{32} & K_{33} & K_{34} & 0 \\ 0 & 0 & K_{43} & K_{44} & K_{45} \\ 0 & 0 & 0 & K_{54} & K_{55} \end{bmatrix}$$

$$K_{ii} = \int_{x_{i-1}}^{x_{i+1}} E\ A(x) \left(\frac{1}{l_e}\right)^2 dx$$

$$K_{i\ i+1} = -\int_{x_i}^{x_{i+1}} E\ A(x) \left(\frac{1}{l_e}\right)^2 dx$$

Stiffness matrix **K** is banded, i.e. populated about the diagonal. Note that the F.E.M. (compact support) shape functions $N_i(x)$ do not have to be piecewise linear, as calculated above; a wide selection is possible.
We will show here how we can generalize to higher order polynomials.

$$\mathcal{P}(\mathbf{Q}) = \mathcal{P}_{int}(\mathbf{Q}) + \mathcal{P}_{ext}(\mathbf{Q})$$

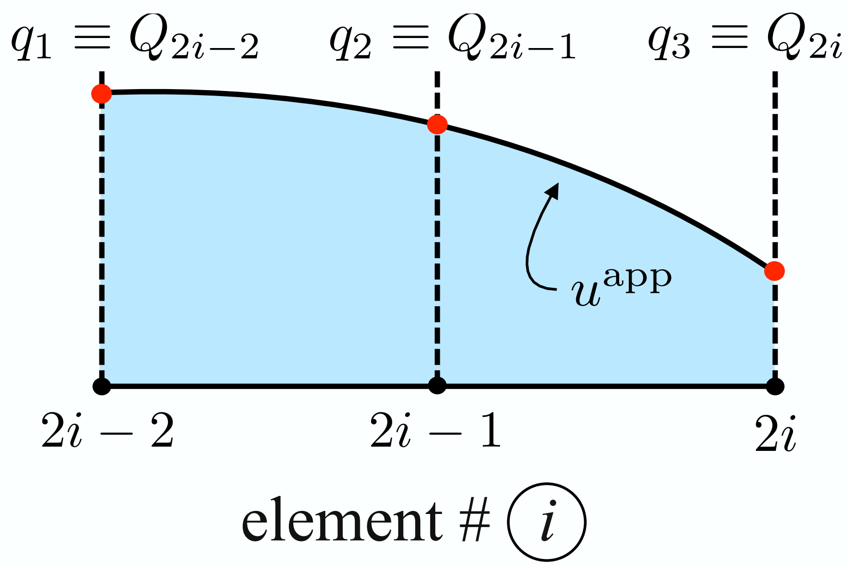
$$\mathcal{P}_{int}(\mathbf{Q}) = \sum_e \mathcal{P}^e_{int} \; ; \quad \mathcal{P}^e_{int} = \int_{l_e} \left[\frac{1}{2} E \; A(x) \left(\sum_{i=1}^{i=m} q_i \frac{dN_i}{dx} \right)^2 \right] dx = \frac{1}{2} \mathbf{q}_e^T \mathbf{k}_e \mathbf{q}_e$$

$$\mathcal{P}_{ext}(\mathbf{Q}) = \sum_e \mathcal{P}^e_{ext} \; ; \quad \mathcal{P}^e_{ext} = -\int_{l_e} \left[\rho g \; A(x) \left(\sum_{i=1}^{i=m} q_i N_i(x) \right) \right] dx = -\mathbf{q}_e^T \mathbf{f}_e$$

$$[\mathbf{k}_e]_{ij} \equiv \int_{l_e} \left[E \; A(x) \left(\frac{dN_i}{dx} \right) \left(\frac{dN_j}{dx} \right) \right] dx \; ; \quad \mathbf{k}_e \text{ element stiffness } (m \times m)$$

$$[\mathbf{f}_e]_i \equiv -\int_{l_e} [\rho g \; A(x) N_i(x)] \, dx \; ; \quad \mathbf{f}_e \text{ element force } (m)$$

Element stiffness $\mathbf{k}_e$ $(m \times m)$ & force $\mathbf{f}_e$ (m) for $(m\text{-}1)$ polynomial interpolation

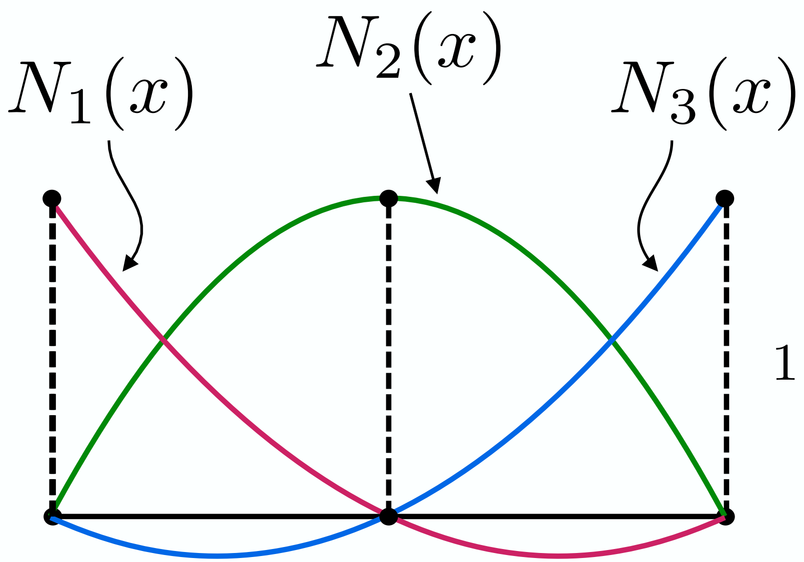


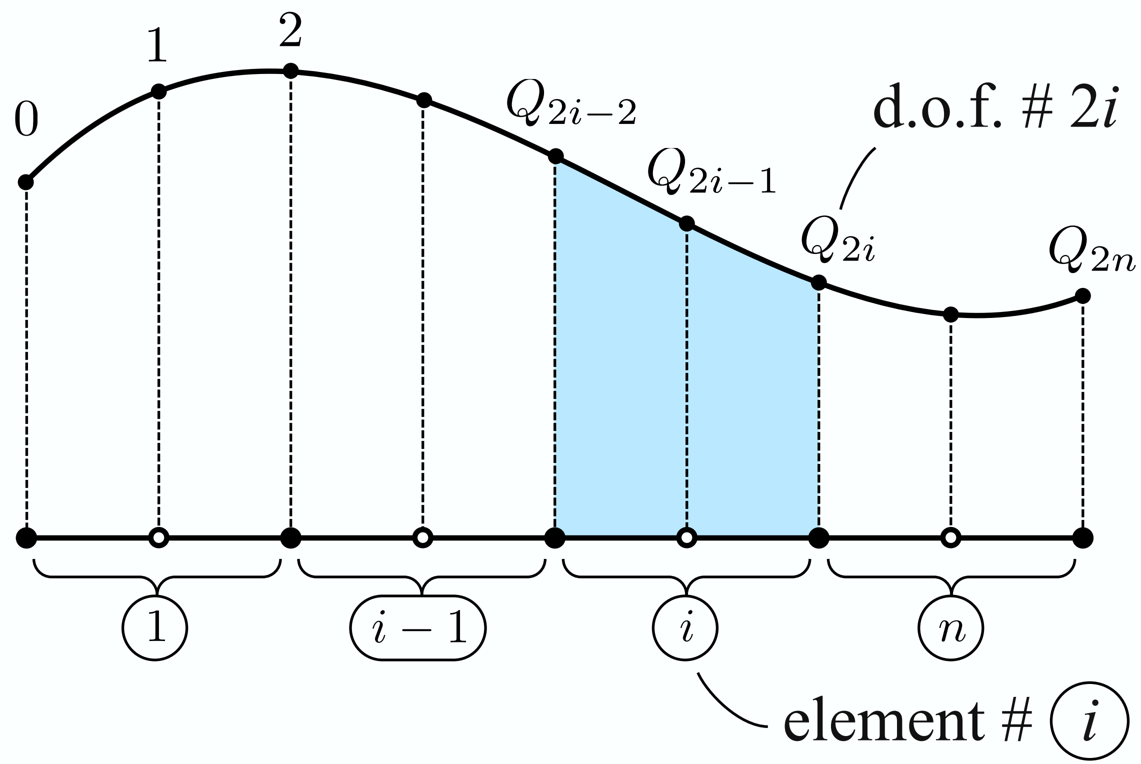
Case *m=3* (2nd order polynomial interp.)

$u^{app}(x) = q_1N_1(x) + q_2N_2(x) + q_2N_2(x)$

Local degree of freedom $\mathbf{q}^T_e = [q_1, q_2, q_3]$

In each element: $u^{app}(x_i) = q_i$ $(i=1,2,3)$





Case of a quadratic polynomial interpolation ($m=3$)

Recall requirement: $u^{app}(x_i) = q_1 N_1(x_i) + q_2 N_2(x_i) + q_2 N_2(x_i) = q_i$

$$N_1(x_1) = 1\ ,\ \ N_1(x_2) = 0\ ,\ \ N_1(x_3) = 0$$
$$N_1(x) = [(x - x_2)(x - x_3)/\left[(x_1 - x_2)(x_1 - x_3)\right]$$

$$N_2(x_1) = 0\ ,\ \ N_2(x_2) = 1\ ,\ \ N_2(x_3) = 0$$
$$N_2(x) = [(x - x_1)(x - x_3)/\left[(x_2 - x_1)(x_2 - x_3)\right]$$

$$N_3(x_1) = 0\ ,\ \ N_3(x_2) = 0\ ,\ \ N_3(x_3) = 1$$
$$N_3(x) = [(x - x_1)(x - x_2)/\left[(x_3 - x_1)(x_3 - x_2)\right]$$

You see now the easy extension to higher order interpolation!

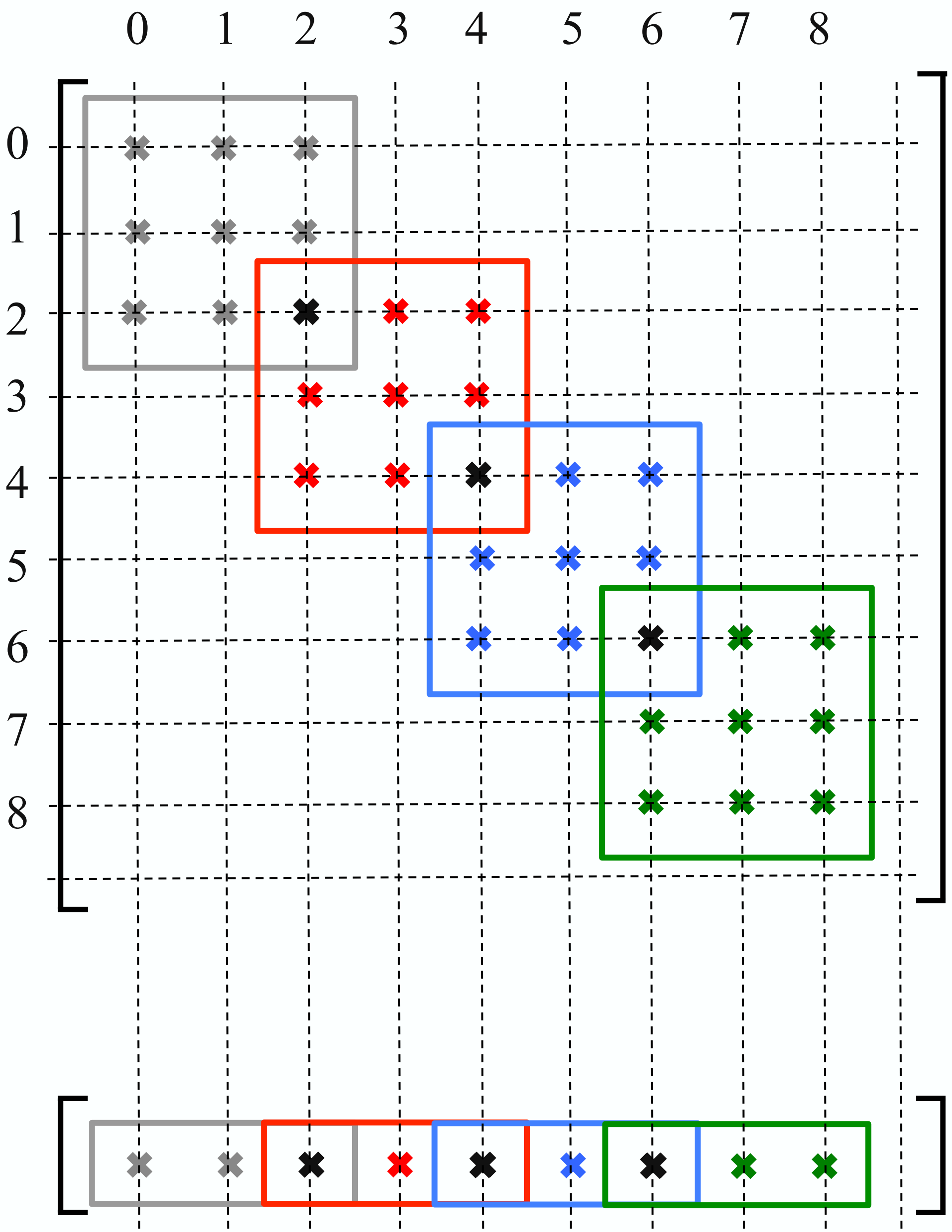
Case of a cubic polynomial interpolation (*m=4*)

$$N_1(x_1) = 1\,,\ \ N_1(x_2) = 0\,,\ \ N_1(x_3) = 0\,,\ \ N_1(x_4) = 0$$
$$N_1(x) = [(x - x_2)(x - x_3)(x - x_4)]\,/\,[(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)]$$

$$N_2(x_1) = 0\,,\ \ N_2(x_2) = 1\,,\ \ N_2(x_3) = 0\,,\ \ N_2(x_4) = 0$$
$$N_2(x) = [(x - x_1)(x - x_3)(x - x_4)]\,/\,[(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)]$$

$$N_3(x_1) = 0\,,\ \ N_3(x_2) = 0\,,\ \ N_3(x_3) = 1\,,\ \ N_3(x_4) = 0$$
$$N_3(x) = [(x - x_1)(x - x_2)(x - x_4)]\,/\,[(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)]$$

$$N_4(x_1) = 0\,,\ \ N_4(x_2) = 0\,,\ \ N_4(x_3) = 0\,,\ \ N_4(x_4) = 1$$
$$N_3(x) = [(x - x_1)(x - x_2)(x - x_3]\,/\,[(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)]$$



Assembling global stiffness matrix K and **global force vector F** from **element stiffness matrix k_e** and **element force vector f_e** : add local components in the appropriate place of global counterparts (for quadratic polynomial interpolation, $m=3$)

local no.➔global no.

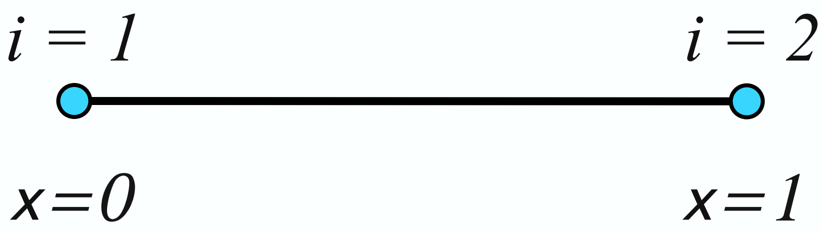
1➔0, 2➔1, 3➔2 for element 1

1➔2, 2➔3, 3➔4 for element 2

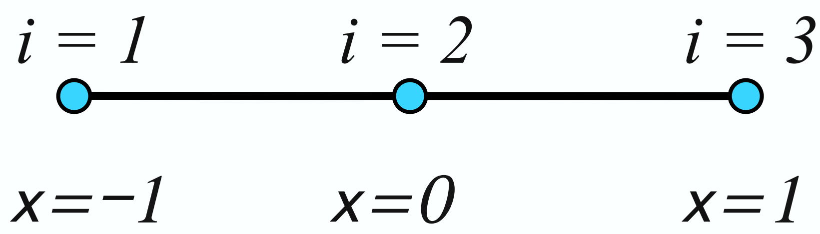
1➔4, 2➔5, 3➔6 for element 3

1➔6, 2➔7, 3➔8 for element 4

It is very convenient to write shape functions in a master element with respect to a normalized coordinate (ξ)



$$m = 2\,; \quad N_1(\xi) = 1 - \xi$$
$$N_2(\xi) = \xi$$



$$m = 3\,; \quad N_1(\xi) = \xi(\xi - 1)/2$$
$$N_2(\xi) = (1 - \xi)(\xi + 1)/2$$
$$N_3(\xi) = \xi(\xi + 1)/2$$

$$[\mathbf{k}_e]_{ij} \equiv \int_\xi \left[E\ A(x(\xi))\,(dN_i/d\xi)\,(dN_j/d\xi)\,(dx/d\xi)^{-1}\right] d\xi$$

$$[\mathbf{f}_e]_i \equiv -\int_\xi \left[\rho g\ A(x(\xi)) N_i(\xi)\,(dx/d\xi)^{-1}\right] d\xi$$

Question: what do we choose for $x(\xi)$?

Answer: (easy) same representation as for displacement!

This type of parametrization that uses the same interpolation scheme for both the displacement and the geometric coordinates is called isoparametric representation and is widely used in F.E.M.

$$u(\xi) = \sum_{i=1}^{i=m} u_i N_i(\xi) \; , \quad u(\xi_i) = u_i \; ; \qquad \text{d.o.f. at node i}$$

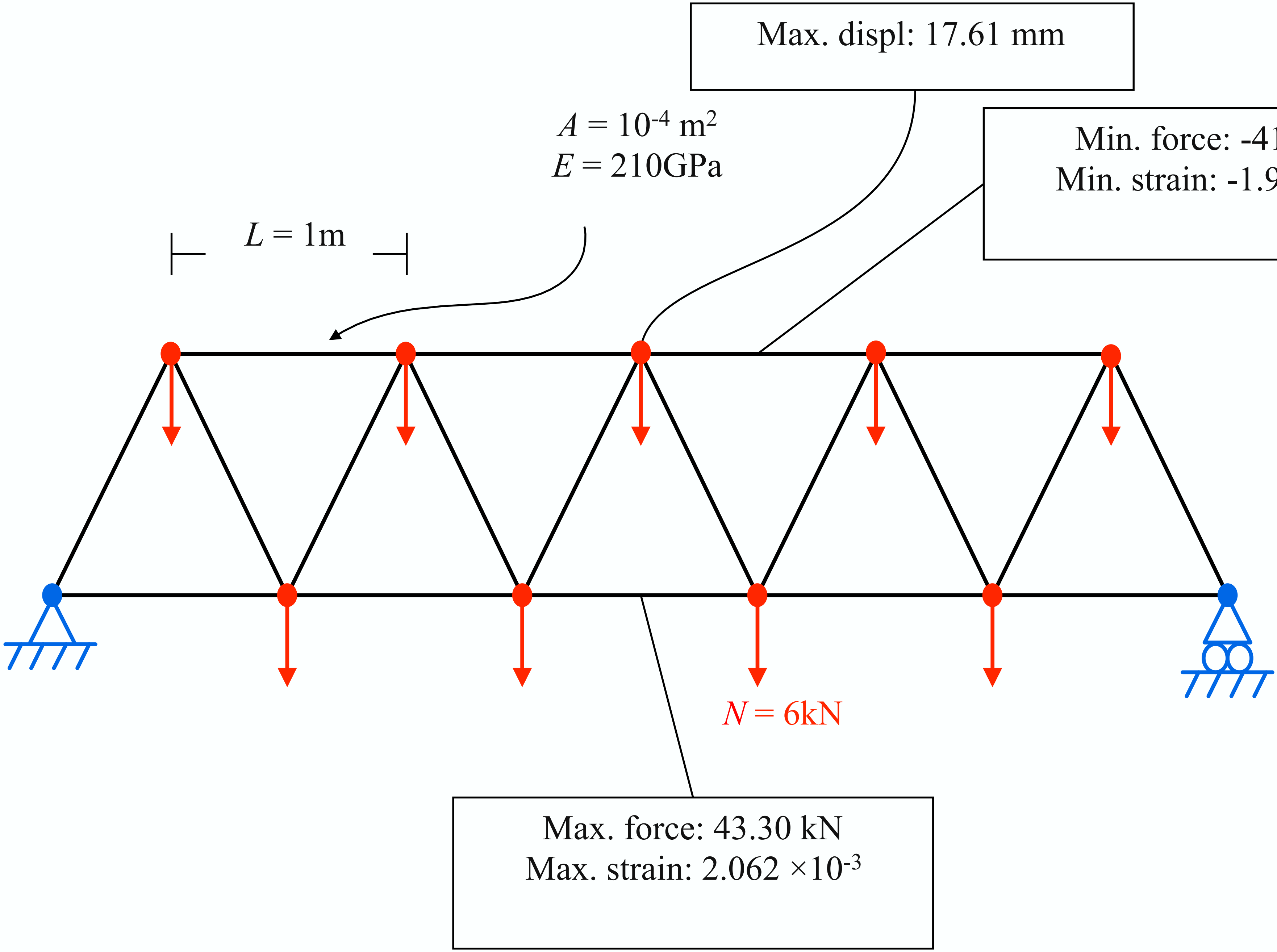
$$x(\xi) = \sum_{i=1}^{i=m} x_i N_i(\xi) \; , \quad x(\xi_i) = x_i \; ; \qquad \text{coordinate at node i}$$

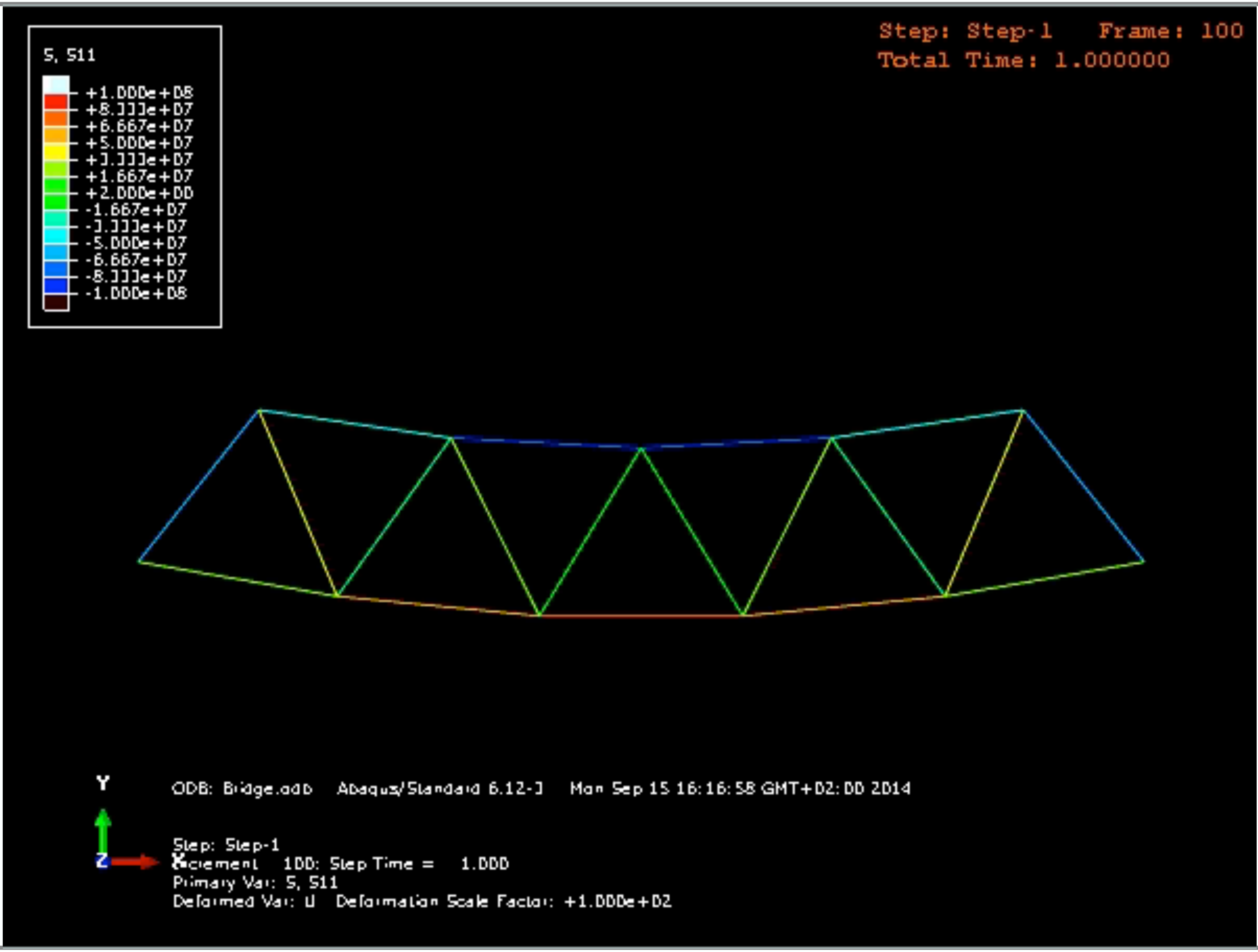
TRUSSES are structures made of bars connected to the rest of the structure by a pin joint at each end; a joint that can transmit forces but cannot transmit moments. As a result, moment equilibrium of the bar dictates that the bar is under axial forces only, since the bars have no distributed loads. Loading is only applied at the nodes of the structure.

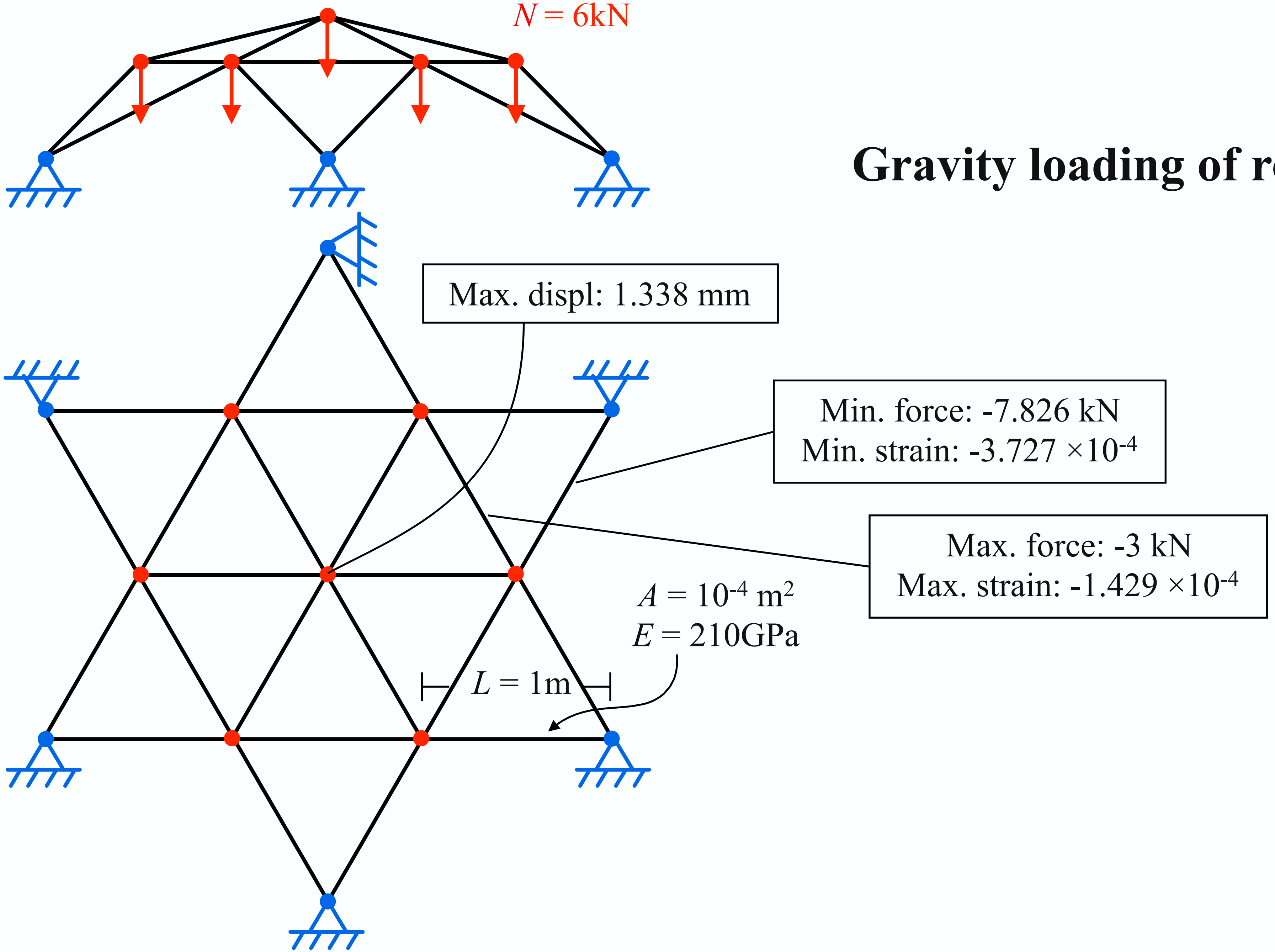
Most engineering structures have a skeleton made of long beams connected to each other and on which a skin is sometimes added (roof structures, tubular design in cars, aircraft fuselage, satellites etc.) and sometimes not (bridges, grid transmission towers – even the Eiffel tower). A truss approximation of these structures (which assumes only axial forces in members and forces on nodes – but no moments – is a very useful first approximation on engineering to do preliminary design studies.

F.E.M. ideas apply here (linear elastic structure, small deformation and displacement linear kinematics) with each bar being an element.

TRUSS EXAMPLES IN 2D AND 3D

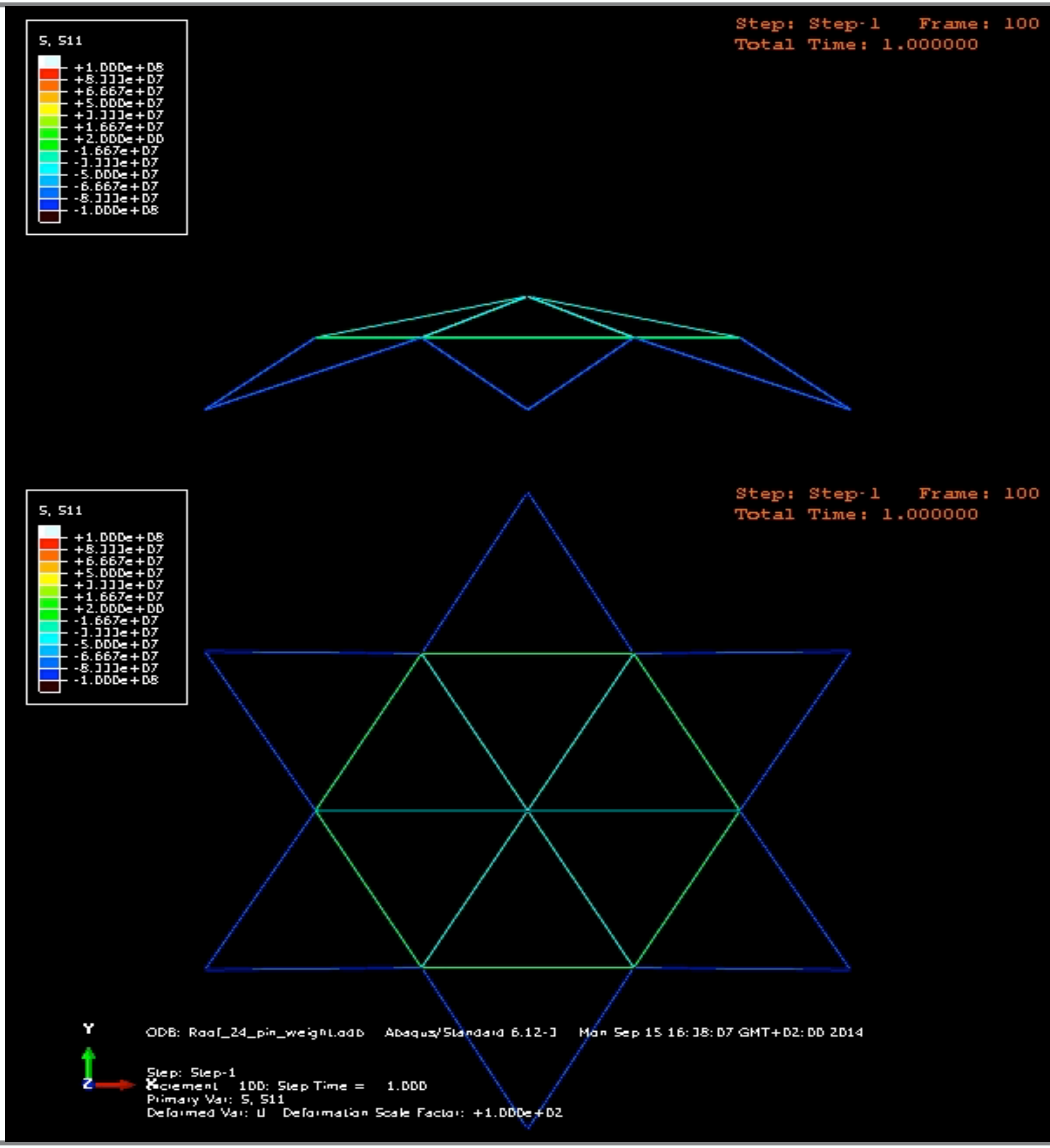

Max. displ: 17.61 mm
$A = 10^{-4}$ m^2
$E = 210$GPa
Min. force: -41.57 kN
Min. strain: -1.979×10^{-3}
$L = 1$m
$N = 6$kN
Max. force: 43.30 kN
Max. strain: 2.062×10^{-3}


S, S11
+1.000e+08
+8.333e+07
+6.667e+07
+5.000e+07
+3.333e+07
+1.667e+07
+2.000e+00
-1.667e+07
-3.333e+07
-5.000e+07
-6.667e+07
-8.333e+07
-1.000e+08
Step: Step-1 Frame: 100
Total Time: 1.000000
Y
Z
ODB: Bridge.odb Abaqus/Standard 6.12-3 Mon Sep 15 16:16:58 GMT+02:00 2014
Step: Step-1
Increment 100: Step Time = 1.000
Primary Var: S, S11
Deformed Var: U Deformation Scale Factor: +1.000e+02

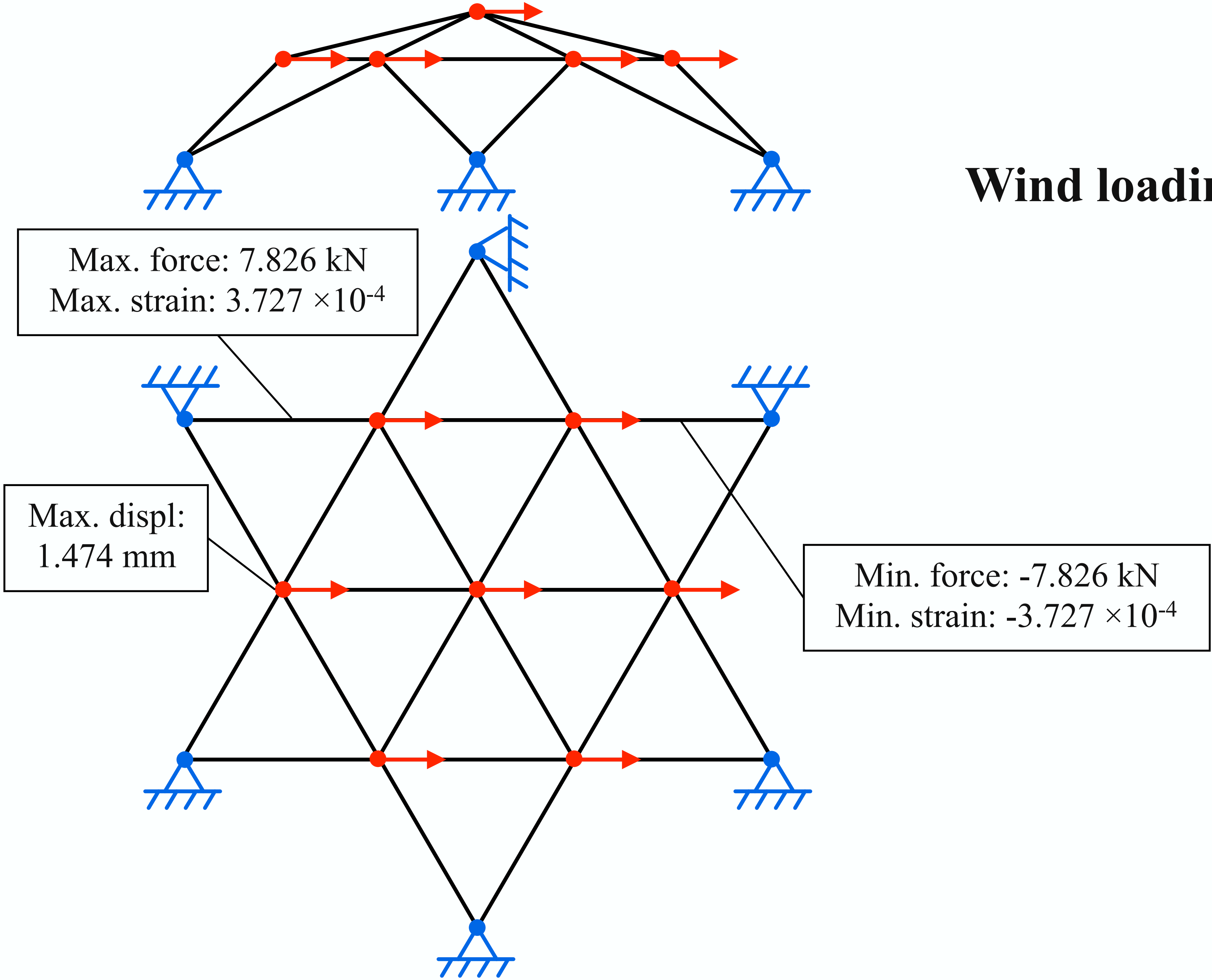


Gravity loading of roof

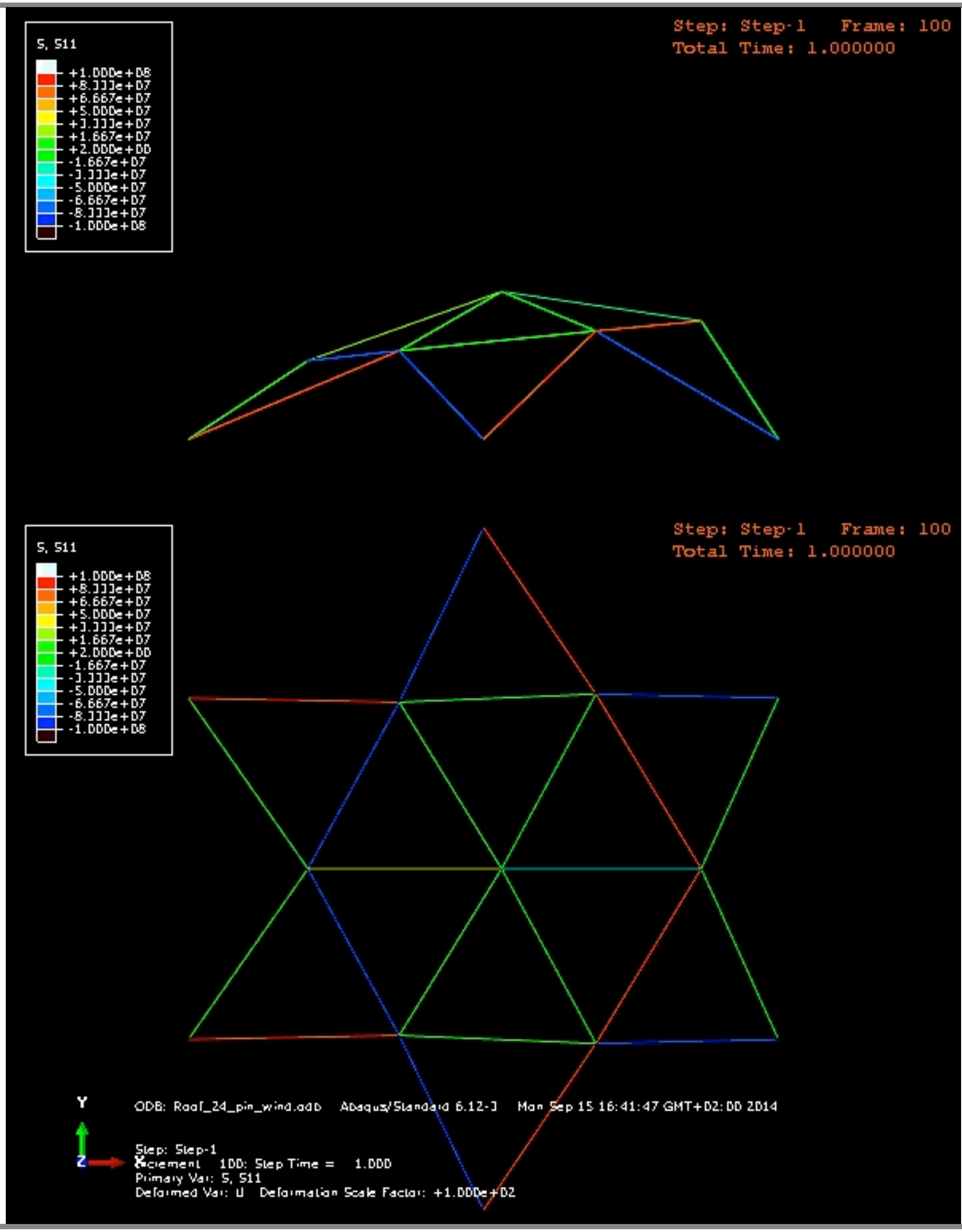
SPACE TRUSSES (AXIALLY LOADED BAR STRUCTURES IN 3D)

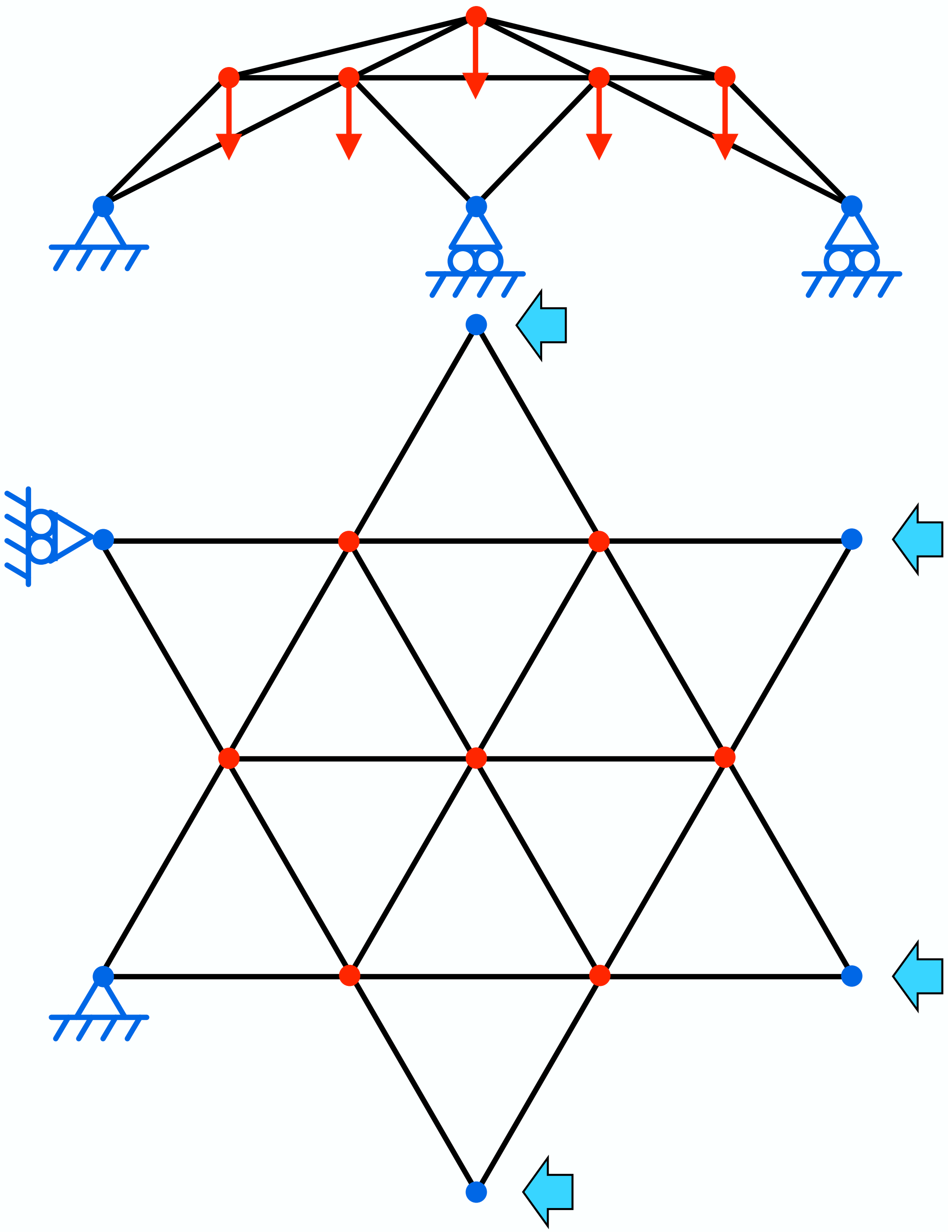


Wind loading of roof



SPACE TRUSSES (AXIALLY LOADED BAR STRUCTURES IN 3D)

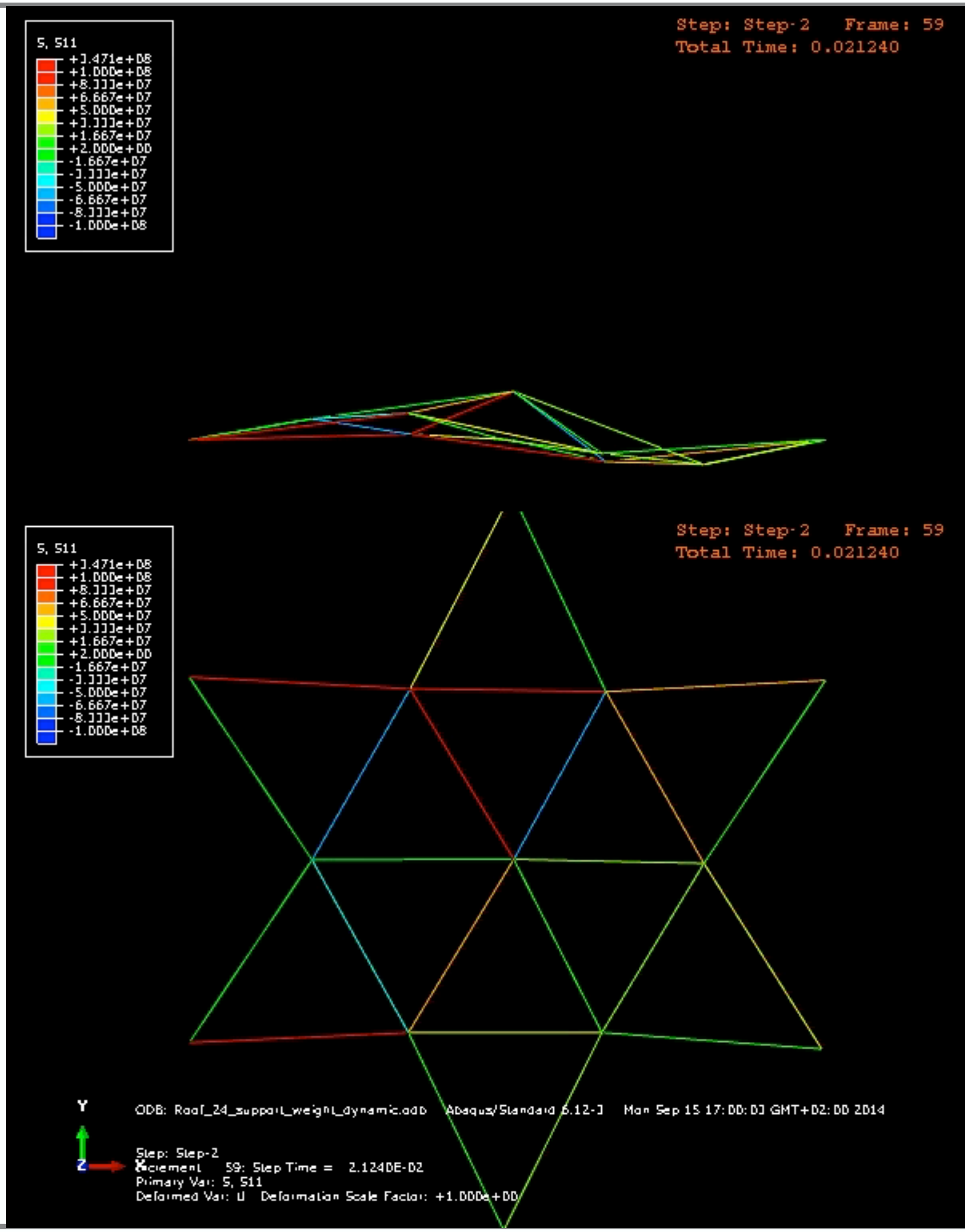




What happens when we change support conditions? (e.g. choose four ground nodes to be on rollers – no in-plane reactions)

Notice that the static problem is ill conditioned (*det* **K** = 0) and cannot be solved

Structure is a mechanism! We can see what happens when we solve the dynamic problem…

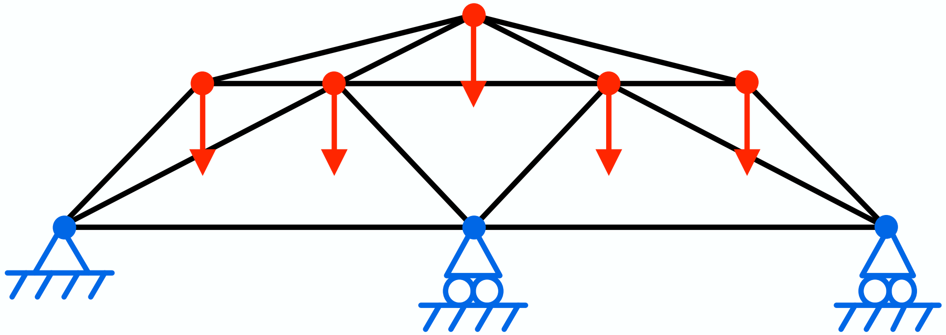


S, S11
+3.471e+08
+1.000e+08
+8.333e+07
+6.667e+07
+5.000e+07
+3.333e+07
+1.667e+07
+2.000e+00
-1.667e+07
-3.333e+07
-5.000e+07
-6.667e+07
-8.333e+07
-1.000e+08
Step: Step-2 Frame: 59
Total Time: 0.021240
S, S11
+3.471e+08
+1.000e+08
+8.333e+07
+6.667e+07
+5.000e+07
+3.333e+07
+1.667e+07
+2.000e+00
-1.667e+07
-3.333e+07
-5.000e+07
-6.667e+07
-8.333e+07
-1.000e+08
Step: Step-2 Frame: 59
Total Time: 0.021240
Y
Z
X
ODB: Roof_24_support_weight_dynamic.odb Abaqus/Standard 6.12-3 Mon Sep 15 17:00:03 GMT+02:00 2014
Step: Step-2
Increment 59: Step Time = 2.1240E-02
Primary Var: S, S11
Deformed Var: U Deformation Scale Factor: +1.000e+00

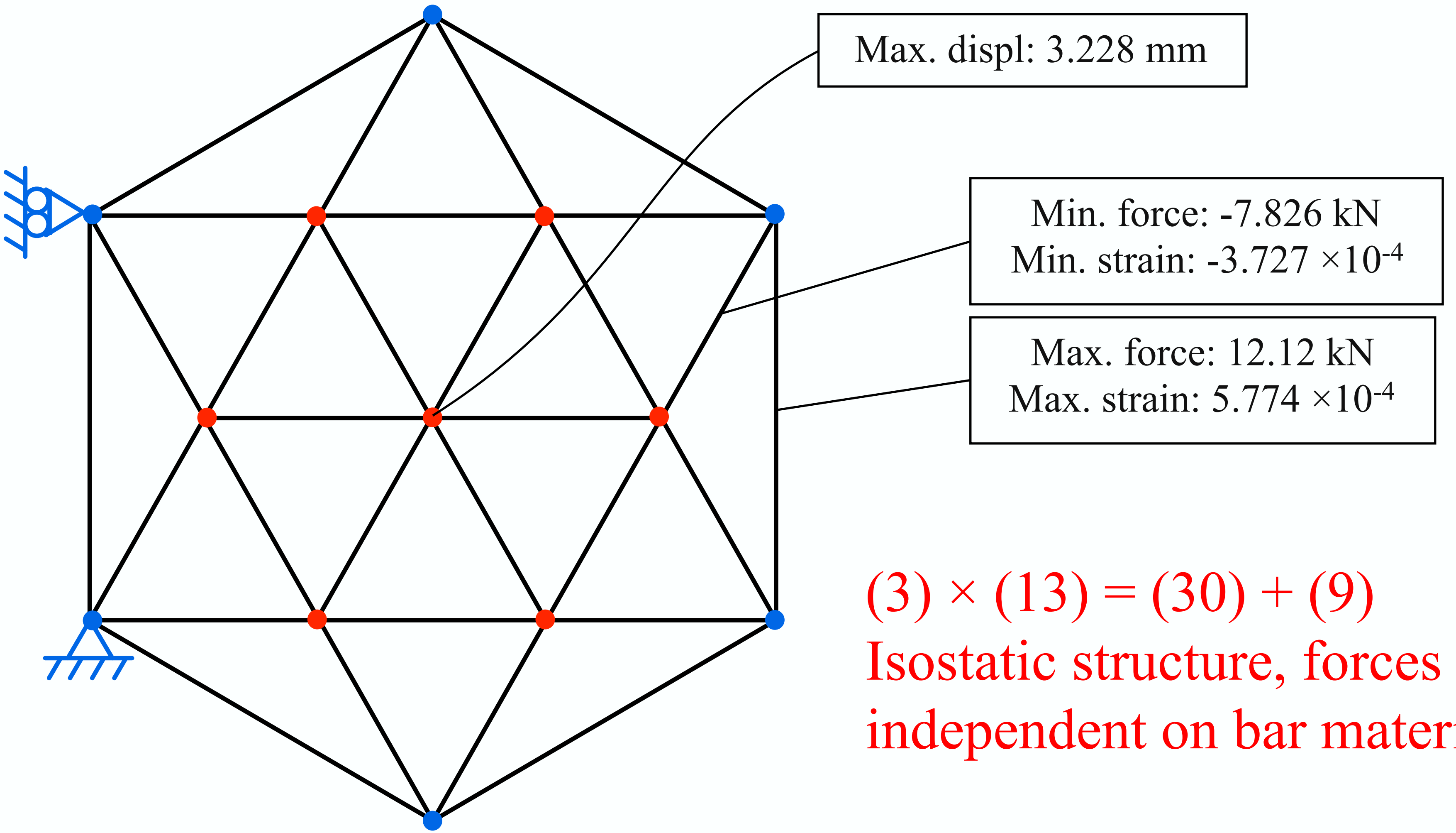
A truss is a **mechanism** (*det* **K** = 0) when the number of available equilibrium equations exceeds the number of unknowns
(spatial dimension) × (nodes) > (bars) + (reactions)

A truss is a **isostatic** (also termed **statically determinate**) structure when the number of available equilibrium equations equals the number of unknowns; in this case you do not need the material properties of the bars, equilibrium equations suffice to solve the problem where bar forces that depend only on geometry!
(spatial dimension) × (nodes) = (bars) + (reactions)

A truss is a **hyperstatic** (also termed **statically indeterminate**) structure when the number of available equilibrium equations is less than the number of unknowns; in this case bar forces depend on material properties
(spatial dimension) × (nodes) < (bars) + (reactions)

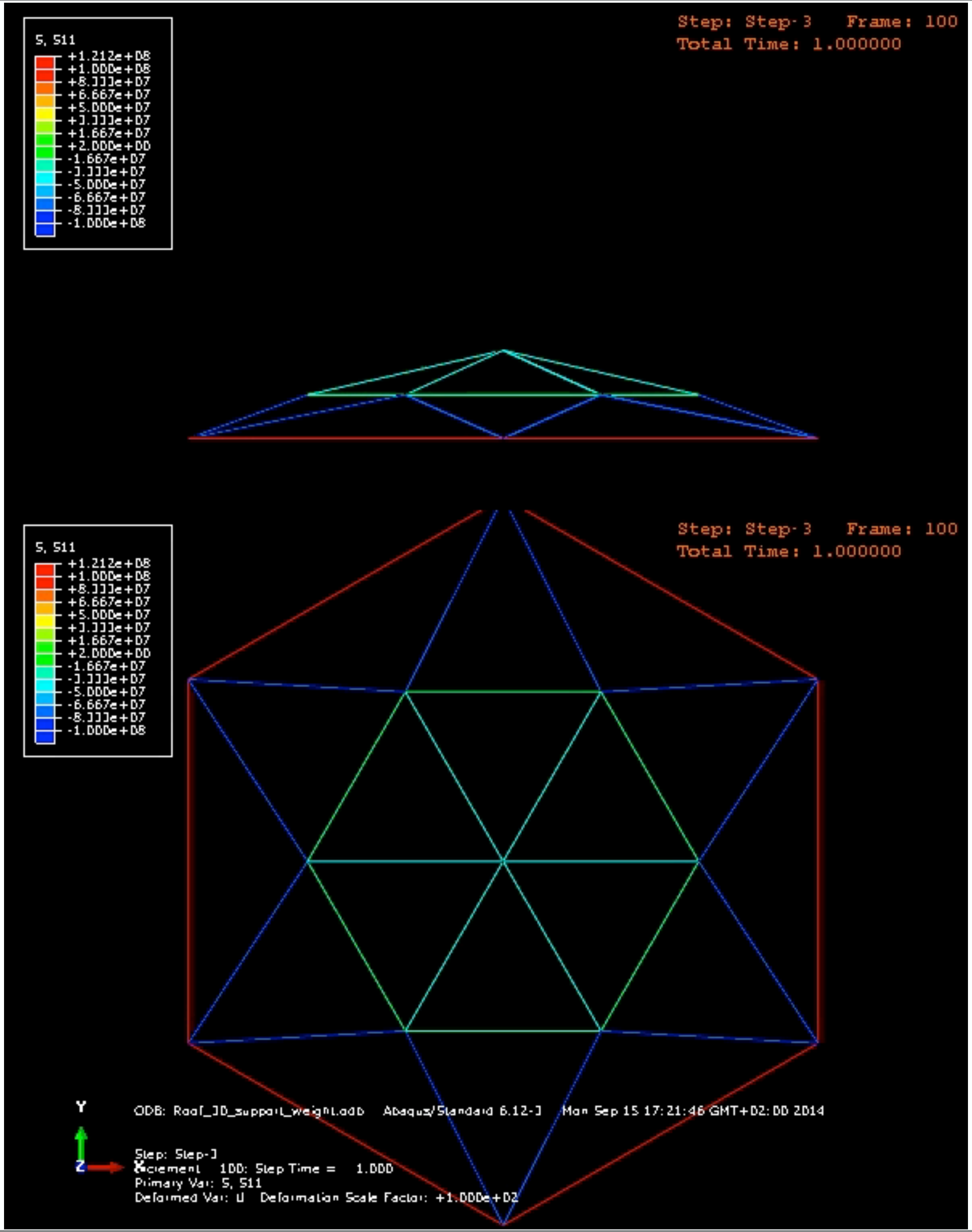


Gravity loading of roof
Modified structure that allows roller supports



$(3) \times (13) = (30) + (9)$

Isostatic structure, forces independent on bar material!


S, S11
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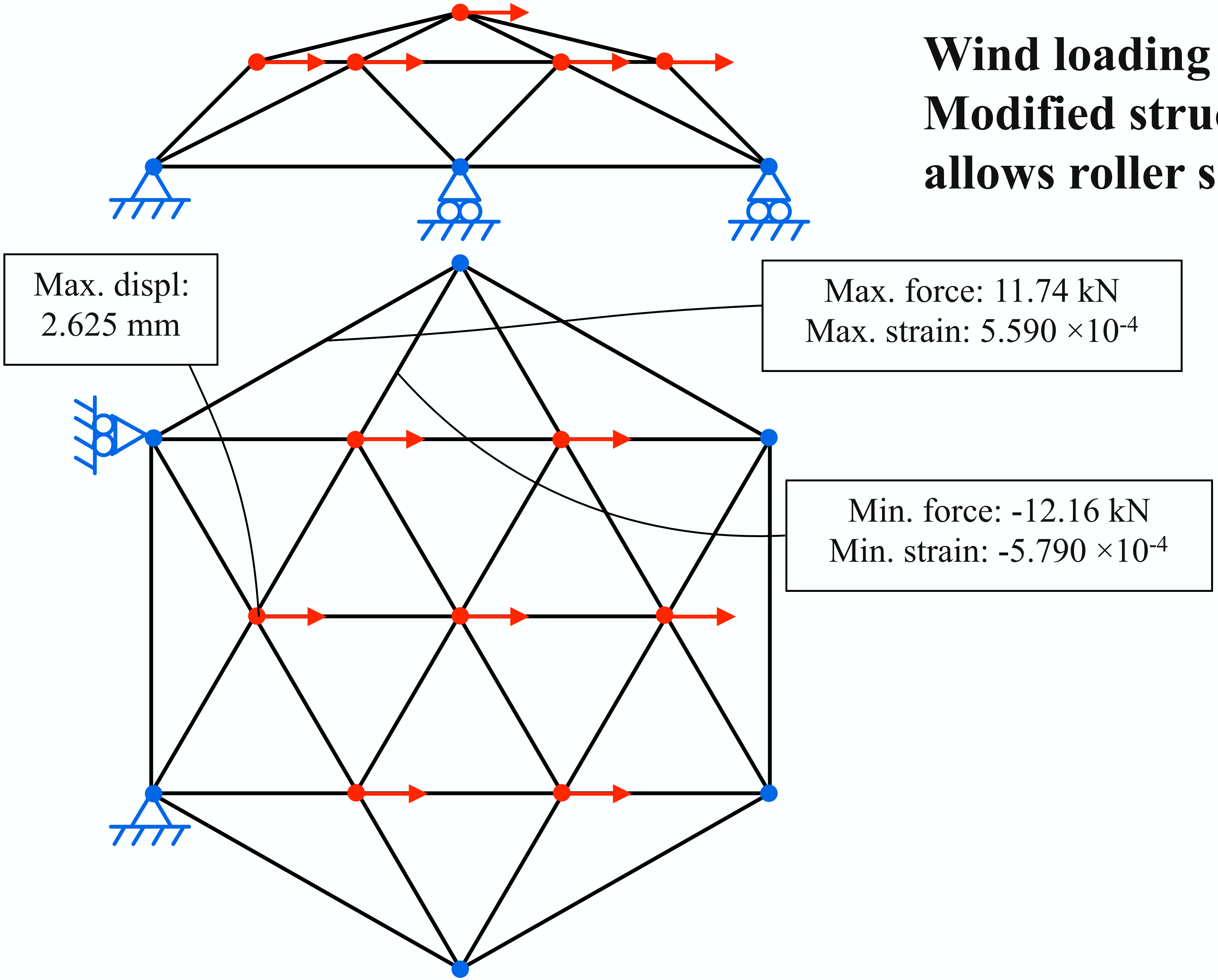
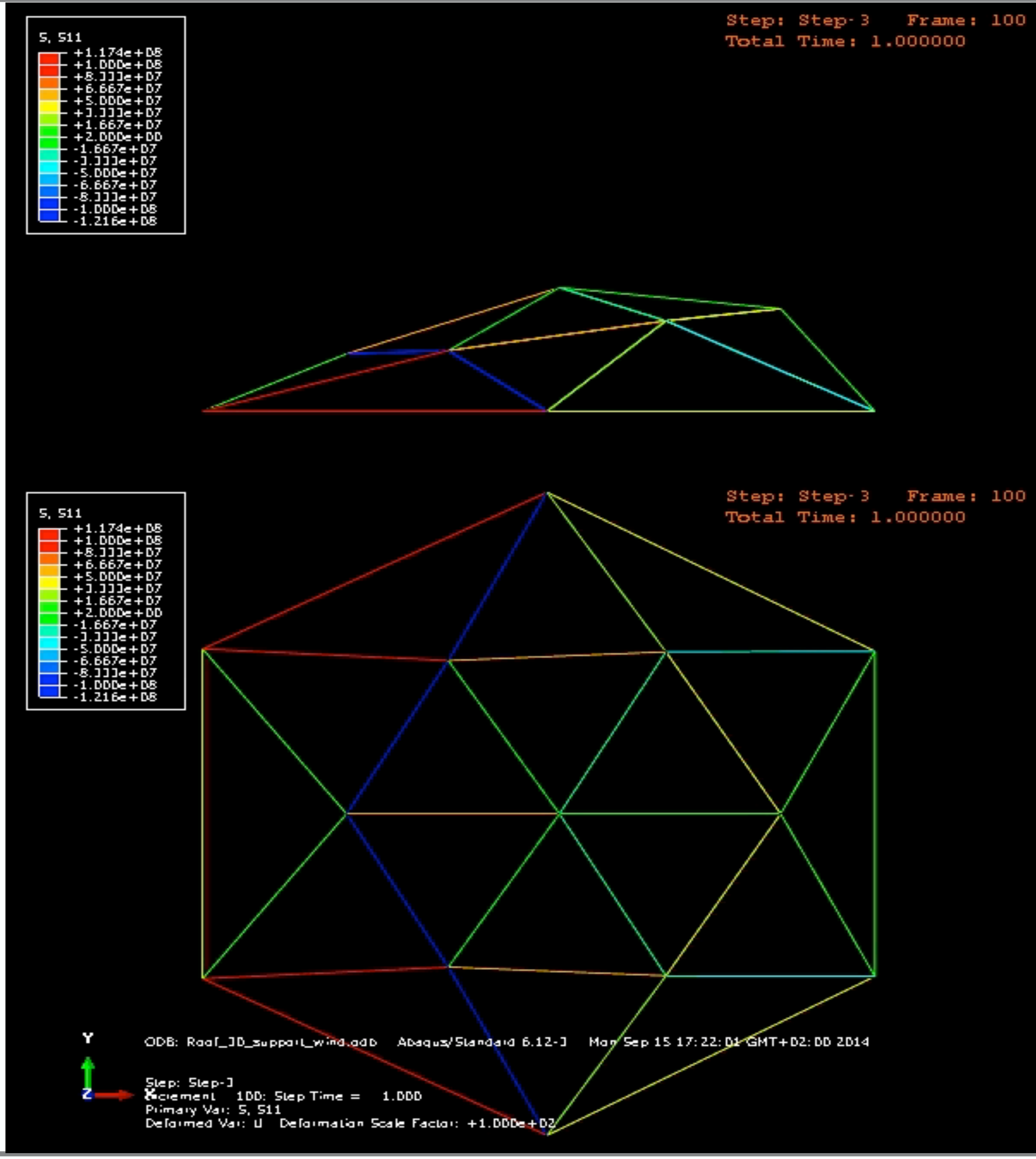
+3.333e+07
+1.667e+07
+2.000e+00
-1.667e+07
-3.333e+07
-5.000e+07
-6.667e+07
-8.333e+07
-1.000e+08
Step: Step-3 Frame: 100
Total Time: 1.000000
Y
Z
X
ODB: Roof_3D_support_weight.odb Abaqus/Standard 6.12-3 Mon Sep 15 17:21:46 GMT+02:00 2014
Step: Step-3
Increment 100: Step Time = 1.000
Primary Var: S, S11
Deformed Var: U Deformation Scale Factor: +1.000e+02

Wind loading of roof
Modified structure that allows roller supports




S, S11
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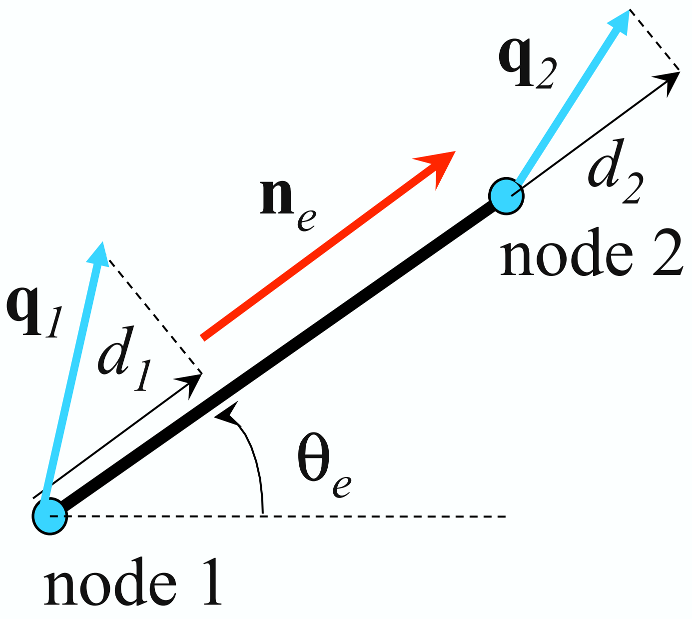
+1.000e+08
+8.333e+07
+6.667e+07
+5.000e+07
+3.333e+07
+1.667e+07
+2.000e+00
-1.667e+07
-3.333e+07
-5.000e+07
-6.667e+07
-8.333e+07
-1.000e+08
-1.216e+08
Step: Step-3 Frame: 100
Total Time: 1.000000
Y
Z
X
ODB: Roof_3D_support_wind.odb Abaqus/Standard 6.12-3 Mon Sep 15 17:22:01 GMT+02:00 2014
Step: Step-3
Increment 100: Step Time = 1.000
Primary Var: S, S11
Deformed Var: U Deformation Scale Factor: +1.000e+02

FORMULATION OF THE TRUSS PROBLEM



d_1 , d_2 , axial displacements of bar

bar elongation: $d_2 - d_1$

d_i projection of displacement $\mathbf{q}_i$ at node i

axial strain of element e : $\varepsilon_e = (d_2 - d_1)/l_e$

$\mathbf{n}_e$ unit vector of element e

$\mathbf{q}_e^T = [q_{1x}, q_{1y}, q_{2x}, q_{2y}]$, $\mathbf{n}_e = [n_{1x}, n_{1y}] = [\cos(\theta_e), \sin(\theta_e)]$ in 2D

$\mathbf{q}_e^T = [q_{1x}, q_{1y}, q_{1z}, q_{2x}, q_{2y}, q_{2z}]$, $\mathbf{n}_e = [n_{1x}, n_{1y}, n_{1z}]$ in 3D

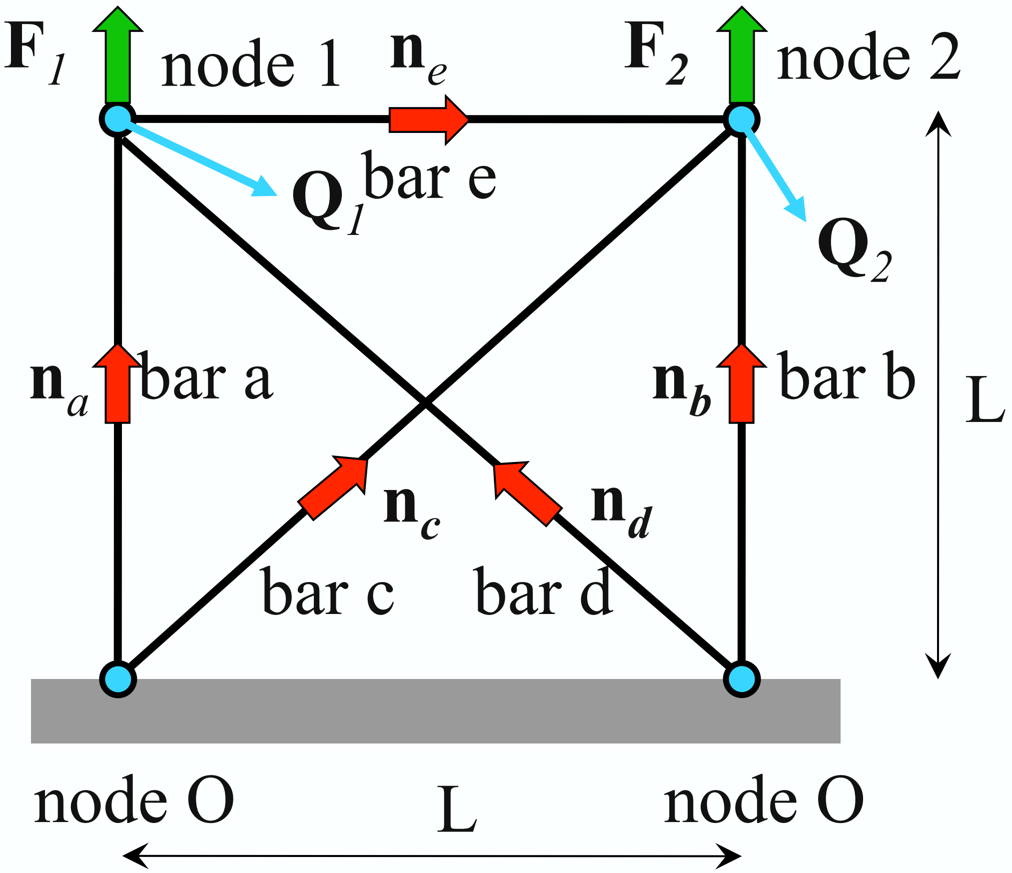
$$\mathcal{P} = \mathcal{P}_{int} + \mathcal{P}_{ext}$$

$$\mathcal{P}_{int} = \sum_{e} \mathcal{P}^{e}_{int} = \frac{1}{2}\mathbf{Q}^T\mathbf{K}\mathbf{Q} \; ; \quad \mathcal{P}^{e}_{int} = \int_{l_e} \left[\frac{1}{2} E \; A_e \epsilon_e^2\right] dx = \frac{1}{2}\mathbf{q}_e^T \mathbf{k}_e \mathbf{q}_e$$

$$\epsilon_e = du/dx = (d_2 - d_1)/l_e \; ; \quad d_i = \mathbf{q}_i^T \mathbf{n}_e \; , \; (i = 1, \; 2) \; , \; \mathbf{q}_e^T = [\mathbf{q}_1^T, \mathbf{q}_2^T]$$

$$\mathbf{k}_e \equiv \begin{bmatrix} \mathbf{k}'_e & -\mathbf{k}'_e \\ -\mathbf{k}'_e & \mathbf{k}'_e \end{bmatrix} \; ; \quad \mathbf{k}'_e \equiv \mathbf{n}_e \mathbf{n}_e^T \; , \; \mathbf{n}_e \text{ element orientation}$$

$$\mathcal{P}_{ext} = - \sum_{nodes} \mathbf{q}_n^T \mathbf{F}_n = \mathbf{Q}^T \mathbf{F} \; ; \quad \mathbf{q}_n \; , \mathbf{F}_n : \text{ displ., force at node n}$$



Uniform section bars (same EA)

$\theta_a = \pi/2$ for element a

$\theta_b = \pi/2$ for element b

$\theta_c = \pi/4$ for element c

$\theta_d = 3\pi/4$ for element d

$\theta_e = 0$ for element e

$$\begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{k}'_a + \mathbf{k}'_d + \mathbf{k}'_e & -\mathbf{k}'_e \\ -\mathbf{k}'_e & \mathbf{k}'_b + \mathbf{k}'_c + \mathbf{k}'_e \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{bmatrix}, \quad (\mathbf{F} = \mathbf{K}\mathbf{Q})$$

PLANAR TRUSS EXAMPLE – ELEMENT STIFFNESSES

$$\mathbf{k}'_e = \frac{EA}{l_e}\begin{bmatrix} \cos(\theta_e)\cos(\theta_e) & \cos(\theta_e)\sin(\theta_e) \\ \sin(\theta_e)\cos(\theta_e) & \sin(\theta_e)\sin(\theta_e) \end{bmatrix} \quad \text{for each element}$$

$$\mathbf{k}'_a = \mathbf{k}'_b = \frac{EA}{L}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{k}'_e = \frac{EA}{L}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{k}'_c = \frac{EA}{L\sqrt{2}}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \mathbf{k}'_d = \frac{EA}{L\sqrt{2}}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1+\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -1 & 0 \\ -\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & 0 & 0 \\ -1 & 0 & 1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & 0 & \frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} Q_{1x} \\ Q_{1y} \\ Q_{2x} \\ Q_{2y} \end{bmatrix}$$

for loading : $\mathbf{F}^T = [0,\ 1,\ 0, 1]$

displacements are : $\mathbf{Q}^T = \frac{PL}{EA}\left[\frac{1}{3+4\sqrt{2}},\ \frac{1+4\sqrt{2}}{3+4\sqrt{2}},\ -\frac{1}{3+4\sqrt{2}},\ \frac{1+4\sqrt{2}}{3+4\sqrt{2}}\right]$

Axial forces in each bar $N_e = EA(d_2\text{-}d_1)/l_e$

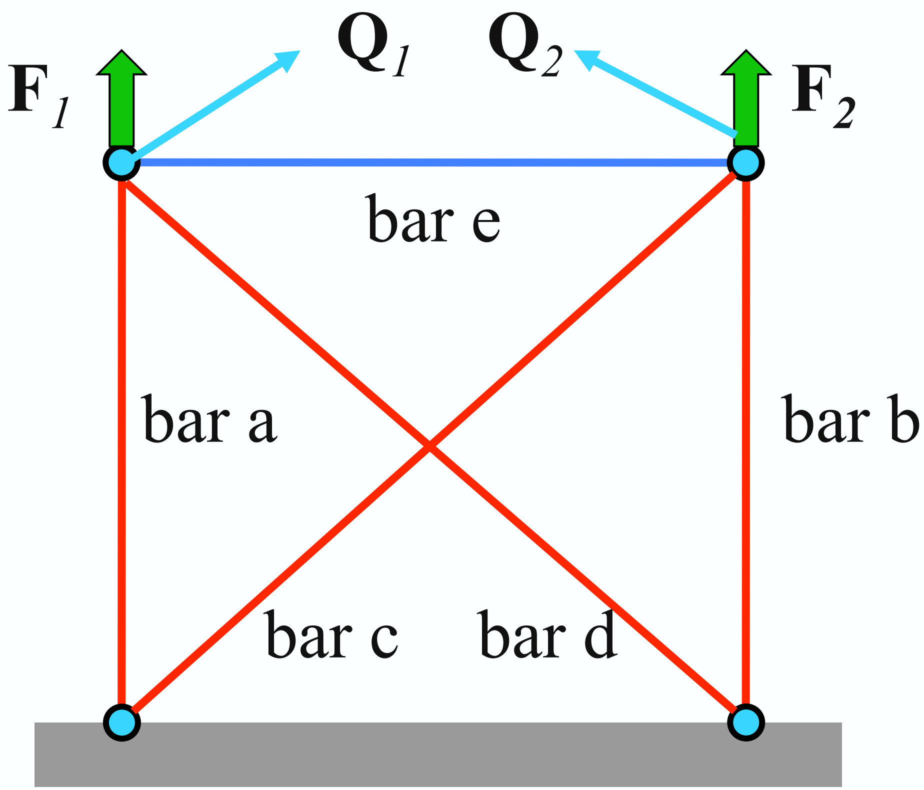
$N_a = P\,(4\sqrt{2}+1)/(4\sqrt{2}+3)$ tensile

$N_b = P\,(4\sqrt{2}+1)/(4\sqrt{2}+3)$ tensile

$N_c = 2\sqrt{2}\,P\,/(4\sqrt{2}+3)$ tensile

$N_d = 2\sqrt{2}\,P\,/(4\sqrt{2}+3)$ tensile

$N_e = -2P\,/(4\sqrt{2}+3)$ compressive



NOTE : Symmetric loading produces symmetric deformation & forces

NOTE : Taking into account thermal loading

$$\epsilon_e = du/dx - \alpha \Delta T = (d_2 - d_1)/l_e - \alpha \Delta T \; ; \quad d_i = \mathbf{q}_i^T \mathbf{n}_e$$

$$\mathcal{P}_{int}^e = \int_{l_e} \left[\frac{1}{2} E \; A_e \epsilon_e^2 \right] dx = \frac{1}{2} \mathbf{q}_e^T \mathbf{k}_e \mathbf{q}_e - \mathbf{q}_e^T \mathbf{f}_e$$

$$\mathbf{k}_e \equiv \begin{bmatrix} \mathbf{k}'_e & -\mathbf{k}'_e \\ -\mathbf{k}'_e & \mathbf{k}'_e \end{bmatrix} \; ; \quad \mathbf{k}'_e \equiv \mathbf{n}_e \mathbf{n}_e^T \; , \; \mathbf{n}_e \text{ element orientation}$$

$$\mathbf{f}_e^T \equiv E \; A_e \alpha \Delta T \left[-\mathbf{n}_e^T \; , \mathbf{n}_e^T \right]$$

α: thermal expansion coefficient, ΔT: temperature change

ANNOUNCEMENT

For students in PA in Mechanics (MEC 592, MEC 595), I will be at your disposal on October 05 at 11:am in Amphi MONGE to talk to you about internships in Mechanics.

I will also inform you about a dual MS degree program with Caltech (Departement de Mecanique at X and Aerospace Engineering at Caltech) which concerns students interested in pursuing a Doctorate degree in either Fluid Mechanics or Solid Mechanics back here (LadHyx or LMS) and talk about similar possibilities exist with University of Minnesota for Aerospace and Civil Engineering (LMS)