WHAT IS THE FINITE ELEMENT METHOD?

RALEIGH-RITZ NUMERICAL SOLUTION TECHNIQUE IN APPLIED MATHEMATICS:

- IDEA STARTED WITH VIBRATION THEORY: FOR CONTINUUM PROBLEMS WITH AN ENERGY, USE SHAPE FUNCTIONS TO CONVERT INFINITE DIMENSIONAL PROBLEM TO A DISCRETE ONE THAT CAN BE SOLVED WITH MATRIX ALGEBRA (1909)

- BY ABOUT 1970's PEOPLE REALIZED THAT THE APPROXIMATE ENGINEERING F.E.M. TECHNIQUE WAS A RALEIGH-RITZ METHOD WITH INGENIOUS SHAPE FUNCTIONS OF COMPACT SUPPORT

THE REST IS THE HISTORY OF ONE OF THE GREATEST CONTRIBUTIONS OF MECHANICS AND APPLIED MATHEMATICS TO MODERN EGINEERING TECHNOLOGY

- APPROACH THAT STARTED WITH LINEAR ELASTICITY WAS EXTENDED TO THE MOST GENERAL TYPE OF NONLINEAR, INELASTIC SOLIDS & STRUCTURES

- METHOD IS APPLICABLE TO A WIDE CLASS OF BOUNDARY PROBLEMS BUT IS BEST SUITED FOR ELLIPTIC PROBLEMS

- FINITE ELEMENTS TECHNOLOGY IS ONE OF THE MOST IMPORTANT CONTRIBUTIONS OF MECHANICS THAT REVOLUTIONIZED ENGINEERING TECHNOLOGY

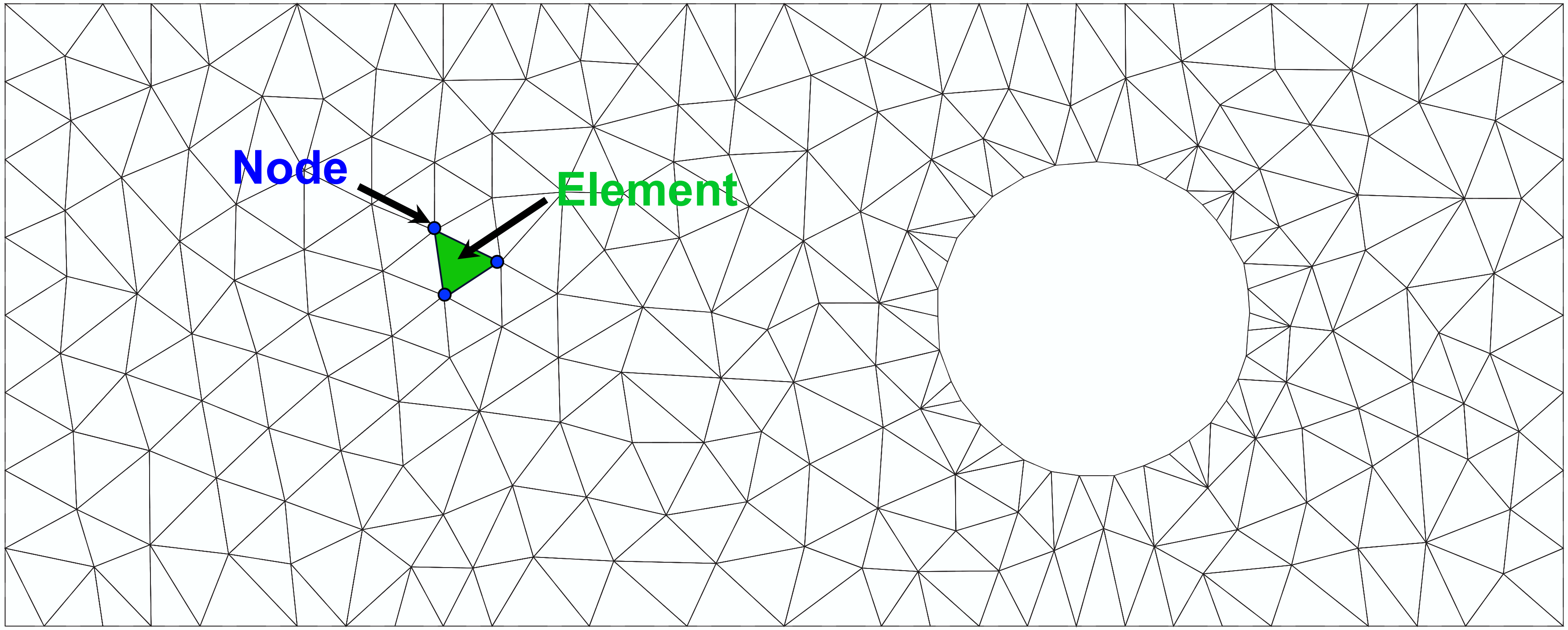
TOPICS COVERED IN THIS CLASS

1. INTRODUCTION TO THE FINITE ELEMENT METHOD USING 1-D MODELS.
2. CHOLESKY METHOD FOR SOLVING LINEAR SYSTEMS.
3. TRUSSES AND FRAMES IN 2D AND 3D.
4. VARIATIONAL FORMULATION FOR LINEAR ELASTICITY B.V.P.
5. PLANE STRESS/STRAIN PROBLEMS USING CONSTANT STRAIN TRIANGLES.
6. ISOPARAMETRIC ELEMENTS FOR 2D PROBLEMS.
7. NUMERICAL INTEGRATION, GENERALIZATION TO 3D PROBLEMS.
8. HIGHER ORDER GRADIENT ENERGIES: BEAMS (1D) AND PLATES (2D).
9. LOCKING PHENOMENA AND SOLUTION PROCEDURES.
10. OTHER PHYSICS PROBLEMS (ELECTROSTATICS, HEAT TRANSFER).
11. TIME-DEPENDENT ANALYSES, EIGENMODES
12. EXTENSION TO NON-LINEAR PROBLEMS (INCREMENTAL NEWTON-RAPHSON)

WHAT IS THE FINITE ELEMENT METHOD?

POWERFULL NUMERICAL METHOD TO SOLVE PROBLEMS IN SOLIDS & STRUCTURES

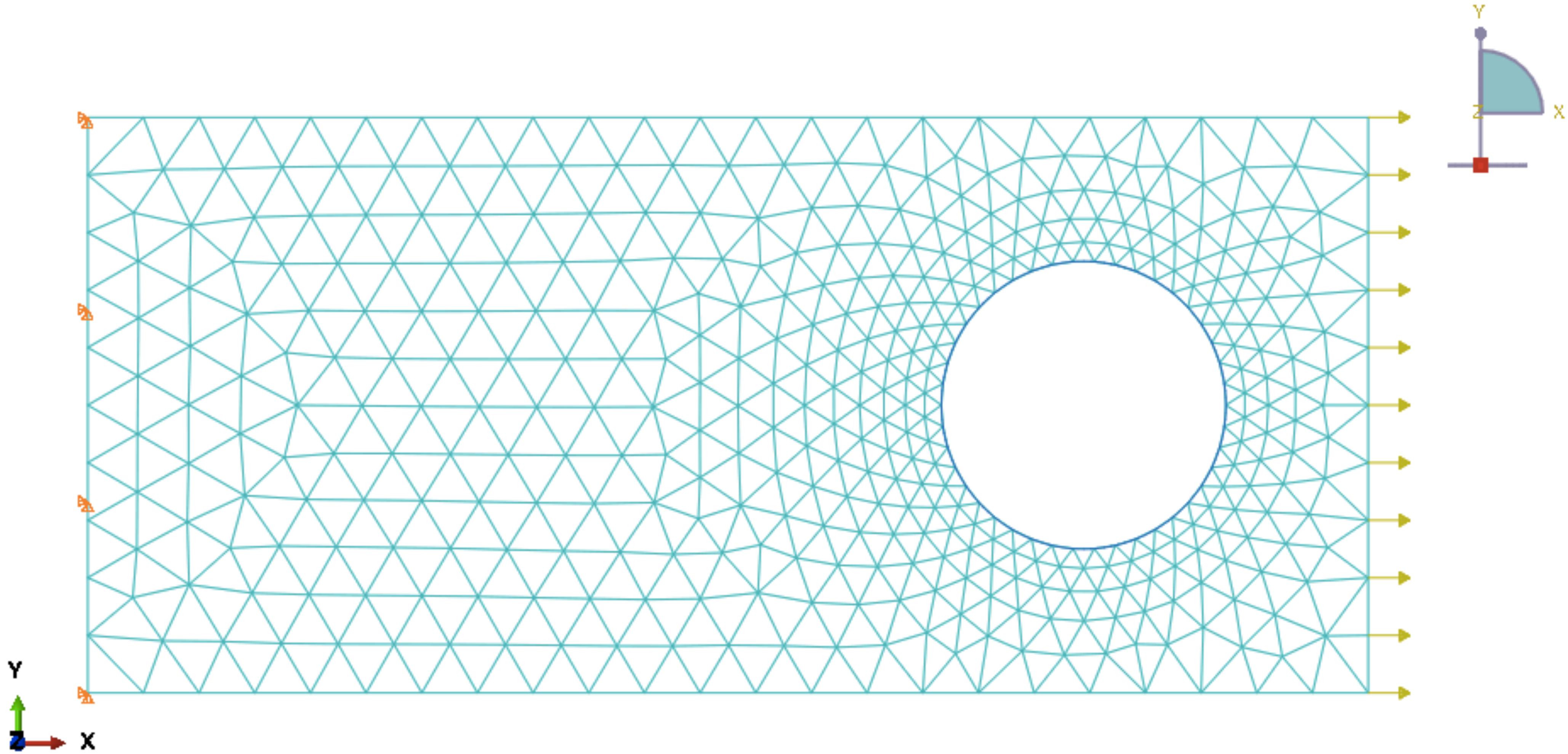
- **STARTED FROM MATRIX PROBLEMS (FRAMES) IN AIRCRAFT STRUCTURES (1940's)**
- **CONTINUED AS A EURISTIC METHOD FOR ELASTIC BODIES, PARTITIONING SOLIDS USING A SIMPLEX GRID (TRIANGLES IN 2D, TETRAHEDRA IN 3D, TERMED FINITE ELEMENTS) AND SATISFYING EQUILIBRIUM THROUGH NODAL FORCES**



FEM SOLUTION OF A LINEARLY ELASTIC PROBLEM

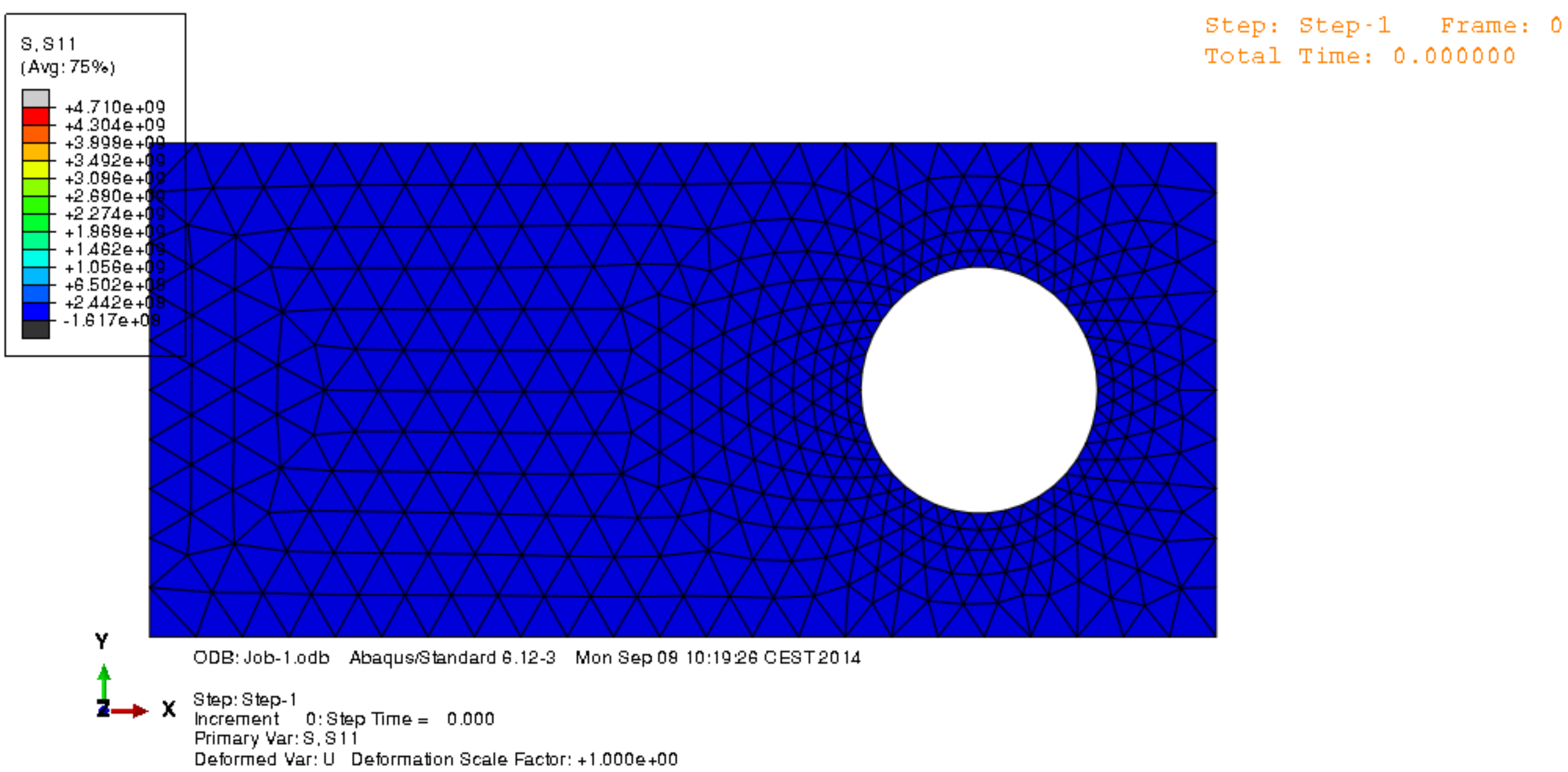
CONSTANT STRAIN TRIANGLE FINITE ELEMENT MESH FOR A PLANE STRAIN PROBLEM

SOLVED USING A COMMERCIAL CODE (ABAQUS – NOT AVAILABLE FOR THIS CLASS)



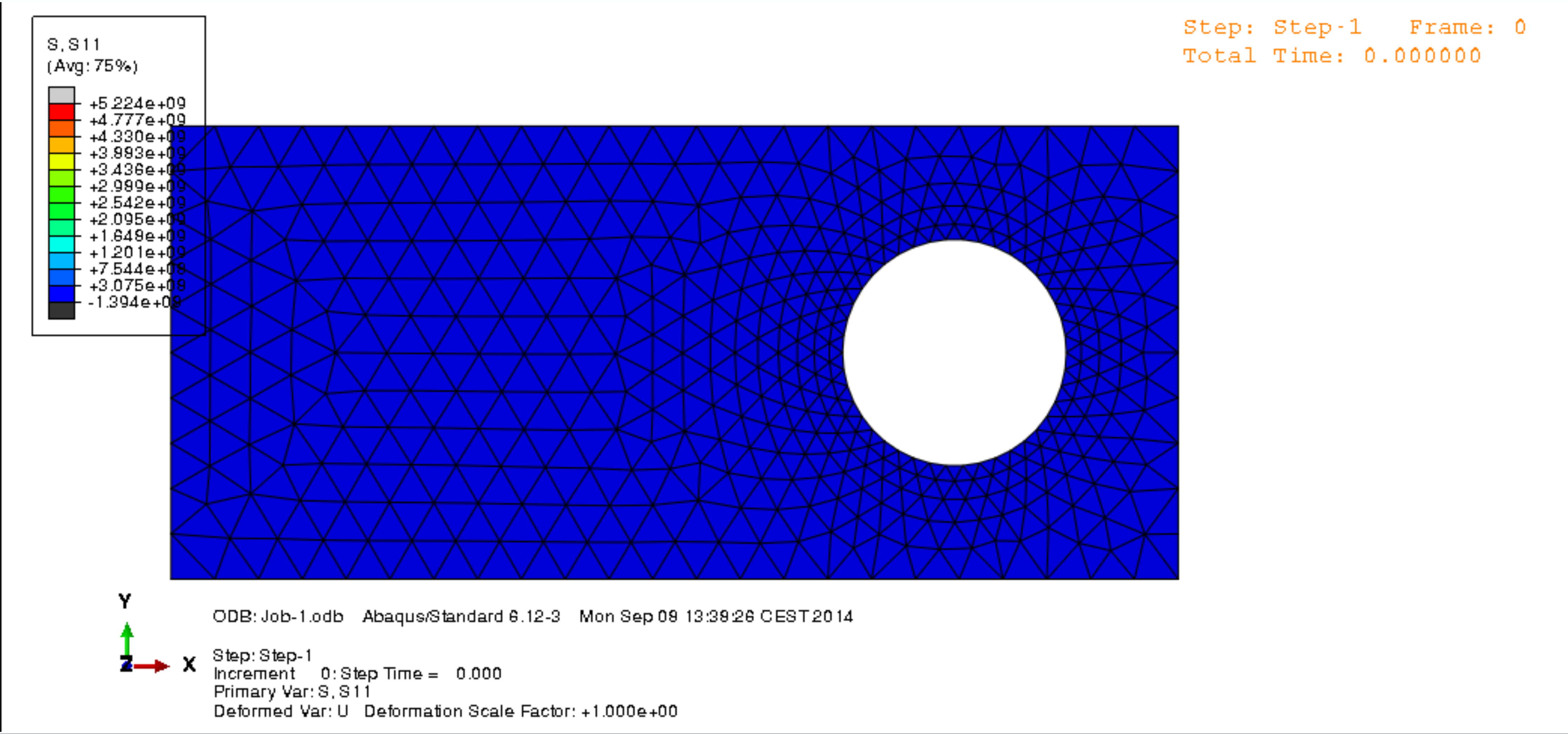
FEM SOLUTION OF A LINEARLY ELASTIC PROBLEM

CLAMPED AT X = 0, NORMAL FORCES PRESCRIBED AT X = L – NORMAL STRESSES σ_{11}



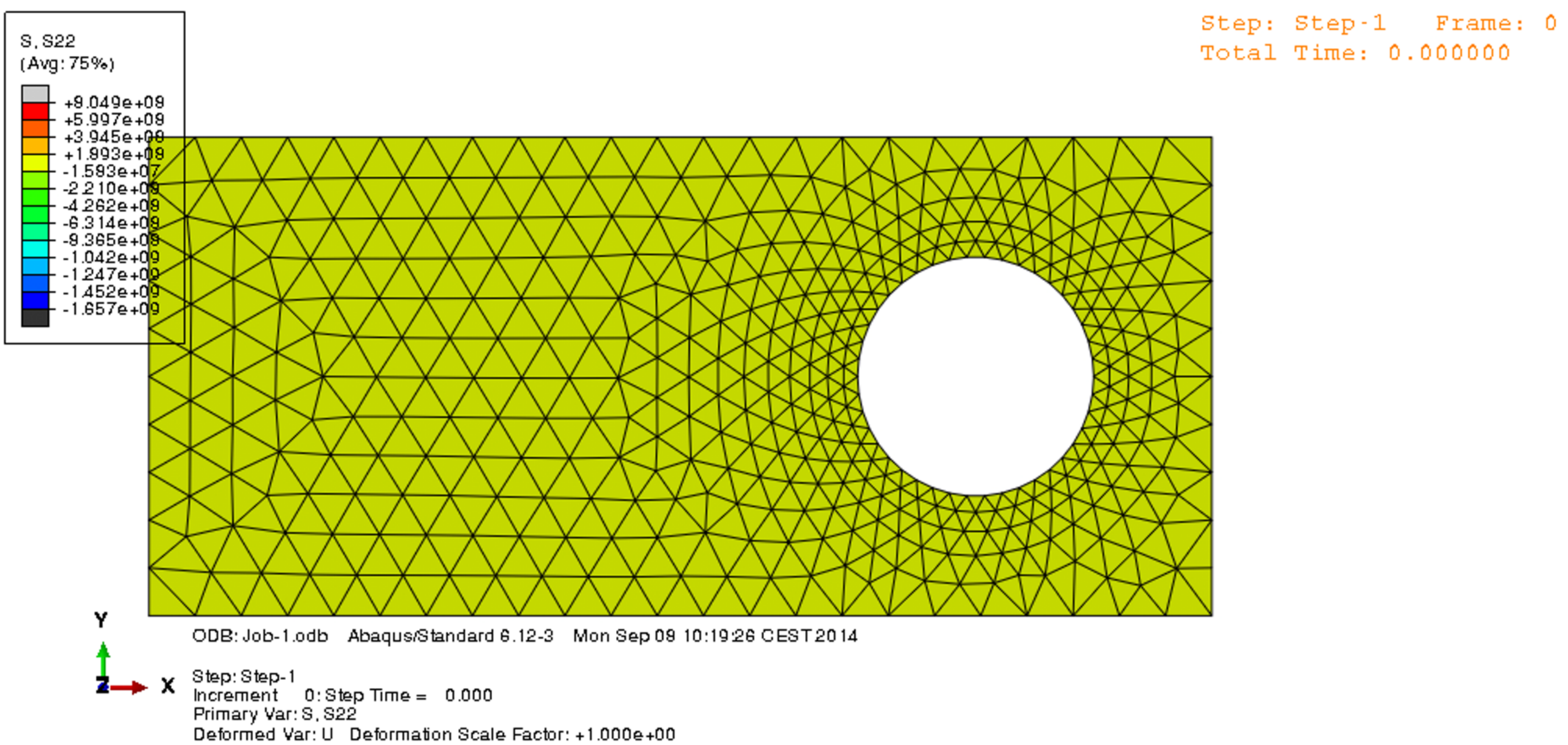
FEM SOLUTION OF A LINEARLY ELASTIC PROBLEM

ROLLERS AT X = 0**, NORMAL DISPLACEMENT PRESCRIBED AT** X = L **– NORMAL STRESSES** σ_{11}



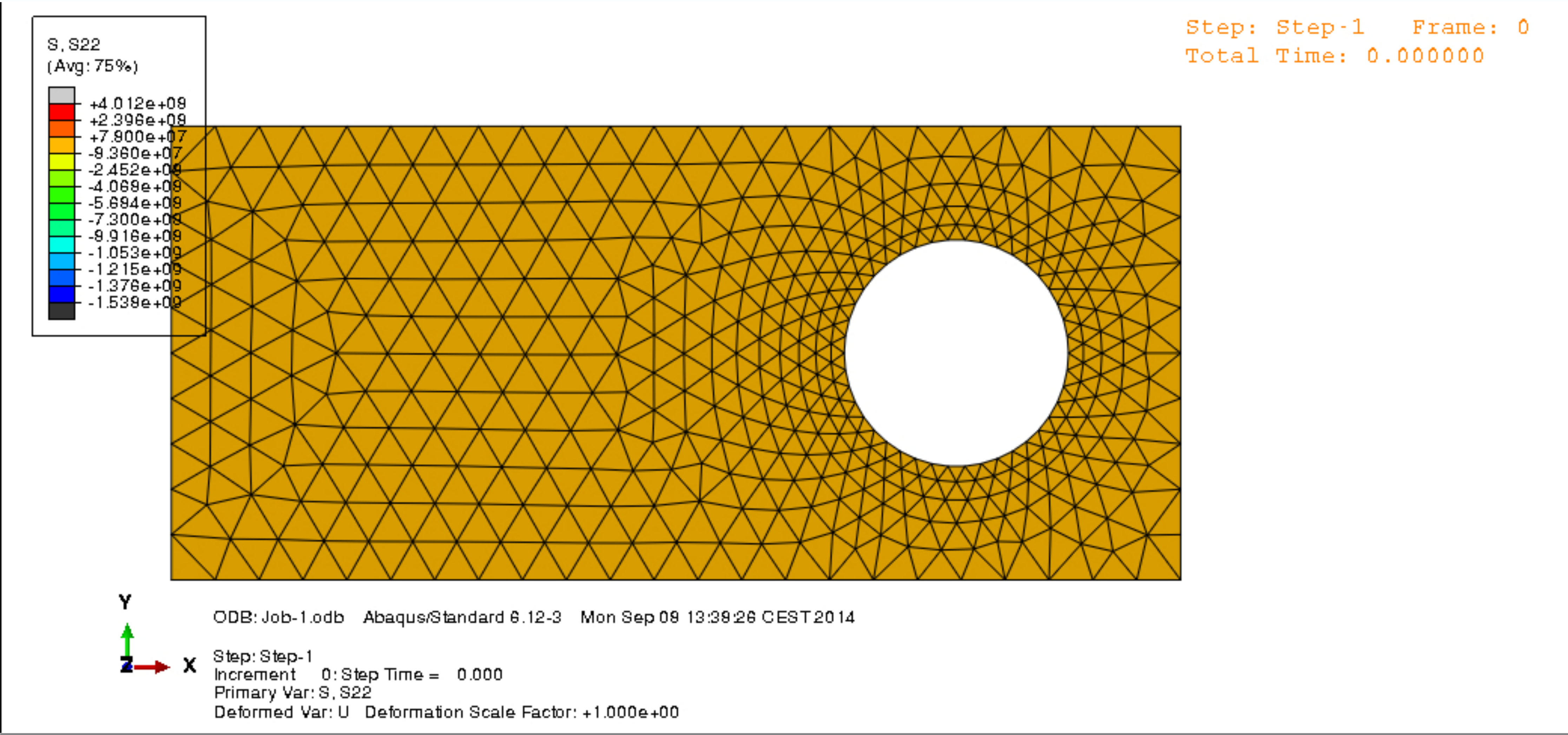
FEM SOLUTION OF A LINEARLY ELASTIC PROBLEM

CLAMPED AT X = 0, **NORMAL FORCES PRESCRIBED AT** X = L **– NORMAL STRESSES** σ_{22}



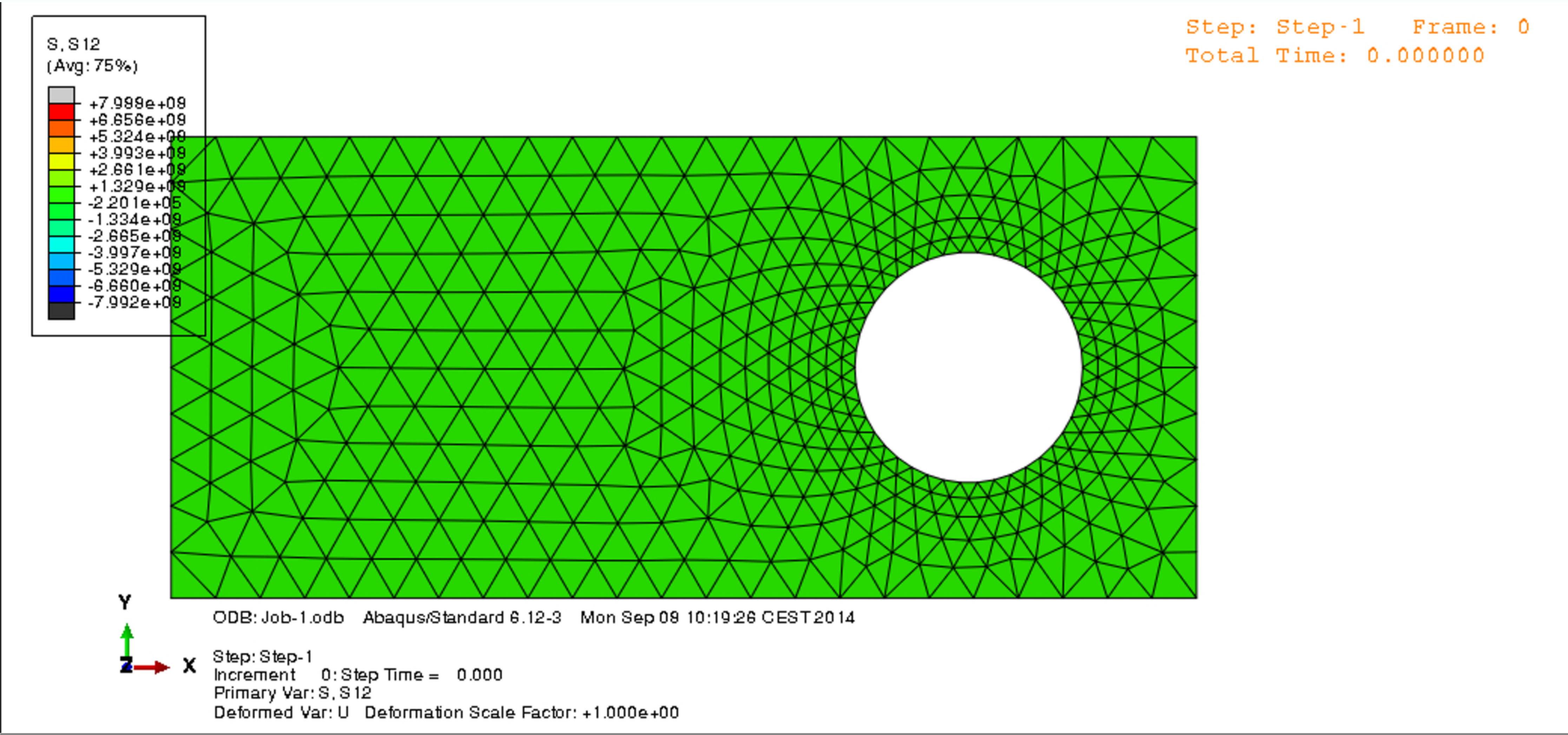
FEM SOLUTION OF A LINEARLY ELASTIC PROBLEM

ROLLERS AT X = 0, NORMAL DISPLACEMENT PRESCRIBED AT X = L – NORMAL STRESSES σ_{22}



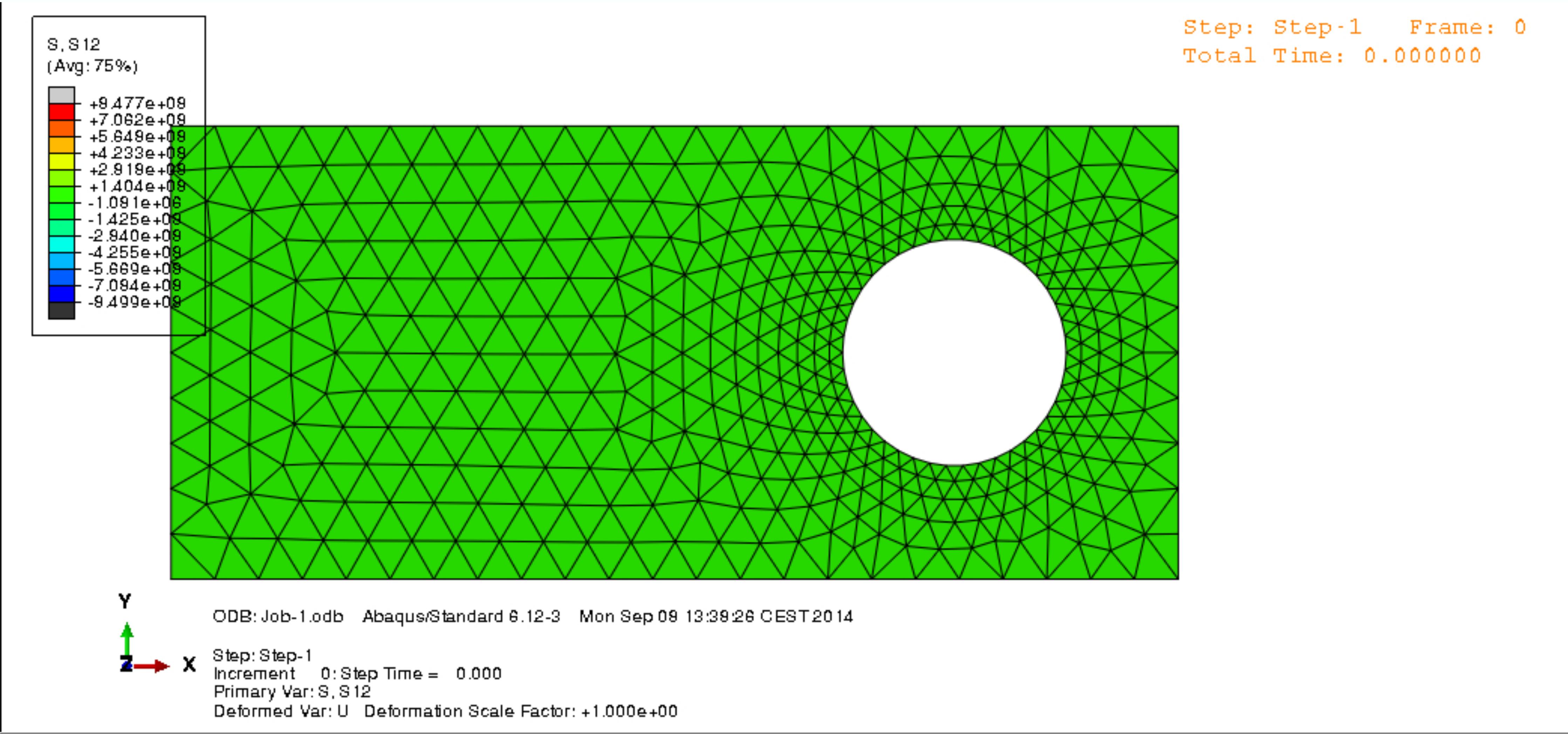
FEM SOLUTION OF A LINEARLY ELASTIC PROBLEM

CLAMPED AT X = 0, **NORMAL FORCES PRESCRIBED AT** X = L **– SHEAR STRESSES** σ_{12}



FEM SOLUTION OF A LINEARLY ELASTIC PROBLEM

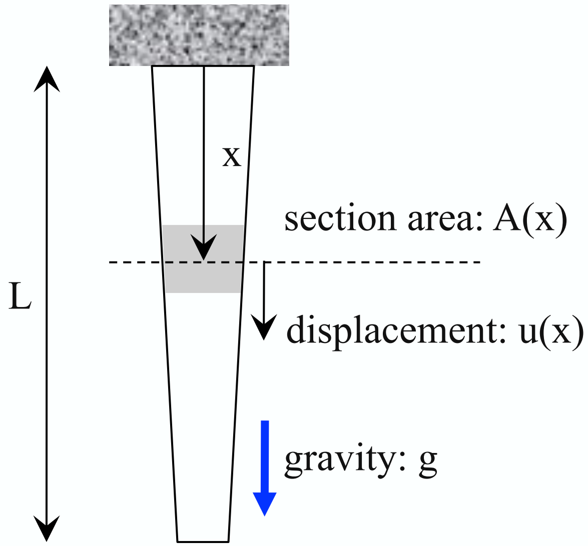
ROLLERS AT X = 0, **NORMAL DISPLACEMENT PRESCRIBED AT** X = L **– SHEAR STRESSES** σ_{12}

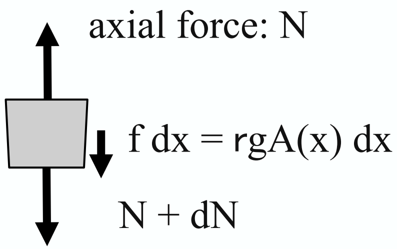


POWERFULL CODES PRODUCE NICE PICTURES, BUT

1. What is the code behind them?
2. Can you trust that simulation is done correctly?
3. You need to understand your model! Very frequently the code gives correct results but you are unable to interpret them…
4. Of course more frequently you translate your math/mechanics problem to a wrong or inadequate algorithm!
5. Goal of the class to help you understand mechanics and how we translate the mathematical model to a functioning algorithm
6. NECESSARY CONDITION: **Understand your mechanics!**

ONE DIMENSIONAL EXAMPLE – DIRECT METHOD





$$\text{Equilibrium}: \quad \frac{dN}{dx} + \rho g A = 0$$

$$\text{Kinematics}: \quad \epsilon(x) = \frac{du}{dx}$$

$$\text{Constitutive}: \quad N(x) = E\,A(x)\,\epsilon(x)$$

$$\text{Governing Eq.}: \quad \frac{d}{dx}\left[E\,A(x)\,\frac{du}{dx}\right] + \rho\,g\,A(x) = 0$$

$$\text{Boundary Cond.}: \quad u(0) = 0\,,\;\; E\,A(L)\frac{du}{dx}(L) = 0$$

Must solve above 2^{nd} order O.D.E. with boundary conditions…

QUESTION: is there more efficient way to formulate & solve this B.V.P.?

ANSWER: use potential energy minimization (variational method)

$$\text{Potential} \quad : \quad \mathcal{P} = \mathcal{P}_{int} + \mathcal{P}_{ext}$$

$$\text{Internal} \quad : \quad \mathcal{P}_{int} = \int_V \frac{1}{2}\sigma(x)\,\epsilon(x)\,dV = \frac{1}{2}\int_0^L E\,A(x)\left(\frac{du}{dx}\right)^2 dx$$

$$\text{External} \quad : \quad \mathcal{P}_{ext} = -\int_V \rho\,g\,u(x)\,dV = -\int_0^L \rho\,g\,A(x)\,u(x)\,dx$$

CLAIM: of all admissible displacement fields *u(x)*, i.e. continuous functions that satisfy the essential boundary condition: $u(0) = 0$, the actual equilibrium solution minimizes the potential energy functional $\mathcal{P}(u(x))$

QUESTION: how to minimize functional $\mathcal{P}(u(x))$, to find equilibrium $u_{eq}(x)$?

ANSWER: convert to the regular minimization you know! (Gateau derivative)

$$\mathcal{P}(u_{eq} + \epsilon\delta u) \geq \mathcal{P}(u_{eq}) \; ; \qquad u(x) \equiv u_{eq}(x) + \epsilon\delta u(x) \; , \; \epsilon \in \mathbb{R}$$

$$\frac{d}{d\epsilon}\left[\mathcal{P}(u_{eq} + \epsilon\delta u)\right]_{\epsilon=0} = 0 \; ; \qquad \text{extremum (1)}$$

$$\frac{d^2}{d\epsilon^2}\left[\mathcal{P}(u_{eq} + \epsilon\delta u)\right]_{\epsilon=0} > 0 \; ; \qquad \text{minimum (2)}$$

$$(1) \implies \int_0^L \left[EA \frac{du_{eq}}{dx} \frac{d\delta u}{dx} - \rho g A \delta u \right] dx = 0$$

$$\left[EA \frac{du_{eq}}{dx} \delta u \right]_{x=0}^{x=L} - \int_0^L \left\{ \left[\frac{d}{dx} \left(EA \frac{du_{eq}}{dx} \right) + \rho g A \right] \delta u \right\} dx = 0$$

$$\forall \ \delta u(x) \neq 0 \implies \frac{d}{dx} \left(EA \frac{du_{eq}}{dx} \right) + \rho g A = 0 \ ; \quad x \in [0, L]$$

$$\delta u(0) = 0 \implies \frac{du_{eq}}{dx} = 0 \ ; \quad x = L$$

$$(2) \implies \int_0^L \left[EA \left(\frac{d\delta u}{dx} \right)^2 \right] dx > 0$$

WE PROVED: of all admissible displacement fields $u(x)$, i.e. continuous functions that satisfy the essential boundary condition: $u(0) = 0$, the actual equilibrium solution $u_{eq}(x)$ minimizes the potential energy functional $\mathcal{P}(u(x))$.

As expected, we obtained a differential equation (Euler-Lagrange) plus an additional – to the essential – boundary condition (natural bound. cond.). We provided a better formulation but we have not solved the problem!

The better formulation gives the key to the numerical solution: instead of minimizing energy in an infinite dimensional space, we should minimize in a finite dimensional space, in which case we end up with an algebraic problem. Thus we use an approximate displacement $u^{app}(x)$ – which involves a finite number of variables $Q_i\ (i=1, \dots n)$ – and minimize $\mathcal{P}(\mathbf{Q})$ with respect to **Q.**

$$\mathcal{P}(u^{app}(x)) = \mathcal{P}(\mathbf{Q}); \quad u^{app} = \sum_{i=1}^{i=n} Q_i N_i(x), \quad \mathbf{Q} \equiv [Q_1, Q_2, \cdots Q_n]$$

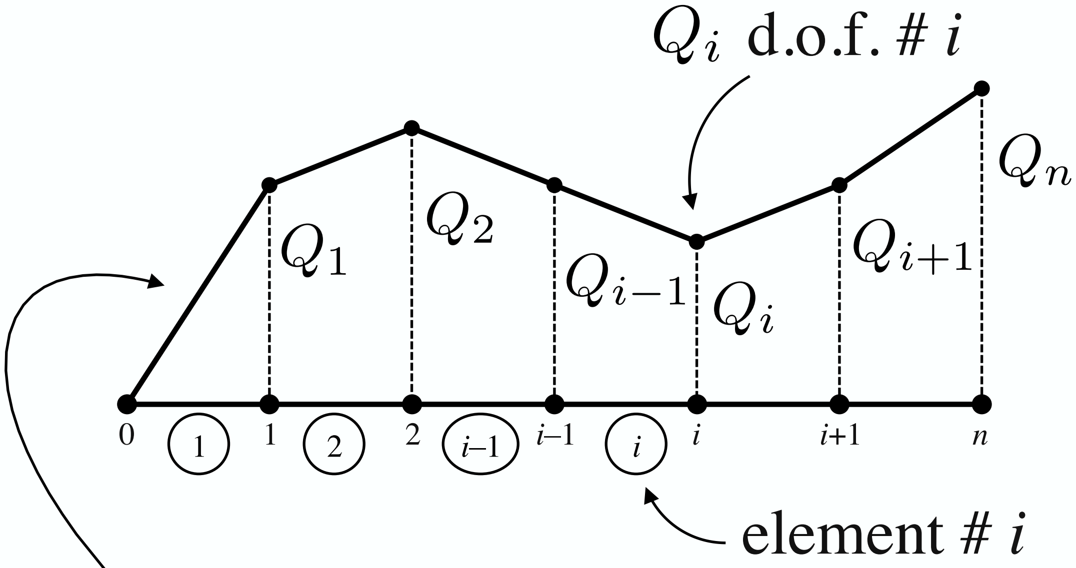
$$\frac{\partial \mathcal{P}(\mathbf{Q})}{\partial Q_i} = \int_0^L \left[EA \frac{du^{app}}{dx} \frac{\partial}{\partial Q_i}\left(\frac{du^{app}}{dx}\right) - \rho g A \frac{\partial u^{app}}{\partial Q_i}\right] dx = 0$$

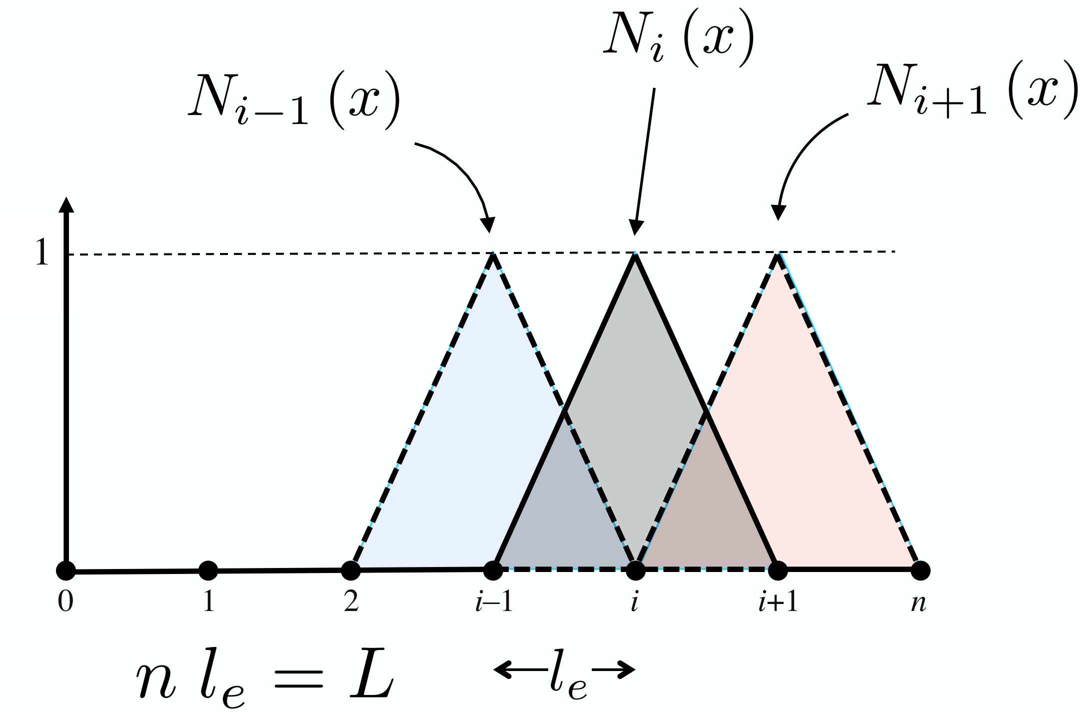
$$\sum_{j=1}^{j=n} \left\{ \int_0^L \left[EA \frac{dN_i}{dx}\frac{dN_j}{dx}\right] dx \right\} Q_j - \int_0^L \left[\rho g A N_i\right] dx = 0$$

$$\sum_{j=1}^{j=n} K_{ij} Q_j - F_i = 0 \; ; \quad (\text{in compact form} : \mathbf{KQ} = \mathbf{F})$$

$$K_{ij} \equiv \int_0^L \left[EA \frac{dN_i}{dx}\frac{dN_j}{dx}\right] dx \, , F_i \equiv \int_0^L \left[\rho g A N_i\right] dx$$

Stiffness matrix: **K**, Force vector: **F**, Degrees of Freedom: **Q**





$$u^{\text{app}}(x) = \sum_i Q_i N_i(x) \qquad \text{Shape function } N_i(x)$$

Easy physical interpretation of d.o.f. (degree of freedom) Q_i at node x_i: due to its construction, $u^{app}(x_i) = Q_i$

Shape functions $N_i(x)$ have compact support: $N_i(x_i) = 1,\ N_i(x_{i-1}) = N_i(x_{i+1}) = 0$.
Compactness of support of shape function great advantage of FEM

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & 0 \\ 0 & K_{32} & K_{33} & K_{34} & 0 \\ 0 & 0 & K_{43} & K_{44} & K_{45} \\ 0 & 0 & 0 & K_{54} & K_{55} \end{bmatrix}$$

$$K_{ii} = \int_{x_{i-1}}^{x_{i+1}} E\ A(x) \left(\frac{1}{l_e}\right)^2 dx$$

$$K_{i\ i+1} = -\int_{x_i}^{x_{i+1}} E\ A(x) \left(\frac{1}{l_e}\right)^2 dx$$

Stiffness matrix **K** is banded, i.e. populated about the diagonal. This structure, due to the compactness of shape functions, has great advantages in both solution time and storage requirements. An efficient algorithm, under the name of Cholesky (André-Louis Cholesky X-1895) decomposition, takes advantage of the banded structure of **K**.

ONE DIMENSIONAL EXAMPLE – CHOLESKY DECOMPOSITION

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & 0 & K_{43} & K_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ 0 & L_{32} & 1 & 0 \\ 0 & 0 & L_{43} & 1 \end{bmatrix} \begin{bmatrix} D_{11} & 0 & 0 & 0 \\ 0 & D_{22} & 0 & 0 \\ 0 & 0 & D_{33} & 0 \\ 0 & 0 & 0 & D_{44} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & 0 & 0 \\ 0 & 1 & U_{23} & 0 \\ 0 & 0 & 1 & U_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Cholesky decomposition (unique), **K = LDU** (lower triangular, diagonal, upper triangular matrices) that have same skyline structure as **K**. Method valid for arbitrary matrices. When $\mathbf{K} = \mathbf{K}^{\mathrm{T}}$, then $\mathbf{L}^{\mathrm{T}} = \mathbf{U}$ (half storage needed)

$$D_{11} = K_{11}$$

$$D_{22} = K_{22} - L_{21}D_{11}U_{12} \;, \quad L_{21} = K_{21}/D_{11} \;, \quad U_{12} = K_{12}/D_{11}$$

$$D_{33} = K_{33} - L_{32}D_{22}U_{23} \;, \quad L_{32} = K_{32}/D_{22} \;, \quad U_{23} = K_{23}/D_{22}$$

$$D_{44} = K_{44} - L_{43}D_{33}U_{34} \;, \quad L_{43} = K_{43}/D_{33} \;, \quad U_{34} = K_{34}/D_{33}$$

$$\mathbf{KQ} = \mathbf{F} \implies \mathbf{LDL^{\intercal}Q} = \mathbf{F} \text{ is solved in steps}$$

$$\mathbf{LY} = \mathbf{F} \quad \begin{pmatrix}\text{forward}\\ \text{substitution}\end{pmatrix} \quad \begin{aligned} &Y_1 = F_1, \quad Y_2 = F_2 - L_{21}Y_1 \\ &Y_3 = F_3 - L_{32}Y_2, \quad Y_4 = F_4 - L_{43}Y_3 \end{aligned}$$

$$\mathbf{DX} = \mathbf{Y} \quad X_i = Y_i / D_{ii}$$

$$\mathbf{L^{\intercal}Q} = \mathbf{X} \quad \begin{pmatrix}\text{backward}\\ \text{substitution}\end{pmatrix} \quad \begin{aligned} &Q_4 = F_4, \quad Q_3 = F_3 - L_{43}Q_4 \\ &Q_2 = F_2 - L_{32}Q_3, \quad Q_1 = F_1 - L_{21}Q_2 \end{aligned}$$

NOTE: Cholesky decomposition need only be done once with **L** & **D** components stored in place of **K**! We can then solve the same structure under different loads **F** simply by doing forward & back substitution again & again

Recall Cholesky decomposition algorithm (symmetric case):

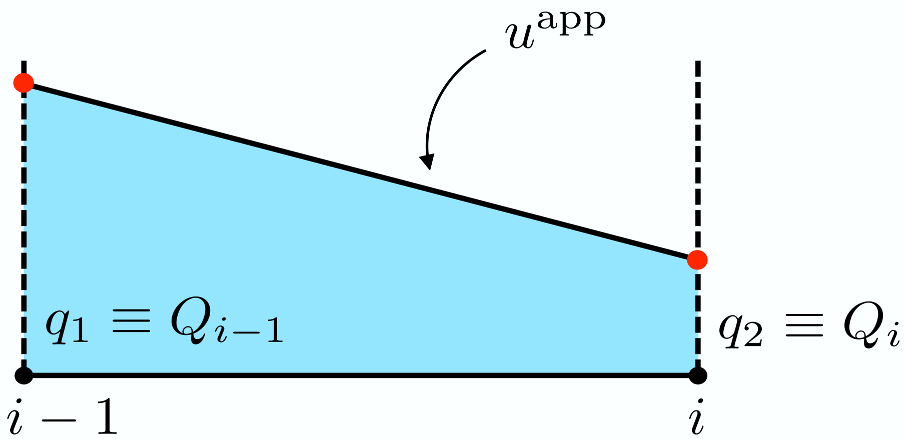
$$
\begin{aligned}
D_{11} &= K_{11} \\
D_{22} &= K_{22} - L_{21} D_{11} L_{21} \,, \quad & L_{21} &= K_{21}/D_{11} \\
D_{33} &= K_{33} - L_{32} D_{22} L_{32} \,, \quad & L_{32} &= K_{32}/D_{22} \\
D_{44} &= K_{44} - L_{43} D_{33} L_{43} \,, \quad & L_{43} &= K_{43}/D_{33}
\end{aligned}
$$

As algorithm progresses, new entries L_{ij} and D_{ii} are stored in K_{ij} and K_{ii} slots respectively since these elements of stiffness matrix will not be needed again:

$$
\begin{aligned}
& D_{11} \longrightarrow K_{11} \\
& L_{12} \longrightarrow K_{21} \,, \quad && D_{22} \longrightarrow K_{22} \\
& L_{32} \longrightarrow K_{32} \,, \quad && D_{33} \longrightarrow K_{33} \\
& L_{34} \longrightarrow K_{43} \,, \quad && D_{44} \longrightarrow K_{44}
\end{aligned}
$$

Check: A positive definite **K** (correct linear elasticity b.v.p.) results in $D_{ii} > 0$

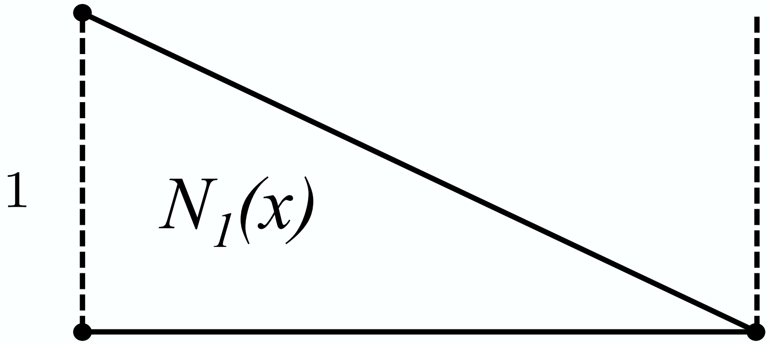
ONE DIMENSIONAL EXAMPLE – ELEMENT STIFFNESS, FORCE

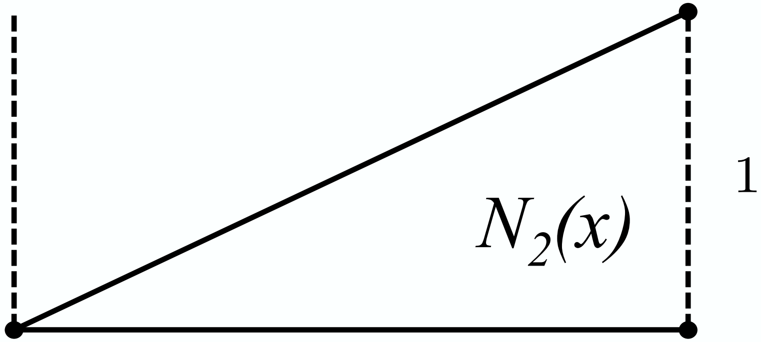


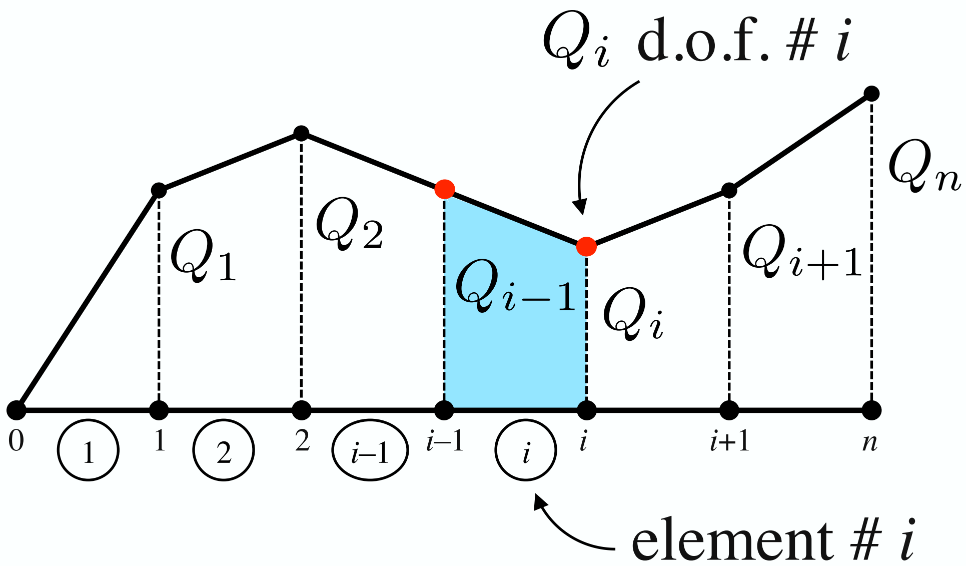
In element i: $u^{app}(x) = q_1 N_1(x) + q_2 N_2(x)$

Local degree of freedom $\mathbf{q}^{\mathrm{T}}_{\mathrm{e}} = [q_1, q_2]$

We find element contribution to global stiffness matrix **K** and force vector **F**







$$\mathcal{P}(\mathbf{Q}) = \mathcal{P}_{int}(\mathbf{Q}) + \mathcal{P}_{ext}(\mathbf{Q})$$

$$\mathcal{P}_{int}(\mathbf{Q}) = \sum_e \mathcal{P}^e_{int} \; ; \quad \mathcal{P}^e_{int} = \int_{l_e} \left[\frac{1}{2} E \, A(x) \left(q_1 \frac{dN_1}{dx} + q_2 \frac{dN_2}{dx} \right)^2 \right] dx = \frac{1}{2} \mathbf{q}_e^T \mathbf{k}_e \mathbf{q}_e$$

$$\mathcal{P}_{ext}(\mathbf{Q}) = \sum_e \mathcal{P}^e_{ext} \; ; \quad \mathcal{P}^e_{ext} = -\int_{l_e} \left[\rho g \, A(x) \left(q_1 N_1(x) + q_2 N_2(x) \right) \right] dx = -\mathbf{q}_e^T \mathbf{f}_e$$

$$[\mathbf{k}_e]_{ij} \equiv \int_{l_e} \left[E \, A(x) \left(\frac{dN_i}{dx} \right) \left(\frac{dN_j}{dx} \right) \right] dx \; ; \quad \mathbf{k}_e \text{ element stiffness matrix}$$

$$[\mathbf{f}_e]_i \equiv -\int_{l_e} \left[\rho g \, A(x) N_i(x) \right] dx \; ; \quad \mathbf{f}_e \text{ element force vector}$$

Finding element stiffness matrix $\mathbf{k}_e$ and element force vector $\mathbf{f}_e$ in the structure

$$\mathcal{P}_{int} = \frac{1}{2}\mathbf{Q}^{\intercal}\mathbf{K}\mathbf{Q} = \sum_e \mathbf{q}_e^{\intercal}\mathbf{k}_e\mathbf{q}_e$$

$$\mathcal{P}_{ext} = -\mathbf{Q}^{\intercal}\mathbf{F} = -\sum_e \mathbf{q}_e^{\intercal}\mathbf{f}_e$$

Relating element stiffness matrix $\mathbf{k}_e$ to global stiffness matrix $\mathbf{K}$ and element force vector $\mathbf{f}_e$ to global force vector $\mathbf{F}$

$\mathbf{q}_e$ local d.o.f. vector
$\mathbf{Q}$ global d.o.f. vector

$$\mathbf{k}_e = \begin{bmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{bmatrix} , \quad k \equiv \int_{l_e} E\ A(x)\frac{1}{(l_e)^2}dx ,$$

$$k_{11}^e = k_{22}^e = k$$

$$k_{12}^e = k_{21}^e = -k$$

$$\mathbf{f}_e = \begin{bmatrix} f_1^e \\ f_2^e \end{bmatrix} , \quad f_1^e = \int_{l_e} \rho g\ A(x)\left(\frac{x_2 - x}{l_e}\right) dx , \quad f_2^e = \int_{l_e} \rho g\ A(x)\left(\frac{x - x_1}{l_e}\right) dx$$

$$
\begin{array}{c}
\begin{matrix} & & i & & j & \end{matrix} \\
\begin{matrix} i \\ \\ j \end{matrix}
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & k^e_{11} & \cdots & k^e_{12} & \cdots \\
 & \vdots & & \vdots & \\
\cdots & k^e_{21} & \cdots & k^e_{22} & \cdots \\
 & \vdots & & \vdots &
\end{bmatrix}
\end{array}
$$

$$
\begin{bmatrix} \cdots & f^e_1 & \cdots & f^e_2 & \cdots \end{bmatrix}
$$

Assembling global stiffness matrix **K** and global force vector **F** from element stiffness matrix $\mathbf{k}_e$ and element force vector $\mathbf{f}_e$

RULE: for each element *e* add to global stiffness matrix & force vector the components in the appropriate places recalling local to global numbering

1➔i, 2➔j for this 2-node element

$$k_{11}^e = k_{22}^e = \frac{EA}{l_e} = \frac{4EA}{L} = k \,, \qquad k_{12}^e = k_{21}^e = -\frac{EA}{l_e} = -\frac{4EA}{L} = -k \,,$$

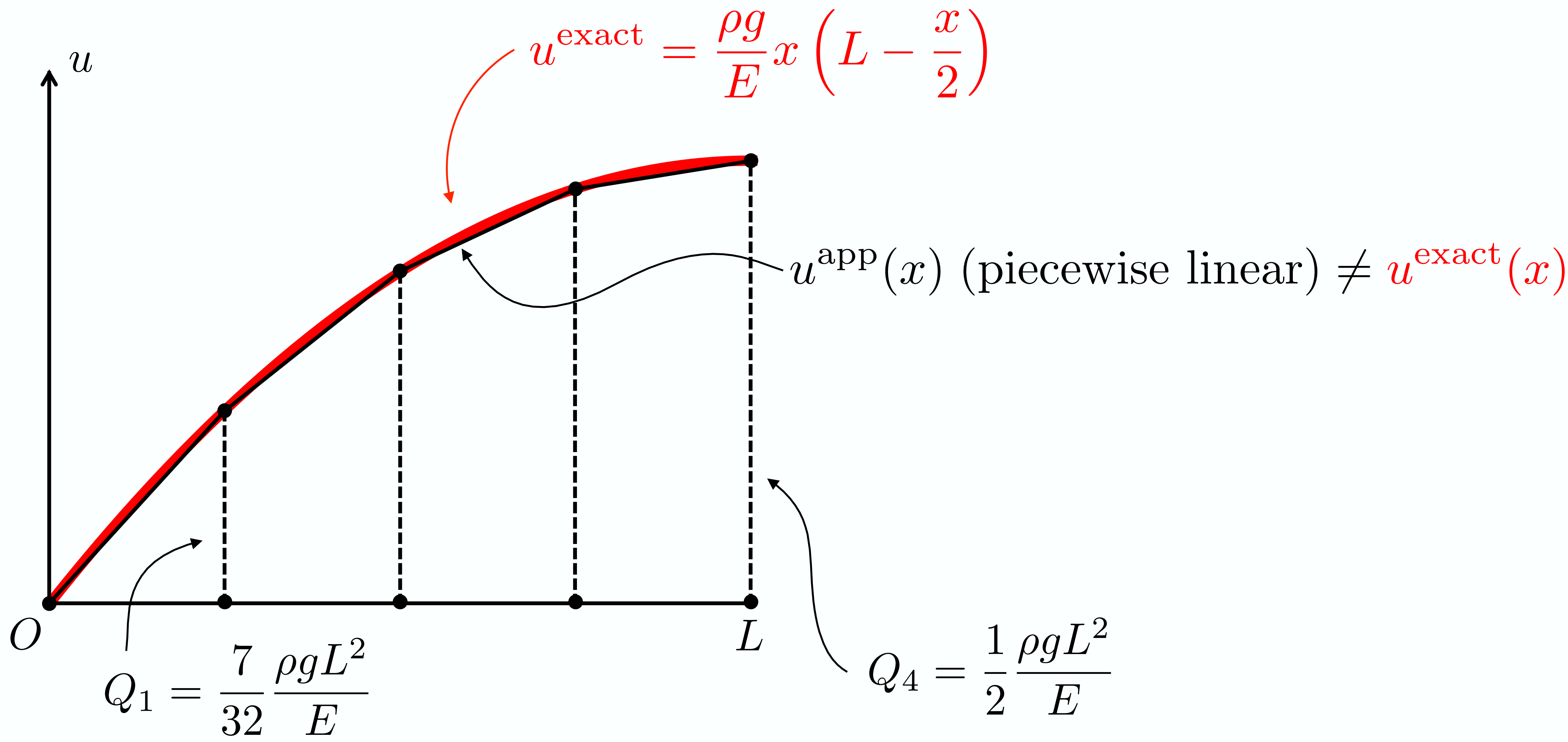
$$f_1^e = f_2^e = \rho g A \frac{l_e}{2} = \frac{\rho g A L}{8} = f \,.$$

Uniform section bar has 4 equal length elements ($l_e = L/4$)

$$\frac{4EA}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \frac{\rho g A L}{8}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$Q_0 = 0$, solve 4×4 system

ONE DIMENSIONAL EXAMPLE – FEM SOLUTION



$$\mathbf{Q}^{\mathsf{T}} = \frac{\rho g L^2}{32E} \begin{bmatrix} 7 & 12 & 15 & 16 \end{bmatrix}$$

Nodal values Q_i are correct (just here)