Computation of the interlayer boundary conditions for the construction of TransferMatrix in `MultiLayerIndentation` Package

Authors: Andrei Constantinescu, Alexander Korsunsky, Olivier Pison, Abdelbacet Ouslati

Source for the `MultiLayerIndentation` Package

A. Constantinescu and A.M. Korsunsky - Elasticity with Mathematica (r)
Cambridge University Press, 2007

A.M. Korsunsky, A. Constantinescu - The influence of indenter bluntness on the apparent contact stiffness of thin coatings,
Thin Solid Films 517 (2009) 4835 4844

A. Constantinescu, A.M. Korsunsky, O. Pison, A. Oueslati - Symbolic and numerical solution of the axisymmetric indentation problem of a multilayered elastic solid
submitted to Int.J. Solids and structures, 2013

H.Y. Yu, S.C. Sanday, B.B. Rath, The effect of substrate on the elastic properties of films determined by the indentation test — axisymmetric Boussinesq problem,
J. Mech. Phys. Solids 38 (6) (1990) 745.

N.N. Lebedev, I.S. Uflyand, Prikladnaya Matematica Mehanica 22 (1958) 320

Load the Tensor2Analysis.m Package

The Tensor2Analysis.m is an ressource file for the book *A. Constantinescu and A.M. Korsunsky - Elasticity with Mathematica, Cambridge University Press, 2007* and can be downloaded from the webpages of the publisher:

http://www.cambridge.org/aus/catalogue/catalogue.asp?isbn=9780521842013&ss=res

or the webpage of authors.

```
SetDirectory["/mydata/constant/Elastica/ElasticaBook/Packages"]
<< Tensor2Analysis.m
```

```
/mydata/constant/Elastica/ElasticaBook/Packages
```

Define coordinate systems

```
Clear["Global`*"]
```

```
SetCoordinates[Cylindrical[r, t, z]]
CoordinatesToCartesian[{r, t, z}]
```

```
Cylindrical[r, t, z]
```

```
{r Cos[t], r Sin[t], z}
```

Displacement and stresses in the layers from Papkovich-Neuber Potentials

```
ψ[i_] = { A1[i] ,  A2[i]} . {Cosh[ λ ( z - h[i - 1] )] , Sinh[ λ ( z - h[i - 1] )] }
  BesselJ[0, λ r ] /  Sinh[ λ ( h[i] - h[i - 1] )]
ϕ[i_] = { A4[i] ,  A3[i]} . {Cosh[ λ ( z - h[i - 1] )] , Sinh[ λ ( z - h[i - 1] )] }
  BesselJ[0, λ r ] / λ /  Sinh[ λ ( h[i] - h[i - 1] )]
```

```
BesselJ[0, r λ] Csch[λ (-h[-1 + i] + h[i])]
 (A1[i] Cosh[λ (z - h[-1 + i])] + A2[i] Sinh[λ (z - h[-1 + i])])
```

$$\frac{1}{\lambda}\text{BesselJ}[0,\ r\,\lambda]\ \text{Csch}[\lambda\,(-h[-1+i]+h[i])]\ (A4[i]\ \text{Cosh}[\lambda\,(z-h[-1+i])]+A3[i]\ \text{Sinh}[\lambda\,(z-h[-1+i])])$$

```
FullSimplify[ Laplacian[ ψ[i] ] ]
FullSimplify[ Laplacian[ ϕ[i] ] ]
```

```
0

0
```

```
psi[i_] = {0, 0, 4 (-1 + ν[i]) ψ[i]}
phi[i_] = 4 (-1 + ν[i]) ϕ[i]
pos = {r, 0, z}
```

```
{0, 0, 4 BesselJ[0, r λ] Csch[λ (-h[-1 + i] + h[i])]
   (A1[i] Cosh[λ (z - h[-1 + i])] + A2[i] Sinh[λ (z - h[-1 + i])]) (-1 + ν[i])}
```

$$\frac{1}{\lambda}4\ \text{BesselJ}[0,\ r\,\lambda]\ \text{Csch}[\lambda\,(-h[-1+i]+h[i])]\ (A4[i]\ \text{Cosh}[\lambda\,(z-h[-1+i])]+A3[i]\ \text{Sinh}[\lambda\,(z-h[-1+i])])\ (-1+\nu[i])$$

```
{r, 0, z}
```

```
u[i_] = 1 / (2 μ[i]) FullSimplify[psi[i] - 1 / (4 (1 - ν[i])) Grad[Dot[pos, psi[i]] + phi[i]]]
```

$$\Big\{-\frac{1}{2\,\mu[i]}\text{BesselJ}[1,\ r\,\lambda]\ \text{Csch}[\lambda\,(-h[-1+i]+h[i])]\ ((z\,\lambda\,A1[i]+A4[i])\ \text{Cosh}[\lambda\,(z-h[-1+i])]+(z\,\lambda\,A2[i]+A3[i])\ \text{Sinh}[\lambda\,(z-h[-1+i])]),$$
$$0,\ \frac{1}{2\,\mu[i]}\text{BesselJ}[0,\ r\,\lambda]\ \text{Csch}[\lambda\,(-h[-1+i]+h[i])]\ (\text{Cosh}[\lambda\,(z-h[-1+i])]\ (z\,\lambda\,A2[i]+A3[i]+A1[i]\ (-3+4\,\nu[i]))+\text{Sinh}[\lambda\,(z-h[-1+i])]\ (z\,\lambda\,A1[i]+A4[i]+A2[i]\ (-3+4\,\nu[i])))\Big\}$$

```
Lphi = Laplacian[ϕ[i]] == 0;
Lpsi = Laplacian[ψ[i]] == 0;
DrLphi = D[Laplacian[ϕ[i]], r] == 0;
DzLphi = D[Laplacian[ϕ[i]], z] == 0;
DrLpsi = D[Laplacian[ψ[i]], r] == 0;
DzLpsi = D[Laplacian[ψ[i]], z] == 0;
```

```
eps[i_] =
  FullSimplify[1 / 2 ( Grad[u[i]] + Transpose[Grad[u[i]]] ) , Assumptions → {Lphi, Lpsi} ];
```

```
sig[i_] = 2 μ[i] (ν[i] / (1 - 2 ν[i]) Tr[eps[i]] IdentityMatrix[3] + eps[i]);
```

```
FullSimplify[ Div[ sig[i] ] , Assumptions → {Lphi, DrLphi , DzLphi , Lpsi, DrLpsi, DzLpsi}]
```

```
{0, 0, 0}
```

Boundary conditions between layers: bonded

```
lh = FullSimplify[{ (u[i][[1]] - u[i + 1][[1]] ) / BesselJ[1, r λ] ,
    (u[i][[3]] - u[i + 1][[3]] ) / BesselJ[0, r λ]  ,
    ( sig[i][[3, 3]] - sig[i + 1][[3, 3]] ) / (λ BesselJ[0, r λ] ) ,
    ( sig[i][[1, 3]] - sig[i + 1][[1, 3]] ) / (λ BesselJ[1, r λ] ) } /. z → h[i]]
```

$$\Big\{\frac{1}{2\,\mu[i]\,\mu[1+i]}\,(\mathrm{Csch}[\lambda\,(-h[i]+h[1+i])]\,(A4[1+i]+\lambda\,A1[1+i]\,h[i])\,\mu[i]-(A3[i]+\lambda\,A2[i]\,h[i]+\mathrm{Coth}[\lambda\,(-h[-1+i]+h[i])]\,(A4[i]+\lambda\,A1[i]\,h[i]))\,\mu[1+i]),$$

$$\frac{1}{2}\left(\frac{1}{\mu[i]}\,(A4[i]+A3[i]\,\mathrm{Coth}[\lambda\,(-h[-1+i]+h[i])]+A1[i]\,(\lambda\,h[i]+\mathrm{Coth}[\lambda\,(h[-1+i]-h[i])]\,(3-4\,\nu[i]))+A2[i]\,(-3+\lambda\,\mathrm{Coth}[\lambda\,(-h[-1+i]+h[i])]\,h[i]+4\,\nu[i]))+\frac{\mathrm{Csch}[\lambda\,(h[i]-h[1+i])]\,(A3[1+i]+\lambda\,A2[1+i]\,h[i]+A1[1+i]\,(-3+4\,\nu[1+i]))}{\mu[1+i]}\right),$$

$$A3[i]+A4[i]\,\mathrm{Coth}[\lambda\,(-h[-1+i]+h[i])]+A2[i]\,(\lambda\,h[i]-2\,\mathrm{Coth}[\lambda\,(h[-1+i]-h[i])]\,(-1+\nu[i]))+A1[i]\,(-2+\lambda\,\mathrm{Coth}[\lambda\,(-h[-1+i]+h[i])]\,h[i]+2\,\nu[i])+\mathrm{Csch}[\lambda\,(h[i]-h[1+i])]\,(A4[1+i]+\lambda\,A1[1+i]\,h[i]+2\,A2[1+i]\,(-1+\nu[1+i])),$$

$$-A4[i]+A3[i]\,\mathrm{Coth}[\lambda\,(h[-1+i]-h[i])]+A1[i]\,(-\lambda\,h[i]+\mathrm{Coth}[\lambda\,(-h[-1+i]+h[i])]\,(1-2\,\nu[i]))+A2[i]\,(1+\lambda\,\mathrm{Coth}[\lambda\,(h[-1+i]-h[i])]\,h[i]-2\,\nu[i])-\mathrm{Csch}[\lambda\,(h[i]-h[1+i])]\,(A3[1+i]+\lambda\,A2[1+i]\,h[i]+A1[1+i]\,(-1+2\,\nu[1+i]))\Big\}$$

```
FullSimplify[Solve[Thread[lh == {0, 0, 0, 0}], {A1[i], A2[i], A3[i], A4[i]}]]
```

```
{{A1[i] → 1/(4 μ[1 + i] (-1 + ν[i])) Csch[λ (h[i] - h[1 + i])]
    ((A3[1 + i] + λ A2[1 + i] h[i]) Sinh[2 λ (h[-1 + i] - h[i])] (μ[i] - μ[1 + i]) +
     2 A4[1 + i] Sinh[λ (h[-1 + i] - h[i])]^2 (-μ[i] + μ[1 + i]) - 2 A2[1 + i] μ[1 + i] (-1 + ν[1 + i]) +
     2 A2[1 + i] Cosh[2 λ (h[-1 + i] - h[i])] μ[1 + i] (-1 + ν[1 + i]) +
     A1[1 + i] (2 λ h[i] Sinh[λ (h[-1 + i] - h[i])]^2 (-μ[i] + μ[1 + i]) +
        Sinh[2 λ (h[-1 + i] - h[i])] (μ[1 + i] (1 - 2 ν[1 + i]) + μ[i] (-3 + 4 ν[1 + i])))),
  A2[i] → 1/(4 μ[1 + i] (-1 + ν[i])) Csch[λ (h[i] - h[1 + i])] (-(A3[1 + i] + λ A2[1 + i] h[i])
      (μ[i] - μ[1 + i]) + Cosh[2 λ (h[-1 + i] - h[i])] (A3[1 + i] + λ A2[1 + i] h[i]) (μ[i] - μ[1 + i]) +
     Sinh[2 λ (h[-1 + i] - h[i])] (A4[1 + i] (-μ[i] + μ[1 + i]) + 2 A2[1 + i] μ[1 + i] (-1 + ν[1 + i])) +
     A1[1 + i] (3 μ[i] - μ[1 + i] + λ h[i] Sinh[2 λ (h[-1 + i] - h[i])] (-μ[i] + μ[1 + i]) -
        4 μ[i] ν[1 + i] + 2 μ[1 + i] ν[1 + i] +
        Cosh[2 λ (h[-1 + i] - h[i])] (μ[1 + i] (1 - 2 ν[1 + i]) + μ[i] (-3 + 4 ν[1 + i])))),
  A3[i] → 1/(4 μ[1 + i] (-1 + ν[i])) Csch[λ (h[i] - h[1 + i])]
    (A4[1 + i] (λ h[i] Sinh[2 λ (-h[-1 + i] + h[i])] (-μ[i] + μ[1 + i]) -
        2 μ[i] (-1 + ν[i]) + 2 Cosh[2 λ (-h[-1 + i] + h[i])] μ[i] (-1 + ν[i])) +
     λ Cosh[2 λ (-h[-1 + i] + h[i])] h[i] (-(A3[1 + i] + λ A2[1 + i] h[i]) (μ[i] - μ[1 + i]) +
        A1[1 + i] (μ[i] (1 + 2 ν[i] - 4 ν[1 + i]) + μ[1 + i] (-1 + 2 ν[1 + i]))) +
     λ h[i] ((A3[1 + i] + λ A2[1 + i] h[i]) (μ[i] - μ[1 + i]) +
        A1[1 + i] (μ[1 + i] (1 - 2 ν[1 + i]) + μ[i] (-1 - 2 ν[i] + 4 ν[1 + i]))) +
     Sinh[2 λ (-h[-1 + i] + h[i])] (A3[1 + i] (μ[1 + i] (3 - 4 ν[i]) + μ[i] (-1 + 2 ν[i])) +
        λ A2[1 + i] h[i] (μ[i] (-1 + 2 ν[i]) + μ[1 + i] (1 - 4 ν[i] + 2 ν[1 + i])) +
        A1[1 + i] (μ[1 + i] (λ^2 h[i]^2 - (-3 + 4 ν[i]) (-1 + 2 ν[1 + i])) -
           μ[i] (λ^2 h[i]^2 - (-1 + 2 ν[i]) (-3 + 4 ν[1 + i]))))),
  A4[i] → 1/(4 μ[1 + i] (-1 + ν[i])) Csch[λ (h[i] - h[1 + i])]
    (2 A4[1 + i] (λ h[i] Sinh[λ (h[-1 + i] - h[i])]^2 (μ[i] - μ[1 + i]) +
        Sinh[2 λ (h[-1 + i] - h[i])] μ[i] (-1 + ν[i])) + A3[1 + i] (-μ[i] + 3 μ[1 + i] +
        λ h[i] Sinh[2 λ (h[-1 + i] - h[i])] (-μ[i] + μ[1 + i]) + 2 μ[i] ν[i] - 4 μ[1 + i] ν[i] +
        Cosh[2 λ (h[-1 + i] - h[i])] (μ[i] (1 - 2 ν[i]) + μ[1 + i] (-3 + 4 ν[i]))) + λ A2[1 + i] h[i]
      (-μ[i] (λ h[i] Sinh[2 λ (h[-1 + i] - h[i])] + 2 Sinh[λ (h[-1 + i] - h[i])]^2 (-1 + 2 ν[i])) +
        μ[1 + i] (λ h[i] Sinh[2 λ (h[-1 + i] - h[i])] +
           2 Sinh[λ (h[-1 + i] - h[i])]^2 (-1 + 4 ν[i] - 2 ν[1 + i]))) +
     A1[1 + i] (μ[1 + i] (-2 λ^2 h[i]^2 Sinh[λ (h[-1 + i] - h[i])]^2 + λ h[i] Sinh[2 λ (h[-1 + i] - h[i])]
           (-1 + 2 ν[1 + i]) + 2 Sinh[λ (h[-1 + i] - h[i])]^2 (-3 + 4 ν[i]) (-1 + 2 ν[1 + i])) + μ[i]
        (2 λ^2 h[i]^2 Sinh[λ (h[-1 + i] - h[i])]^2 + λ h[i] Sinh[2 λ (h[-1 + i] - h[i])] (1 + 2 ν[i] -
              4 ν[1 + i]) - 2 Sinh[λ (h[-1 + i] - h[i])]^2 (-1 + 2 ν[i]) (-3 + 4 ν[1 + i]))))}}
```

Boundary conditions between layers: sliding

```
lhs = FullSimplify[{ ( u[i][[3]] - u[i + 1][[3]] ) / BesselJ[0, r λ] ,
     ( sig[i][[3, 3]] - sig[i + 1][[3, 3]] ) / (λ BesselJ[0, r λ] ) ,
    sig[i][[1, 3]] / (λ BesselJ[1, r λ] ),
    sig[i + 1][[1, 3]] / (λ BesselJ[1, r λ] ) } /. z → h[i]]
```

```
{1/2 (1/μ[i] (A4[i] + A3[i] Coth[λ (-h[-1 + i] + h[i])] +
      A1[i] (λ h[i] + Coth[λ (h[-1 + i] - h[i])] (3 - 4 ν[i])) +
      A2[i] (-3 + λ Coth[λ (-h[-1 + i] + h[i])] h[i] + 4 ν[i])) +
    (Csch[λ (h[i] - h[1 + i])] (A3[1 + i] + λ A2[1 + i] h[i] + A1[1 + i] (-3 + 4 ν[1 + i])))/μ[1 + i]),
 A3[i] + A4[i] Coth[λ (-h[-1 + i] + h[i])] +
  A2[i] (λ h[i] - 2 Coth[λ (h[-1 + i] - h[i])] (-1 + ν[i])) +
  A1[i] (-2 + λ Coth[λ (-h[-1 + i] + h[i])] h[i] + 2 ν[i]) +
  Csch[λ (h[i] - h[1 + i])] (A4[1 + i] + λ A1[1 + i] h[i] + 2 A2[1 + i] (-1 + ν[1 + i])),
 Csch[λ (-h[-1 + i] + h[i])]
  (-Cosh[λ (-h[-1 + i] + h[i])] (A3[i] + λ A2[i] h[i] + A1[i] (-1 + 2 ν[i])) +
    Sinh[λ (h[-1 + i] - h[i])] (A4[i] + λ A1[i] h[i] + A2[i] (-1 + 2 ν[i]))),
 -Csch[λ (-h[i] + h[1 + i])] (A3[1 + i] + λ A2[1 + i] h[i] + A1[1 + i] (-1 + 2 ν[1 + i]))}
```

```
FullSimplify[ Solve[ Thread[ lhs == {0, 0, 0, 0} ], {A1[i], A2[i], A3[i], A4[i]} ] ]
```

```
Solve::svars : Equations may not give solutions for all "solve" variables. >>
```

```
$Aborted
```

```
Normal[CoefficientArrays[ lhs , {A1[i], A2[i], A3[i], A4[i]} ][[1]] ] // MatrixForm
```

$$\begin{pmatrix} \frac{\text{Csch}[\lambda (h[i]-h[1+i])] (A3[1+i]+\lambda A2[1+i] h[i]+A1[1+i] (-3+4 \nu[1+i]))}{2 \mu[1+i]} \\ \text{Csch}[\lambda (h[i]-h[1+i])] (A4[1+i]+\lambda A1[1+i] h[i]+2 A2[1+i] (-1+\nu[1+i])) \\ 0 \\ -\text{Csch}[\lambda (-h[i]+h[1+i])] (A3[1+i]+\lambda A2[1+i] h[i]+A1[1+i] (-1+2 \nu[1+i])) \end{pmatrix}$$

```
Normal[CoefficientArrays[ lhs , {A1[i], A2[i], A3[i], A4[i]} ][[2]] ] // MatrixForm
```

$$\begin{pmatrix} \frac{\lambda h[i]+\text{Coth}[\lambda (h[-1+i]-h[i])] (3-4 \nu[i])}{2 \mu[i]} \\ -2+\lambda \text{Coth}[\lambda (-h[-1+i]+h[i])] h[i]+2 \nu[i] \\ \lambda \text{Csch}[\lambda (-h[-1+i]+h[i])] h[i] \text{Sinh}[\lambda (h[-1+i]-h[i])]-\text{Coth}[\lambda (-h[-1+i]+h[i])] (-1+2 \nu[i]) \\ 0 \end{pmatrix}$$

Displacement and stresses in the halfspace

```
ψf = A5 Exp[ - λ ( z - hf )] BesselJ[0, λ r ]
ϕf = A6 Exp[ - λ ( z - hf )] BesselJ[0, λ r ] / λ
```

$$A5\, e^{-(-hf+z)\lambda} \text{BesselJ}[0, r\lambda]$$

$$\frac{A6\, e^{-(-hf+z)\lambda} \text{BesselJ}[0, r\lambda]}{\lambda}$$

```
FullSimplify[ Laplacian[ ψf ] ]
FullSimplify[ Laplacian[ ϕf ] ]
```

0

0

```
psif = {0, 0, 4 (-1 + νf) ψf}
phif = 4 (-1 + νf) ϕf
pos = {r, 0, z}
```

$$\left\{0, 0, 4\,\mathrm{A5}\, e^{-(-\mathrm{hf}+z)\lambda} (-1 + \nu\mathrm{f})\, \mathrm{BesselJ}[0, r\lambda]\right\}$$

$$\frac{4\,\mathrm{A6}\, e^{-(-\mathrm{hf}+z)\lambda} (-1 + \nu\mathrm{f})\, \mathrm{BesselJ}[0, r\lambda]}{\lambda}$$

{r, 0, z}

```
uf = 1 / (2 μf) FullSimplify[psif - 1 / (4 (1 - νf)) Grad[Dot[pos, psif] + phif]]
```

$$\left\{-\frac{e^{(\mathrm{hf}-z)\lambda} (\mathrm{A6} + \mathrm{A5}\, z\, \lambda)\, \mathrm{BesselJ}[1, r\lambda]}{2\,\mu\mathrm{f}}, 0, -\frac{e^{(\mathrm{hf}-z)\lambda} (\mathrm{A6} + \mathrm{A5}\, (3 + z\,\lambda - 4\,\nu\mathrm{f}))\, \mathrm{BesselJ}[0, r\lambda]}{2\,\mu\mathrm{f}}\right\}$$

```
Lphi = Laplacian[ϕf] == 0;
Lpsi = Laplacian[ψf] == 0;
DrLphi = D[Laplacian[ϕf], r] == 0;
DzLphi = D[Laplacian[ϕf], z] == 0;
DrLpsi = D[Laplacian[ψf], r] == 0;
DzLpsi = D[Laplacian[ψf], z] == 0;
```

```
epsf = FullSimplify[1 / 2 ( Grad[uf] + Transpose[Grad[uf]] ) , Assumptions → {Lphi, Lpsi} ];
```

```
sigf = 2 μf (νf / (1 - 2 νf) Tr[epsf] IdentityMatrix[3] + epsf);
```

```
FullSimplify[ Div[ sigf ] , Assumptions → {Lphi, DrLphi , DzLphi , Lpsi, DrLpsi, DzLpsi}]
```

{0, 0, 0}

Boundary conditions last layer to half space: bonded

```
lhf = FullSimplify[ {( u[i][[1]] - uf[[1]]  )  / BesselJ[1, r λ]  ,
      ( u[i][[3]] - uf[[3]]  )  / BesselJ[0, r λ]  ,
      ( sig[i][[3, 3]] - sigf[[3, 3]]  ) / (λ BesselJ[0, r λ] ) ,
     ( sig[i][[1, 3]] - sigf[[1, 3]]  ) / (λ BesselJ[1, r λ] )  } /. z → hf /. h[i] → hf ]
```

$$\left\{\frac{-\mu\mathrm{f}\, (\mathrm{hf}\,\lambda\, \mathrm{A2}[i] + \mathrm{A3}[i] + (\mathrm{hf}\,\lambda\, \mathrm{A1}[i] + \mathrm{A4}[i])\, \mathrm{Coth}[\lambda\, (\mathrm{hf} - h[-1 + i])]) + (\mathrm{A6} + \mathrm{A5}\, \mathrm{hf}\,\lambda)\, \mu[i]}{2\,\mu\mathrm{f}\, \mu[i]},\right.$$

$$\frac{1}{2\,\mu\mathrm{f}\,\mu[i]} (\mu\mathrm{f}\, \mathrm{A4}[i] + \mu\mathrm{f}\, \mathrm{A3}[i]\, \mathrm{Coth}[\lambda\, (\mathrm{hf} - h[-1 + i])] +$$
$$(\mathrm{A6} + \mathrm{A5}\, (3 + \mathrm{hf}\,\lambda - 4\,\nu\mathrm{f}))\, \mu[i] + \mu\mathrm{f}\, \mathrm{A2}[i]\, (-3 + \mathrm{hf}\,\lambda\, \mathrm{Coth}[\lambda\, (\mathrm{hf} - h[-1 + i])] + 4\,\nu[i]) +$$
$$\mu\mathrm{f}\, \mathrm{A1}[i]\, (\mathrm{hf}\,\lambda - 3\, \mathrm{Coth}[\lambda\, (\mathrm{hf} - h[-1 + i])] + 4\, \mathrm{Coth}[\lambda\, (\mathrm{hf} - h[-1 + i])]\, \nu[i])),$$
$$-2\,\mathrm{A5} - \mathrm{A6} - \mathrm{A5}\, \mathrm{hf}\,\lambda + 2\,\mathrm{A5}\,\nu\mathrm{f} + \mathrm{hf}\,\lambda\, \mathrm{A2}[i] + \mathrm{A3}[i] + \mathrm{Coth}[\lambda\, (\mathrm{hf} - h[-1 + i])]$$
$$(\mathrm{A4}[i] + 2\,\mathrm{A2}[i]\, (-1 + \nu[i])) + \mathrm{A1}[i]\, (-2 + \mathrm{hf}\,\lambda\, \mathrm{Coth}[\lambda\, (\mathrm{hf} - h[-1 + i])] + 2\,\nu[i]),$$
$$-\mathrm{A5} - \mathrm{A6} - \mathrm{A5}\, \mathrm{hf}\,\lambda + 2\,\mathrm{A5}\,\nu\mathrm{f} + \mathrm{A2}[i] - \mathrm{A4}[i] - (\mathrm{hf}\,\lambda\, \mathrm{A2}[i] + \mathrm{A3}[i])\, \mathrm{Coth}[\lambda\, (\mathrm{hf} - h[-1 + i])] -$$
$$\left. 2\,\mathrm{A2}[i]\,\nu[i] + \mathrm{A1}[i]\, (-\mathrm{hf}\,\lambda + \mathrm{Coth}[\lambda\, (\mathrm{hf} - h[-1 + i])] - 2\, \mathrm{Coth}[\lambda\, (\mathrm{hf} - h[-1 + i])]\, \nu[i])\right\}$$

```
FullSimplify[ Solve[ Thread[ lhf == {0, 0, 0, 0} ], {A1[i], A2[i], A3[i], A4[i]} ] ]
```

$$\Big\{\Big\{A1[i] \to \frac{1}{2\,\mu f\,(-1+\nu[i])}\,\mathrm{Sinh}[\lambda\,(hf-h[-1+i])]^2\\
(-\mu f\,(A6+A5\,(2+hf\,\lambda-2\,\nu f))+\mu f\,(A5+A6+A5\,hf\,\lambda-2\,A5\,\nu f)\,\mathrm{Coth}[\lambda\,(hf-h[-1+i])]+\\
(A6+A5\,hf\,\lambda-(A6+A5\,(3+hf\,\lambda-4\,\nu f))\,\mathrm{Coth}[\lambda\,(hf-h[-1+i])])\,\mu[i]),$$

$$A2[i] \to \frac{1}{2\,\mu f\,(-1+\nu[i])}\,\mathrm{Sinh}[\lambda\,(hf-h[-1+i])]^2\\
(-\mu f\,(A5+A6+A5\,hf\,\lambda-2\,A5\,\nu f)+\mu f\,(A6+A5\,(2+hf\,\lambda-2\,\nu f))\,\mathrm{Coth}[\lambda\,(hf-h[-1+i])]+\\
(A6+A5\,(3+hf\,\lambda-4\,\nu f)-(A6+A5\,hf\,\lambda)\,\mathrm{Coth}[\lambda\,(hf-h[-1+i])])\,\mu[i]),$$

$$A3[i] \to \frac{1}{2\,\mu f\,(-1+\nu[i])}\,\mathrm{Sinh}[\lambda\,(hf-h[-1+i])]^2\\
(\mu f\,(hf\,\lambda\,(A5+A6+A5\,hf\,\lambda-2\,A5\,\nu f)+(A6\,(3-hf\,\lambda)+A5\,(3-6\,\nu f+hf\,\lambda\,(1-hf\,\lambda+2\,\nu f)))\\
\mathrm{Coth}[\lambda\,(hf-h[-1+i])]-4\,(A5+A6+A5\,hf\,\lambda-2\,A5\,\nu f)\,\mathrm{Coth}[\lambda\,(hf-h[-1+i])]\,\nu[i])+\\
\mu[i]\,(A6\,(2-hf\,\lambda)-A5\,hf\,\lambda\,(1+hf\,\lambda-4\,\nu f)+(A6\,(-1+hf\,\lambda)+A5\,(-3+hf\,\lambda\,(-1+hf\,\lambda)+4\,\nu f))\\
\mathrm{Coth}[\lambda\,(hf-h[-1+i])]+\\
2\,(-A6-A5\,hf\,\lambda+(A6+A5\,(3+hf\,\lambda-4\,\nu f))\,\mathrm{Coth}[\lambda\,(hf-h[-1+i])])\,\nu[i])),$$

$$A4[i] \to \frac{1}{2\,\mu f\,(-1+\nu[i])}\,\mathrm{Sinh}[\lambda\,(hf-h[-1+i])]\,(-hf\,\lambda\,\mu f\,(A5+A6+A5\,hf\,\lambda-2\,A5\,\nu f)\\
\mathrm{Cosh}[\lambda\,(hf-h[-1+i])]+\mu f\,\mathrm{Sinh}[\lambda\,(hf-h[-1+i])]\\
(A6\,(-3+hf\,\lambda)+A5\,(-3+hf\,\lambda\,(-1+hf\,\lambda-2\,\nu f)+6\,\nu f)+4\,(A5+A6+A5\,hf\,\lambda-2\,A5\,\nu f)\,\nu[i])+\\
\mu[i]\,(\mathrm{Cosh}[\lambda\,(hf-h[-1+i])]\,(A6\,(-2+hf\,\lambda)+A5\,hf\,\lambda\,(1+hf\,\lambda-4\,\nu f)+2\,(A6+A5\,hf\,\lambda)\,\nu[i])+\\
\mathrm{Sinh}[\lambda\,(hf-h[-1+i])]\\
(A6-A6\,hf\,\lambda+A5\,(3+hf\,\lambda\,(1-hf\,\lambda)-4\,\nu f)-2\,(A6+A5\,(3+hf\,\lambda-4\,\nu f))\,\nu[i])))\Big\}\Big\}$$

■ Boundary conditions last layer to half space: sliding

```
lhs = FullSimplify[{ ( u[i][[3]] - uf[[3]] ) / BesselJ[0, r λ]  ,
    ( sig[i][[3, 3]] - sigf[[3, 3]] ) / (λ BesselJ[0, r λ]) ,
   sig[i][[1, 3]] / (λ BesselJ[1, r λ] ),
   sigf[[1, 3]] / (λ BesselJ[1, r λ] ) } /. z → h[i]]
```

$$\Big\{\frac{1}{2\,\mu f\,\mu[i]}\\
\Big(e^{\lambda\,(hf-h[i])}\,(A6+A5\,(3-4\,\nu f)+A5\,\lambda\,h[i])\,\mu[i]+\mu f\,(A4[i]+A3[i]\,\mathrm{Coth}[\lambda\,(-h[-1+i]+h[i])]+\\
A1[i]\,(\lambda\,h[i]+\mathrm{Coth}[\lambda\,(h[-1+i]-h[i])]\,(3-4\,\nu[i]))+\\
A2[i]\,(-3+\lambda\,\mathrm{Coth}[\lambda\,(-h[-1+i]+h[i])]\,h[i]+4\,\nu[i]))\Big),\\
A3[i]+A4[i]\,\mathrm{Coth}[\lambda\,(-h[-1+i]+h[i])]-e^{\lambda\,(hf-h[i])}\,(A6-2\,A5\,(-1+\nu f)+A5\,\lambda\,h[i])+\\
A2[i]\,(\lambda\,h[i]-2\,\mathrm{Coth}[\lambda\,(h[-1+i]-h[i])]\,(-1+\nu[i]))+\\
A1[i]\,(-2+\lambda\,\mathrm{Coth}[\lambda\,(-h[-1+i]+h[i])]\,h[i]+2\,\nu[i]),\\
\mathrm{Csch}[\lambda\,(-h[-1+i]+h[i])]\\
(-\mathrm{Cosh}[\lambda\,(-h[-1+i]+h[i])]\,(A3[i]+\lambda\,A2[i]\,h[i]+A1[i]\,(-1+2\,\nu[i]))+\\
\mathrm{Sinh}[\lambda\,(h[-1+i]-h[i])]\,(A4[i]+\lambda\,A1[i]\,h[i]+A2[i]\,(-1+2\,\nu[i]))),\\
e^{\lambda\,(hf-h[i])}\,(A5+A6-2\,A5\,\nu f+A5\,\lambda\,h[i])\Big\}$$

```
Normal[CoefficientArrays[ lhs , {A1[i], A2[i], A3[i], A4[i]} ][[2]] ] // MatrixForm
```

$$\begin{pmatrix}
\frac{\lambda\,h[i]+\mathrm{Coth}[\lambda\,(h[-1+i]-h[i])]\,(3-4\,\nu[i])}{2\,\mu[i]}\\
-2+\lambda\,\mathrm{Coth}[\lambda\,(-h[-1+i]+h[i])]\,h[i]+2\,\nu[i]\\
\lambda\,\mathrm{Csch}[\lambda\,(-h[-1+i]+h[i])]\,h[i]\,\mathrm{Sinh}[\lambda\,(h[-1+i]-h[i])]-\mathrm{Coth}[\lambda\,(-h[-1+i]+h[i])]\,(-1+2\,\nu[i])\\
0
\end{pmatrix}$$

```
a6 = FullSimplify[ Solve[ Thread[ lhs[[4]] == 0], {A6} ][[1]] ]
```

$$\{A6 \to A5\,(-1 + 2\,\nu f - \lambda\, h[i])\}$$

```
lhss = FullSimplify[lhs /. a6]
```

$$\Big\{\frac{1}{2\,\mu f\,\mu[i]}\Big(-2\,A5\,e^{\lambda\,(hf - h[i])}\,(-1 + \nu f)\,\mu[i] + \mu f\,(A4[i] + A3[i]\,\mathrm{Coth}[\lambda\,(-h[-1 + i] + h[i])] + A1[i]\,(\lambda\, h[i] + \mathrm{Coth}[\lambda\,(h[-1 + i] - h[i])]\,(3 - 4\,\nu[i])) + A2[i]\,(-3 + \lambda\,\mathrm{Coth}[\lambda\,(-h[-1 + i] + h[i])]\,h[i] + 4\,\nu[i]))\Big),$$
$$-A5\,e^{\lambda\,(hf - h[i])} + A3[i] + A4[i]\,\mathrm{Coth}[\lambda\,(-h[-1 + i] + h[i])] + A2[i]\,(\lambda\, h[i] - 2\,\mathrm{Coth}[\lambda\,(h[-1 + i] - h[i])]\,(-1 + \nu[i])) + A1[i]\,(-2 + \lambda\,\mathrm{Coth}[\lambda\,(-h[-1 + i] + h[i])]\,h[i] + 2\,\nu[i]),$$
$$\mathrm{Csch}[\lambda\,(-h[-1 + i] + h[i])]\,(-\mathrm{Cosh}[\lambda\,(-h[-1 + i] + h[i])]\,(A3[i] + \lambda\, A2[i]\,h[i] + A1[i]\,(-1 + 2\,\nu[i])) + \mathrm{Sinh}[\lambda\,(h[-1 + i] - h[i])]\,(A4[i] + \lambda\, A1[i]\,h[i] + A2[i]\,(-1 + 2\,\nu[i]))),\ 0\Big\}$$

```
a4 = Solve[ lhss[[3]] == 0, A4[i] ][[1]]
```

$$\{A4[i] \to \mathrm{Csch}[\lambda\,(h[-1 + i] - h[i])]\,\mathrm{Sinh}[\lambda\,(-h[-1 + i] + h[i])]\,(-A1[i]\,\mathrm{Coth}[\lambda\,(-h[-1 + i] + h[i])] + A3[i]\,\mathrm{Coth}[\lambda\,(-h[-1 + i] + h[i])] + \lambda\, A2[i]\,\mathrm{Coth}[\lambda\,(-h[-1 + i] + h[i])]\,h[i] + A2[i]\,\mathrm{Csch}[\lambda\,(-h[-1 + i] + h[i])]\,\mathrm{Sinh}[\lambda\,(h[-1 + i] - h[i])] - \lambda\, A1[i]\,\mathrm{Csch}[\lambda\,(-h[-1 + i] + h[i])]\,h[i]\,\mathrm{Sinh}[\lambda\,(h[-1 + i] - h[i])] + 2\,A1[i]\,\mathrm{Coth}[\lambda\,(-h[-1 + i] + h[i])]\,\nu[i] - 2\,A2[i]\,\mathrm{Csch}[\lambda\,(-h[-1 + i] + h[i])]\,\mathrm{Sinh}[\lambda\,(h[-1 + i] - h[i])]\,\nu[i])\}$$

```
lhsss = FullSimplify[ lhss /. a4 ]
```

$$\Big\{-\frac{A5\,e^{\lambda\,(hf - h[i])}\,(-1 + \nu f)}{\mu f} + \frac{(A2[i] - A1[i]\,\mathrm{Coth}[\lambda\,(h[-1 + i] - h[i])])\,(-1 + \nu[i])}{\mu[i]},$$
$$-A5\,e^{\lambda\,(hf - h[i])} - A1[i] + A2[i]\,\mathrm{Coth}[\lambda\,(h[-1 + i] - h[i])] - \mathrm{Csch}[\lambda\,(h[-1 + i] - h[i])]^2\,(A3[i] + \lambda\, A2[i]\,h[i] + A1[i]\,(-1 + 2\,\nu[i])),\ 0,\ 0\Big\}$$