

On the identification of elastic moduli from displacement-force boundary measurements

Andrei Constantinescu

To cite this article: Andrei Constantinescu (1995) On the identification of elastic moduli from displacement-force boundary measurements, *Inverse Problems in Engineering*, 1:4, 293-313, DOI: [10.1080/174159795088027587](https://doi.org/10.1080/174159795088027587)

To link to this article: <http://dx.doi.org/10.1080/174159795088027587>



Published online: 24 Oct 2007.



Submit your article to this journal [↗](#)



Article views: 23



View related articles [↗](#)



Citing articles: 17 View citing articles [↗](#)

ON THE IDENTIFICATION OF ELASTIC MODULI FROM DISPLACEMENT-FORCE BOUNDARY MEASUREMENTS

ANDREI CONSTANTINESCU

Laboratoire de Mécanique des Solides (CNRS (URA 317), Polytechnique, Mines, Ponts et Chaussées) Ecole Polytechnique, 91128 Palaiseau Cedex, France, e-mail: constant@athena.polytechnique.fr

(Received 4 March 1994; in final form 28 July 1994)

Communicated by G. S. Dulikravich

This paper addresses an identification problem for a linear elastic anisotropic body. We suppose that we can measure the displacement distribution on the boundary of an elastic body induced by a known applied static load. This represents a partial knowledge of the Dirichlet to Neumann data map. From such displacement-force boundary data pairs we reconstruct the interior distribution of the elastic moduli by minimizing an error-functional based on the constitutive equation. The decomposition of this error-functional using 'eigenelastic moduli' and 'eigentensors' will indicate the limitation of identification in the anisotropic case and will be the key point in the process of minimization. Numerical results for cubic material symmetry are finally used to validate the feasibility of the proposed method.

KEY WORDS: Elastic moduli, boundary measurements, error on constitutive law

1 INTRODUCTION

The mechanical behavior of structures is governed by three different types of equations. Two of them derive from general laws of nature, the equations of dynamics and of kinematics, and one characterizes the mechanical properties of the material considered, the constitutive law.

Provided the first two equations, one of the problems in solid mechanics is therefore the identification of the constitutive law. The identification is usually accomplished in two steps. First the nature of the constitutive law: elastic, elasto-plastic, etc. is chosen, and then the coefficients intervening in the equations are determined. The second step is generally based on the implicit hypothesis that the studied body is homogeneous. Consequently classical experiments and their interpretation fail for inhomogeneous bodies.

The goal of this article is to explore a method to solve the identification problem for a linear elastic anisotropic body with an inhomogeneous interior distribution of the elastic moduli. We suppose that the displacement distribution on the boundary induced by a known applied static load is measurable. From such displacement-force boundary data pairs the interior distribution of the elastic moduli is reconstructed. This

is an inverse problem in linear elasticity. A complete review of inverse problems in mechanics is extensively presented by Bui in [5].

The numerical reconstruction presented here minimizes over kinematically and statically admissible fields an error-functional, the error on constitutive law. This functional was previously used in elasticity [10, 11] for error-localization in structures. In order to change this property into an explicit identification property of the elastic moduli, the strain and stress tensor fields are decomposed on an well suited orthogonal basis. In the case of isotropy, presented in [2], the projections of tensor fields on this basis were the spherical and deviatorical parts of the respective tensor field. For anisotropic elasticity, the case presented here, spherical and deviatorical parts are transformed into eigentensors, their more general counterparts. The eigentensors and the eigenelastic moduli are 'eigenvectors' respectively 'eigenvalues' for the linear elastic equation. These notions are comprehensively presented in [13, 14].

The separation of the terms containing the eigenelastic moduli in the error on constitutive law, is a key point in our procedure because it permits the minimization of the error-functional with the eigenelastic moduli as explicit variables of the process, and will also define the limitations of this technique.

The paper begins with a presentation of the mathematical identification problem (section 2). A non-identifiability example introduces a conjecture about identifiability for anisotropic materials (section 3). Then (section 4), the error on constitutive law and its decomposition using eigenelastic moduli and eigentensors is discussed. First a short review of the case of isotropic elasticity is examined and afterwards general anisotropic elasticity is introduced. This will also determine the method of reconstruction. In the end (section 5), some numerical examples are presented using synthetic data and the alternating direction implicit (ADI) method for minimization. The examples address the identification of an inclusion in a matrix for an aluminum-copper composite. The material symmetry considered during the reconstruction procedure was cubic symmetry.

2 THE IDENTIFICATION PROBLEM

Let us consider an elastic body occupying in the reference configuration a regular domain Ω , with boundary $\partial\Omega$. Let \mathbf{u} , \mathbf{E} , \mathbf{T} stand for the vector field of displacements, the tensor field of strain and the tensor field of stress respectively.

In the present framework of small deformations and linear elasticity, assuming a stress-free initial state and no body forces, the governing equations on Ω are:

$$\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u}), \quad \mathbf{T} = \mathbf{CE}, \quad \text{div } \mathbf{T} = \mathbf{0}. \quad (1)$$

where \mathbf{C} denotes the fourth rank tensor of the elastic moduli. The elasticity tensor is supposed to be inhomogeneous, that is: $\mathbf{C} = \mathbf{C}(\mathbf{x})$, $\mathbf{x} \in \Omega$. In the sequel dependence of \mathbf{x} will not be shown explicitly unless needed for clarity.

Without choosing for the moment any particular material symmetry¹ for C , we expect that the classical symmetry restrictions hold:

$$C_{ijkl} = C_{jikl}, \quad C_{ijkl} = C_{ijlk}, \quad C_{ijkl} = C_{klij}. \quad (2)$$

These relations follow from the symmetry of the stress tensor, the symmetry of the strain tensor and the existence of an elastic potential, respectively [12]. The existence of the elastic potential also implies that C is positive definite.

With the above conditions, equations (1) form an elliptic system on Ω for the displacement \mathbf{u} :

$$\operatorname{div}(C\nabla\mathbf{u}) = 0 \quad (3)$$

Given the elasticity tensor C and either of the following boundary conditions:

- imposed displacements: $\mathbf{u}|_{\partial\Omega} = \boldsymbol{\xi}$, or
- imposed forces: $\mathbf{T}\mathbf{n}|_{\partial\Omega} = \boldsymbol{\varphi}$ (where \mathbf{n} is the exterior unitary normal on $\partial\Omega$),

one has the classical well-posed boundary value problem of linear elasticity; this is the *direct problem*.

If the elasticity tensor C is to be identified, an *additional unknown* is added to the problem *without bringing an additional equation* to it. One possibility to overcome this difficulty, is to supply more information by prescribing *simultaneously* displacements and forces over the whole boundary $\partial\Omega$.

This defines now an *inverse problem*, where C has to be identified from over-specified boundary conditions. The equations and boundary conditions are:

$$\operatorname{div}(C\nabla\mathbf{u}) = 0, \quad \mathbf{u}|_{\partial\Omega} = \boldsymbol{\xi}, \quad \text{and} \quad \mathbf{T}\mathbf{n}|_{\partial\Omega} = \boldsymbol{\varphi}. \quad (4)$$

3 EXISTENCE AND UNIQUENESS RESULTS

It is natural to begin the investigation of this inverse problem with the question of identifiability: if for two elasticity tensors C and L all boundary measurements are the same, must C equal L ?

The general answer for anisotropic elasticity is *no*, as will be proven by an example. However, if the investigation field is to be reduced to isotropic elasticity non-identifiability is eliminated and the answer is *yes* [8, 15]. These results, as well as the numerical method developed in the sequel, conduct to a conjecture for partial identifiability in the general case of anisotropy, which will be formulated in the second part of this section.

¹We remind that in this paper isotropy, cubic symmetry, ..., anisotropy are understood with the conventions of solid mechanics as local symmetry properties of $C(x)$ in $x \in \Omega$.

3.1 Example of non-identifiability in anisotropic elasticity

The example of non-identifiability is essentially based on the theorem of work and energy and the possibility of ‘hiding’ a change of variables in the elastic moduli. The construction stems from a similar example given by Kohn and Vogelius [7] for the non-identification of anisotropic electric conductivities.

Let us consider two elasticity tensors L and C , and a diffeomorphism $\Psi: \Omega \rightarrow \Omega$ mapping Ω into itself and equal to the identity on the boundary $\partial\Omega(\Psi(\mathbf{x}) = \mathbf{x}, \forall \mathbf{x} \in \partial\Omega)$, the three related by the following relation:

$$L_{ijkl}(\Psi(\mathbf{x})) = |\det \nabla \Psi|^{-1} \sum_{n,m} C_{inkm}(\mathbf{x}) \Psi_{n,j}(\mathbf{x}) \Psi_{m,i}(\mathbf{x}) \tag{5}$$

The preceding formula is similar to a change of variables, but it is obvious that the values and symmetries of the two elasticity tensors L and C are really shifted. The two bodies are therefore different, but it will be proven that their boundary measurements are always the same.

The construction of L , C and Ψ related by (5) is not trivial as in the electric case, because for given C and Ψ , L stemming from the relation (5) does not fulfill all the symmetry requirements of an elasticity tensor. The symmetries:

$$L_{ijkl} = L_{klij}, \quad L_{ijkl} = L_{ilkj}, \tag{6}$$

are respected, but:

$$L_{ijkl} = L_{jilk}, \quad L_{ijkl} = L_{ijlk} \tag{7}$$

are not generally verified.

However, one can show that for a given Ψ there exist at least one elasticity tensor C such that the resulting L respects all the symmetry requirements of an elasticity tensor. Indeed, using (5), the relations (7) transform for a given Ψ , in each point of the domain, into a linear system for the elements of C :

$$\sum_{n,m} C_{inkm} \Psi_{n,j} \Psi_{m,l} - \sum_{n,m} C_{jnkm} \Psi_{n,i} \Psi_{m,l} = 0 \tag{8}$$

This gives a homogeneous linear system with 15 equations and 21 unknowns (the moduli C_{ijkl}), which does always have a solution. Thus for a given Ψ and an elasticity tensor C solution of (7), (5) gives an elasticity tensor L .

The change of variable formula with $y = \Psi(\mathbf{x})$ becomes by the use of (5):

$$\int_{\Omega} (\nabla_x \mathbf{u}) C(\mathbf{x}) (\nabla_x \mathbf{v}) dx = \int_{\Psi(\Omega)} (\nabla_y (\mathbf{u} \circ \Psi^{-1})) L(\mathbf{y}) (\nabla_y (\mathbf{v} \circ \Psi^{-1})) dy \tag{9}$$

for all vector fields of displacements \mathbf{u} and \mathbf{v} . After integration by parts the last equation takes the following form:

Downloaded by [ETH Zurich] at 01:13 08 January 2016

$$\int_{\Omega} \mathbf{v} \operatorname{div}_x \mathbf{T}_C(\mathbf{u}) \, dx - \int_{\partial\Omega} v \mathbf{T}_C(\mathbf{u}) \mathbf{n} \, ds = \int_{\psi(\Omega)} \psi(\Omega) (\mathbf{v} \circ \Psi^{-1}) \operatorname{div}_y \mathbf{T}_L(\mathbf{u} \circ \Psi^{-1}) \, dy - \int_{\partial\Omega} (\mathbf{v} \circ \Psi^{-1}) \mathbf{T}_L(\mathbf{u} \circ \Psi^{-1}) \mathbf{n} \, ds \tag{10}$$

where $\mathbf{T}_C(\mathbf{u}) = \mathbf{C}(\mathbf{x}) \nabla_x \mathbf{u}$ and $\mathbf{T}_L(\mathbf{u}) = \mathbf{L}(\mathbf{x}) \nabla_x \mathbf{u}$. Using the fact that Ψ is the identify on the boundary and taking \mathbf{u} a solution of the elastic problem with the elasticity tensor \mathbf{C} and \mathbf{v} a virtual displacement field the preceding relation becomes:

$$\int_{\Omega} (\mathbf{v} \circ \Psi^{-1}) \operatorname{div}_y \mathbf{T}_L(\mathbf{u} \circ \Psi^{-1}) \, dx = 0$$

This relation implies that $\mathbf{u} \circ \Psi^{-1}$ is solution of an elastic problem with elasticity tensor \mathbf{L} , because $\mathbf{v} \circ \Psi^{-1}$ is still a virtual displacement field for any virtual displacement field v and a diffeomorphism Ψ . As a consequence there is a one-to-one correspondence between the solutions of problems of elasticity tensors \mathbf{C} and \mathbf{L} . Moreover the corresponding solutions have the same global deformation energy and the same boundary values in displacements and forces:

$$\mathbf{u}|_{\partial\Omega} = (\mathbf{u} \circ \Psi^{-1})|_{\partial\Omega}$$

$$\mathbf{T}_C(\mathbf{u})|_{\partial\Omega} = \mathbf{T}_L(\mathbf{u} \circ \Psi^{-1})|_{\partial\Omega}$$

In other words, one can *not* identify anisotropic elastic moduli only by displacement force boundary measurements. The natural conjecture is that this is the only obstruction to identifiability, exactly as in the electrical identification problem [17]. The conjecture proposed in the next part is a weaker result, justified by the variational method which will be proposed for the reconstruction.

3.2 Uniqueness conjecture

The only investigated elastic identification problems are related to the particular case of isotropic elasticity. Ikehata [8, 9], and later Nakamura and Uhlmann [15] proved that, provided certain regularity conditions on the functions are met, the knowledge of *all* possible displacement-force boundary data pairs², i.e. (ξ, φ) , leads to a unique distribution of Lamé moduli (λ, μ) .

It is important to notice that the priori knowledge of the local material symmetry (isotropic elasticity) did eliminate the non-identifiability mentioned in the example before. In order to gain an intuition on the identifiability for the general anisotropic

²The knowledge of all displacement-force boundary data pairs (ξ, φ) is equivalent with the knowledge of the Dirichlet-to-Neumann data map. It is only for simplicity that this application was not introduced in the discussion of this paper

problem, it is interesting to point out the technique used in the proofs [8, 9, 15] for isotropic elasticity. Using the theorem on work and energy and a special class of displacement solutions one can relate the interior distribution of Lamé moduli with the boundary measurements. These displacements transform the energy integral in a Spatial Fourier Transform of the elastic moduli. The identifiability of the Lamé moduli results afterwards from the theorem of work and energy.

In the numerical identification process [2], not Lamé moduli, but bulk and shear modulus were used, because they permit a decomposition of the energy (in spherical and deviatorical parts) with good properties for the reconstruction process. The importance of this fact is hidden in isotropic elasticity, because we have a one-to-one mapping from Lamé moduli to bulk and shear modulus.

If one wants to preserve the properties of bulk and shear modulus in anisotropic elasticity, one has to pass to eigenelastic moduli [13, 14] (for a brief presentation see appendix A). These moduli are the eigenvalues of the elasticity tensor considered as a linear application. The eigentensors are the corresponding eigenvectors. In a tensorial notation the elasticity tensor becomes:

$$\mathbf{C} = \sum_{k=1}^6 c(k) \mathbf{N}^{(k)} \otimes \mathbf{N}^{(k)}$$

with $c(k)$ the eigenelastic moduli and $\mathbf{N}^{(k)}$ the eigentensors (second order symmetric tensors, with $\mathbf{N}^{(k)} \cdot \mathbf{N}^{(k)} = 1$). S , the minimum number of elastic moduli characterizing an anisotropic material can be written as: $S = K + M + C$ ($S \leq 21$), with K ($K \leq 6$) the number of distinct eigenelastic moduli, M ($M \leq 12$) the number of elasticity distributors and C ($C \leq 3$) the number of coordinate orientation parameters [13]. The elasticity distributors are ratios of relative extensions in certain direction and might be considered as generalization of Poisson ratios. They are completely determined by: $\text{tr } \mathbf{N}^{(k)}$ and $\text{tr } \mathbf{N}^{(k)} \mathbf{N}^{(k)} \mathbf{N}^{(k)}$, $k = 1, 6$. The coordinate orientation parameters depend directly of the choice of the coordinate system selected in order to describe material symmetry.

If the energy is decomposed using eigenmoduli and eigentensors, eigenmoduli will explicitly appear in the terms of the decomposition, while elasticity distributors and coordinate orientation parameters will rest hidden in the eigentensors.

Taking into account the role played by energy in the uniqueness results [8, 15] and the before mentioned considerations about eigenelastic moduli it seems to be possible to formulate an identifiability result in the general case of anisotropy. For the moment, this is a conjecture and it will be stated without any proof:

If the eigentensors $\mathbf{N}^{(k)}$, $k = 1, 6$ (or equivalently the elasticity distributors and the coordinate orientation parameters) of an anisotropic elastic material are known in each interior point of the body, the knowledge of all displacement-force boundary data pairs (ξ, φ) will determine the interior distribution of the eigenelastic moduli $c_{(k)}$, $k = 1, 6$.

4 A VARIATIONAL METHOD FOR RECONSTRUCTION

The mathematical results presented in the preceding section demanded the knowledge all possible displacement-force boundary data pairs, which is an impossible demand

from a practical point of view. One can reasonably only expect to know a finite number of data pairs at a finite number of measurement points.

However for the simplicity of the variational formulation the knowledge of continuous vector fields over the whole boundary $\partial\Omega$ (this could be achieved by interpolation) will be assumed. Therefore the measured data will be represented by N displacement-force boundary vector fields pairs: $(\xi_i, \varphi_i)_{i=1,N}$.

The equations and boundary conditions governing this problem are mentioned in (1), and (4). Taking the constitutive law apart, the equilibrium and the geometrical equations can be grouped with the boundary conditions to form *kinematically* and *statically* admissible fields. These are sets of stresses, respectively strains defined as follows:

$$KA(\xi) = \{E | \exists u, E = \frac{1}{2}(\nabla u + \nabla^T u) \text{ on } \Omega \text{ and } u|_{\partial\Omega} = \xi \text{ on } \partial\Omega\}$$

$$SA(\varphi) = \{T | \operatorname{div} T = 0 \text{ on } \Omega \text{ and } Tn|_{\partial\Omega} = \varphi \text{ on } \partial\Omega\}$$

Therefore the practical identification problem can be stated as follows:

Find the interior distribution $C(x)$ of the elastic moduli and the N triplets of displacements, strain and stress fields $(u_i(x), E_i(x), T_i(x))_{i=1,N}$ satisfying the constitutive equations on Ω :

$$T_i = CE_i, \quad (11)$$

and belonging to the sets:

$$E_i \in KA(\xi_i) \quad T_i \in SA(\varphi_i) \quad (12)$$

This suggest to look for a solution by finding a best fit for the constitutive equation over the sets of admissible (i.e. symmetries, positive definiteness) elasticity tensor fields C , kinematically and statically admissible fields. One way to accomplish this is to introduce a norm which measures the error on constitutive law (ECL):

$$I(C, E_1, \dots, E_N, T_1, \dots, T_N) = \sum_{i=1}^N \frac{1}{2} \int_{\Omega} |C^{-1/2} T_i - C^{1/2} E_i|^2 dx \quad (13)$$

which is to be minimized over all arguments subject to the following constraints:

$$E_i \in KA(\xi_i) \quad \text{and} \quad T_i \in SA(\varphi_i), \quad (i = 1, N) \quad (14)$$

³ C is symmetric and positive definite so $C^{1/2}$ and $C^{-1/2}$ are well defined. The proof involves the second rank 6-dimensional tensor representation for which the power function can be defined.

The choice of the integrand³ as $|C_y^{-1/2} T - C_y^{1/2} E|^2$, rather than e.g. $|C^{-1} T - E|^2$ or $|T - CE|^2$, is justified by its physical dimension as a strain energy density.

This error-functional was initially introduced by Ladevèze and Leguillon [10] to study the finite element error. The ECL density gives an accurate spatial localization of the finite element error. This remarkable property has later been used by Ladevèze *et al.* [11] as a criterion to adjust the stiffness or the mass matrices of a finite element model in order to obtain similar frequencies and modal values on the finite element model and the real body.

Later, in electrostatics, the error on constitutive law has been used to determine the interior distribution of the electrical conductivity from boundary voltage-flux measurements. This problem has recently received much attention from both viewpoints, mathematical [7] and technical [16]. A large survey of the existing bibliography on the subject can be found in [6, 7, 1]. The electric and elastic problem are both governed by elliptic equations. Therefore the techniques applied to the scalar electrostatic inverse problem can be transposed to the vector elastic inverse problem.

Returning to elasticity, the decomposition of I will be discussed in the sequel.

Applying to the terms containing $E_i T_i$ the theorem of work and energy, one obtains:

$$I(C, E_1, \dots, E_N, T_1, \dots, T_N) = \sum_{i=1}^N \frac{1}{2} \int_{\Omega} T_i C^{-1} T_i + E_i C E_i dx - \sum_{i=1}^N \int_{\partial\Omega} \xi_i \cdot \varphi_i dx$$

As ξ_i and φ_i are the known boundary displacements and forces, we remark that the only part in I which plays a role in the minimization process is:

$$J(C, E_1, \dots, E_N, T_1, \dots, T_N) = \sum_{i=1}^N \frac{1}{2} \int_{\Omega} T_i C^{-1} T_i + E_i C E_i dx$$

The fact that J is a sum of stress and strain energy allows, with fixed C an independent and simple minimization in T_i and E_i respectively. It would be a great benefit to have the same kind of explicit minimization in C , as in the electric counterpart of the error on the constitutive law [6, 7]. This is not straightforward because in elasticity C is a tensor.

We shall furnish a technique to overcome this difficulty, first by explaining it in the case of isotropy and then extending it to the anisotropic case.

4.1 Isotropic elasticity

For the isotropic elasticity every tensor A can be written as the sum of its spherical part $1/3(\text{tr } A) I$ and his deviatorical part \hat{A} . The bulk and the shear moduli are taken as the two independent elastic constants, with:

$$\eta = 3\lambda + 2\mu \quad \omega = 2\mu \quad (15)$$

In this case J becomes:

$$\begin{aligned}
 J(\eta, \omega, \mathbf{E}_1, \dots, \mathbf{E}_N, \mathbf{T}_1, \dots, \mathbf{T}_N) &= \sum_{i=1}^N \frac{1}{6} \int_{\Omega} \eta^{-1} (\text{tr } \mathbf{T}_i)^2 + \eta (\text{tr } \mathbf{E}_i)^2 dx \\
 &+ \sum_{i=1}^N \frac{1}{2} \int_{\Omega} \omega^{-1} \mathring{\mathbf{T}}_i \cdot \mathring{\mathbf{T}}_i + \omega \mathring{\mathbf{E}}_1 \cdot \mathring{\mathbf{E}}_i dx \quad (16)
 \end{aligned}$$

The partial minimization of J with respect to η and ω has the following explicit solution:

$$\eta^2(\mathbf{x}) = \frac{\sum_{i=1}^N (\text{tr } \mathbf{T}_i)^2}{\sum_{i=1}^N (\text{tr } \mathbf{E}_i)^2} \quad \text{and} \quad \omega^2(\mathbf{x}) = \frac{\sum_{i=1}^N \mathring{\mathbf{T}}_i \cdot \mathring{\mathbf{T}}_i}{\sum_{i=1}^N \mathring{\mathbf{E}}_i \cdot \mathring{\mathbf{E}}_i}$$

There are two important properties which do permit the splitting of J :

- the spherical and deviatorical parts of a tensor are orthogonal in the sense that their doubly contracted tensor product is zero.
- the constitutive law reduces to a proportionality between the spherical (respectively deviatorical) part of stress and strain, with factor η (respectively ω):

$$\frac{1}{3}(\text{tr } \mathbf{T}) \mathbf{I} = \frac{1}{3} \eta (\text{tr } \mathbf{E}) \mathbf{I} \quad \text{and} \quad \mathring{\mathbf{T}} = \omega \mathring{\mathbf{E}} \quad (17)$$

4.2 Anisotropic elasticity

These properties of the spherical and deviatorical parts are inherited in the general case of anisotropy by the eigentensors and eigenelastic moduli of the linear elastic constitutive law. The eigenvectors of the 3-dimensional fourth-rank elasticity tensor \mathbf{C} , considered as a 6-dimensional second-rank tensor, are called eigentensors when projected back in the 3-dimensional tensor space. The corresponding eigenvalues are called eigenelastic moduli. An extended discussion of these concepts is presented in [14]. A concise definition and some examples are given in Appendix A.

If we denote by $\mathbf{A}^{(k)}$, $k = 1, 6$ the projections of a tensor \mathbf{A} on the basis of eigentensors, and by $c(k)$, $k = 1, 6$ the corresponding eigenelastic moduli of the elasticity tensor \mathbf{C} , we have:

- $\mathbf{A}^{(k)} \cdot \mathbf{A}^{(j)} = 0$ if $k \neq j$, and
- the constitutive equations $\mathbf{T} = \mathbf{C}\mathbf{E}$ breaks in 6 uncoupled equations:

$$\mathbf{T}^{(k)} = c_{(k)} \mathbf{E}^{(k)} \quad k = 1, 6 \quad (18)$$

We remark that there are at most 6 different eigenelastic moduli, because there are at most 6 different eigenvalues for a 6 dimensional second-rank tensor, and therefore $K \leq 6$. Using eigentensors and eigenelastic moduli the error on constitutive law I , writes as follows:

$$I(c_{(1)}, \dots, c_{(6)}, \mathbf{E}_1, \dots, \mathbf{E}_N, \mathbf{T}_1, \dots, \mathbf{T}_N) = \sum_{i=1}^N \frac{1}{2} \sum_{k=1}^6 \int_{\Omega} |c_{(k)}^{-1/2} \mathbf{T}_i^{(k)} - c_{(k)}^{1/2} \mathbf{E}_i^{(k)}|^2 dx. \quad (19)$$

Downloaded by [ETH Zurich] at 01:13 08 January 2016

The functional J , analogous to (16) becomes:

$$J(c_{(1)}, \dots, c_{(6)}, \mathbf{E}_1, \dots, \mathbf{E}_N, \mathbf{T}_1, \dots, \mathbf{T}_N) = \sum_{i=1}^N \frac{1}{2} \sum_{k=1}^6 \int_{\Omega} |c_{(k)}|^{-1} \mathbf{T}_i^{(k)} \cdot \mathbf{T}_i^{(k)} + c_{(k)} \mathbf{E}_i^{(k)} \cdot \mathbf{E}_i^{(k)} dx \quad (20)$$

and the explicit minimization of J in the directions of the eigenelastic moduli gives, similar to (4.1), an explicit expression for the optimal choice for the interior distribution of the eigenelastic moduli:

$$c_{(k)} = \frac{\sum_{i=1}^N \mathbf{T}_i^{(k)} \cdot \mathbf{T}_i^{(k)}}{\sum_{i=1}^N \mathbf{E}_i^{(k)} \cdot \mathbf{E}_i^{(k)}} \quad k = 1, 6$$

The form of J defined by (20), can now be minimized applying the alternating direction implicit method (ADI) or the modified Newton (MN) method defined by Kohn and McKenney [6] in the electric identification problem. We give in Appendix B an overview of these two algorithms.

4.3 Limitations of the reconstruction method

The method presented in the section before is based mainly on the decomposition of the error on constitutive law using eigenelastic moduli and eigentensors. This permits a straightforward reconstruction of the eigenelastic moduli. Therefore this method applies especially in the cases of **isotropic** or **cubic material symmetry**, where the minimum number of elastic moduli can be directly related to the eigenelastic constants. In such a case, there are no elastic distributors and the eigentensors are independent of elastic moduli. For other material symmetries as orthotropic or tetragonal symmetry, where the number of elastic moduli exceeds the number of eigenelastic moduli, the eigentensors depend directly of the elastic moduli, and therefore some additional information should be provided in order to identify the interior distribution of all the moduli from displacement-force boundary measurements.

5 TEST CALCULATIONS

The numerical examples present the identification of an inclusion in a rectangular matrix (see Figures 2–5) for cubic elastic symmetry. The materials for the matrix and inclusion were aluminum (isotropic) and copper (cubic symmetry) with the following values for the elastic moduli (Young modulus, Poisson coefficient, respectively shear modulus):

$$E_{Al} = 66 \times 10^9 \text{ Pa} \quad \nu_{Al} = 0.32 \quad G_{Al} = \frac{E_{Al}}{2(1 + \nu_{Al})} = 25 \times 10^9 \text{ Pa}$$

$$E_{Cu} = 66 \times 10^9 \text{ Pa} \quad \nu_{Cu} = 0.42 \quad G_{Cu} = 75 \times 10^9 \text{ Pa} \quad (21)$$

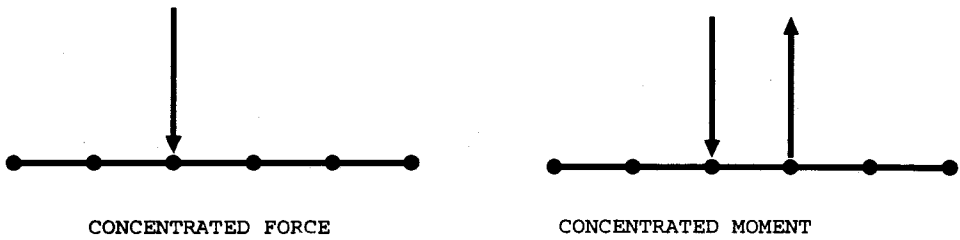


Figure 1 Concentrated forces and moments

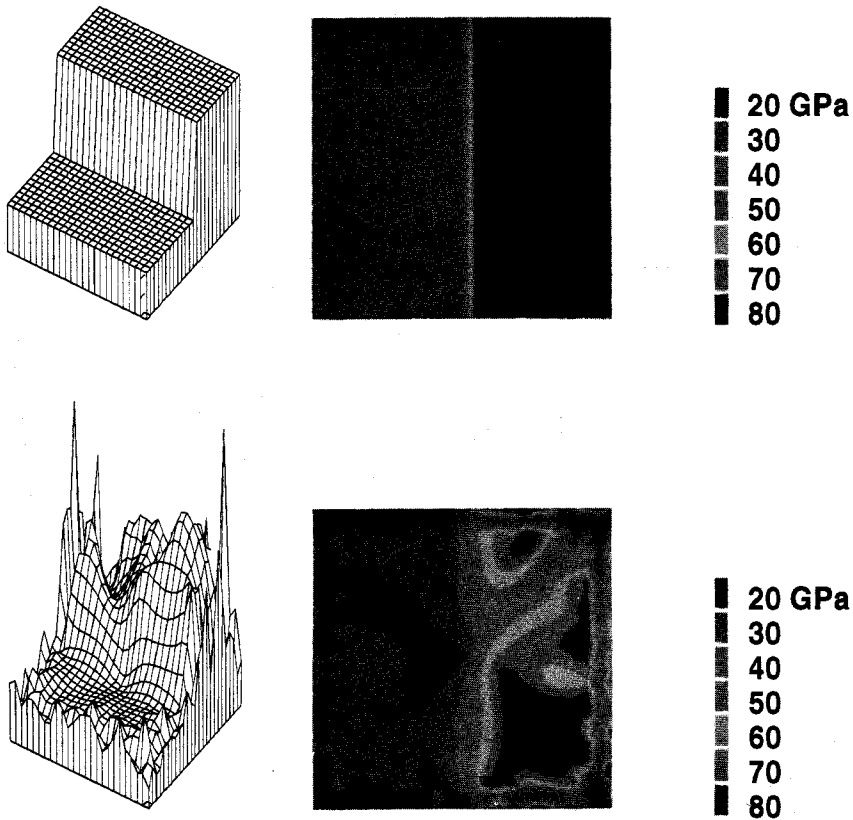


Figure 2 Real (top) and reconstructed (bottom) step inclusion (after 5 iterations) of the shear moduli G with 10% noise on a 24×24 mesh. (See Colour Plate 1)

The corresponding values for the eigenelastic constants are given in Appendix A.

In a first step, displacement-force boundary measurements were generated by direct elastic computations. For a given 'real' distribution of elastic moduli (see Figures 2–5) and a given boundary load (concentrated forces or moments, see Figure 1) a boundary displacement was computed. In a second step, synthetic data-pairs were injected in the

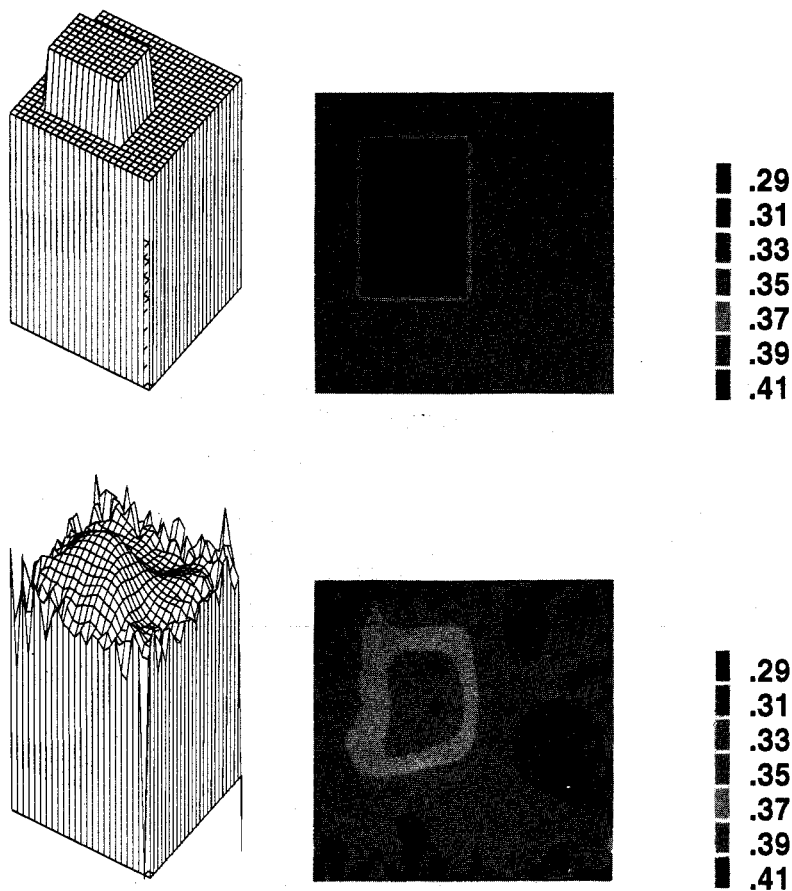


Figure 3 Real (top) and reconstructed (bottom) rectangular inclusion (after 10 iterations) of the Poisson coefficient ν with no noise on a 24×24 mesh. (See Colour Plate II)

alternating direction implicit minimization scheme in order to reconstruct the elastic moduli. The data-pairs have also been perturbed by a white noise with a maximum amplitude of 10%.

All the computations were programmed using the finite element oriented language CASTEM2000, on a HP 720 workstation. The domain, 1 unit \times 1 unit was divided into $n \times n$ quadrangular linear elements. The finest mesh had 48×48 elements. For a 24×24 mesh, 96 direct computation were executed in 6 minutes. However 17 inverse iterations took for the same mesh, 5 hours on the same workstation. The overtime is justified by the internal transfers from eigenmoduli to the E, ν, G form of elastic moduli in the computation of the rigidity.

The reconstruction method considered the body with cubic symmetry. In order to avoid divergence, the local values of elastic moduli were forced to stay in a closed

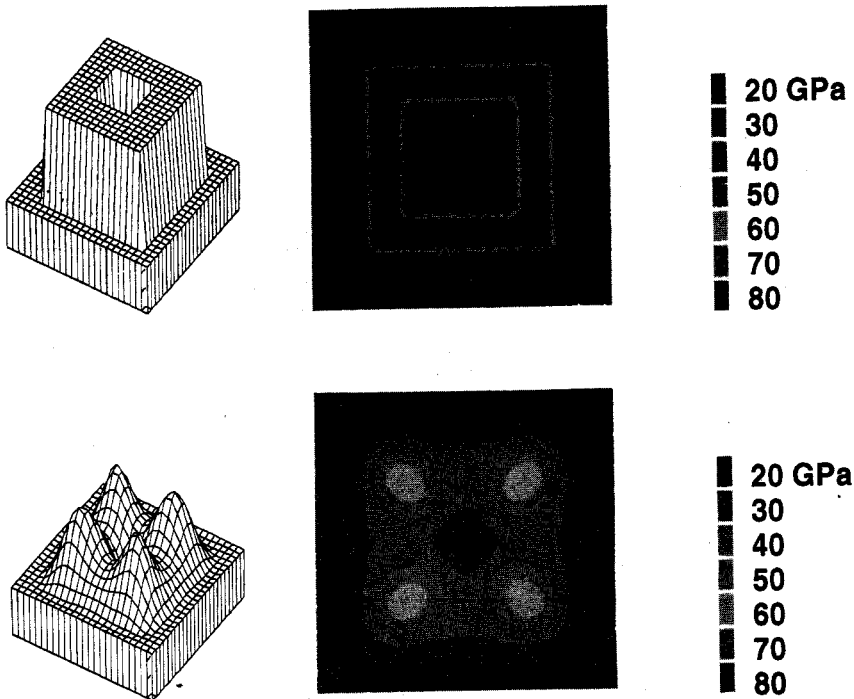


Figure 4 Real (top) and reconstructed (bottom) ring-shaped inclusion (after 32 iterations) of the shear moduli G with no noise on a 24×24 mesh. (See Colour Plate III)

interval by a min-max cutting at each iteration. The permitted values were 5 times greater (or less) than the initial guess for the moduli. This procedure enhanced the stability of the method to noise, compared to the previously published results for isotropic elasticity [2, 4].

No other assumption has been made on the distribution of the elastic moduli. A set of constant distributions was tested as an initial guess for the reconstruction algorithm. The chosen values for the moduli varied up to 10% from real values of the moduli of aluminum or copper. However initial values did not influence the final result.

The results presented in Figures 2–5 correspond to a copper inclusion in an aluminum matrix. They illustrate partly our general conclusions on the reconstruction method for cubic anisotropy:

- Shape, location and average level of the moduli are generally well reconstructed in the first 5–15 iterations (see Figure 5). The relative error and the errors in volume and moments reported to the real moduli depends of the shape of the inclusion, the distribution of the applied load, and the noise level. The encountered values for different errors vary between 0% and 20%. In pathological cases, as for example 10% noise, errors can increase after the 10 iteration.

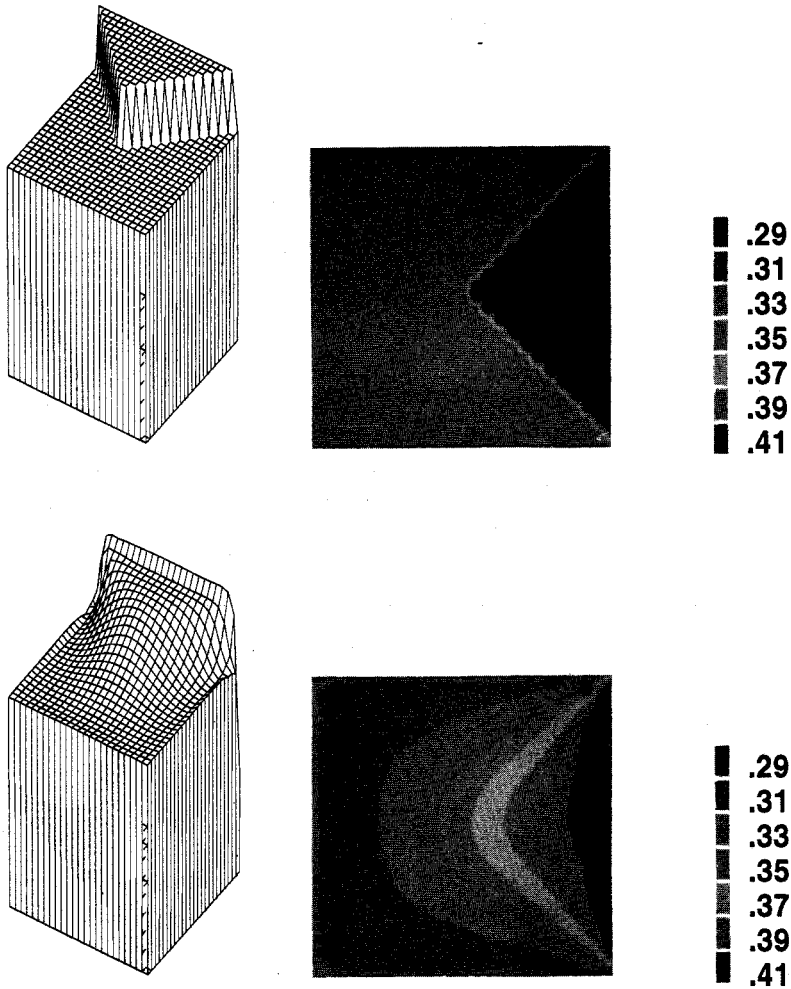


Figure 5 Real (top) and reconstructed (bottom) corner inclusion (after 14 iterations) of the Poisson coefficient ν with no noise on a 24×24 mesh. (See Colour Plate IV)

The relative error decreases below 10% for the Young modulus E and the Poisson ratio ν and at 10–30% for the shear modulus G .

The errors in volume or moment are not so important. Young modulus and Poisson coefficient are below 5% error, and the shear modulus is at 10–20% error after the first 5 iterations. Continuing the iterations brings Young moduli and Poisson coefficient at 2–3% and the shear modulus below 10% (see Figure 4).

The fact that errors in volume and moment are lesser than relative errors implies that beforehand information, such as form of inclusions, or regularization would enhance the reconstruction.

If noise becomes important, 5 and 10% noise for the other tested forms, the reconstructed distributions become highly oscillating after 10 iteration and the scheme diverges. These images do not permit any identification of the real distribution. However the early 5–10 iterations provide even in this cases a fairly accurate localization and the form of the inclusion, even if the values of the elastic moduli are just a rough indication of the real value. This suggest that early stopping of the iterations in noisy cases gives a certain amount of important information. A combination of early stopping and regularization of the results might therefore enhance the accuracy of the identification.

- A comparison between concentrated forces and moments as applied loads shows that forces give a better reconstruction in the first (i.e. 5–8) iterations, whether moments are better after a larger number of iterations.
- A stabilizing effect on the oscillations of the reconstruction is observed when in the reconstruction one employs measurements corresponding to forces applied in the neighborhood of corners, compared with the case where no such measurement are employed.
- Errors didn't have a systematic increase or decrease for coarser or finer meshes. A possible explanation of this fact is that for finer meshes more measurements were used in the reconstruction. However for a $n \times n$ mesh, the number of measurements was proportional to n , and the number of unknown values was proportional to n^2 .
- The values of error showed that the Young modulus and the Poisson coefficient were better reconstructed than the shear modulus. Even though, shear modulus showed up to be more stable to noise, than the other two.
- The error on constitutive law has, as in the case of isotropic elasticity a great descent in the first 5–10 iterations, approaching afterwards a limiting value with little decrease. The limiting value depends essentially of the noise level in the boundary measurements. For high noise levels, 5% for the corner inclusion and 10% for all inclusions the error on constitutive law fails to converge, and becomes oscillating after 10 iterations. Therefore this could be taken as a criterion to stop the reconstruction method.

The numerical problems encountered are related on one side on the transfer passages between different forms of expression of the elastic moduli, and on the other side on the huge gradients in the deformation and stress fields obtained from the concentrated loads. Internal demands of the programming language demanded several interpolation passages from fields defined at the nodes to fields defined at the Gaussian interpolation points of the elements. It is the reason why we preferred the usage of linear elements instead of quadrangular elements, which can conduct to changes of the sign of the fields by interpolation. It is also obvious that this internal constraints of the programming contributed in the general error of the reconstruction.

Some interesting conclusions on the elastic identification problem and the ECL are stemming from a comparison with the electric identification and the reconstruction performed by Kohn and Mckenney [6].

In a first approach one can compare similar images, obtained with different kinds of boundary loads and slightly different minimization methods. The elastic reconstruc-

tion used only concentrated forces and moments as boundary loads and only the IDA method for minimization, while the electrical one [6], used harmonic functions as boundary loads and both methods, IDA and MN, for minimization.

In a second approach one can discuss about minimizing the ECL over isotropic or anisotropic materials. Considering that the material properties are expressed by the values of a second (electricity) or a fourth order tensor (elasticity) one can remark that isotropic materials are completely determined by the eigenvalues of this tensor, whether anisotropic materials need more information for their characterisation.

For the electrical problem, Kohn and Vogelius [7] showed that one can identify the second order conductivity tensor up to its eigenvalues, and that the ECL considered as a functional over anisotropic conductivities is the relaxation of the ELC considered as a functional over isotropic conductivities. This means for example that minimizing the ECL over isotropic conductivities and a fine mesh is in a certain sense equivalent to minimizing the ECL over anisotropic conductivities and a coarse mesh. The numerical results obtained using the ECL over anisotropic conductivities in order to identify an isotropic conductivity were reported to be "disastrous", by Kohn and McKenney [6]. This behavior was explained as a direct effect of the non-uniqueness of the identification of anisotropic conductivities.

For the elastic problem the identification of the elasticity tensor up to its eigenvalues has been conjectured. To our knowledge there exists no rigorous proof of the equivalent relaxation statement. The statement should express that the ECL considered over anisotropic elasticity tensors is the relaxation of the ECL considered over isotropic elasticity tensors. Even without a proof it should be obvious that permitting the minimisation over anisotropic materials, a class including isotropic materials, one also permits the minimum to be smaller.

As a parallel to the electric problem one might therefore expect to encounter when identifying aluminium, an isotropic material, by minimizing the ELC over elasticity tensors with cubic symmetry, a special case of anisotropy, similar problems as the ones reported by Kohn and McKenney [6] in the electric problem.

By extrapolation of the experience in the electric identification problem [6] one might expect to encounter difficulties when identifying aluminium, an isotropic material, by minimizing the ELC over elasticity tensors with cubic symmetry, which is a special case of anisotropy. However this was not the case. A possible reason for this fact is that cubic symmetry is still completely determined by the eigenvalues of the elasticity tensor, as in the isotropic case. As a consequence difficulties are related with the identification of more complex anisotropies, where the elasticity tensor is not completely determined by its eigenvalues.

6 CONCLUSION

Some aspects of an identification problem in anisotropic elasticity have been highlighted. From the mathematical point of view, a uniqueness conjecture has been stated and a variational method for the reconstruction has been proposed. From the practical point of view promising numerical results have been obtained. A reasonably accurate reconstruction was computed after a modest number of iterations. The minimization

process was simple and explicit. However when data was noisy, the picture degraded after sufficiently many iterations.

The method seems to be sufficiently stable, in other words insensitive to noise and interpolation errors in the data. The efficiency could probably be enhanced if additional information (shape, location, ...) is provided. Given accurate data from finite many different measurements a good approximation of the true coefficient has been obtained.

The method should also be tested in other case of material anisotropy, where the number of elastic moduli exceeds the number of eigenelastic moduli (i.e. orthotropic, tetragonal symmetry).

Acknowledgement

The work is part of a doctoral thesis presented at Ecole Polytechnique (Palaiseau, France) under the guidance of Professor H. D. Bui. The author wants to express his gratitude to Professor H. D. Bui, who introduced him to the subject, and provided continuous encouragement during the work. He also thanks M. Bonnet, H. Maigre and D. Rittel for their helpful remarks.

References

- [1] M. Bonnet, H. D. Bui, H. Maigre and J. Planchard, Identification of heat conduction co-efficient, application to non-destructive testing In *IUTAM Symposium on Inverse problems in engineering mechanics* [11–15 May 1992, Tokyo, Japan, ed. M. Tanaka and H. D. Bui, Springer Verlag, 1993].
- [2] A. Constantinescu-A Numerical Investigation of the Elastic Moduli in an Inhomogeneous Body *Inverse Problems in Engineering: Theory and Practice* ed. N. Zabarras et al., ASME-100357, 77 (1993).
- [3] A. Constantinescu-*Sur l'identification des modules élastiques*, Mémoire de thèse de doctorat, Ecole Polytechnique, Palaiseau (France) (1994).
- [4] M. Bonnet, A. Constantinescu-Quelques remarques sur l'identification de modules élastiques à l'aide de mesures sur la frontière, *Actes du 11ème Congrès Français de Mécanique*, Villeneuve d'Ascq, [Presses Universitaires de Lille] 6–10 (1993).
- [5] H. D. Bui-*Introduction aux problèmes inverses en mécanique des matériaux*, Editions Eyrolles, Paris, 1993 (English translation-CRC Press, Boca Raton, Japanese translation-Shokabo, Tokyo, (to appear); Chinese translation-Editions of the University of Harbin, Harbin (to appear)) (1994).
- [6] R. Kohn, A. McKenney-Numerical implementation of a variational method for electric impedance tomography *Inverse problems*, no. 6, 389–414 (1990).
- [7] R. Kohn, M. Vogelius-Relaxation of a Variational Method for Impedance Computed Tomography *Comm. Pure. Appl. Math.*, XL, 745–777 (1987).
- [8] M. Ikehata-Inversion for the linearized problem for an inverse boundary value problem in elastic prospection *SIAM J. Appl. Math.*, 50, 6, 1635–1644 (1990).
- [9] M. Ikehata-An inverse problem for the plate in the Love-Kirchhoff theory *SIAM J. Appl. Math.*, 53, 4, 942–970 (1993).
- [10] P. Ladevèze, D. Leguillon-Error estimates procedures in the finite element method and application *SIAM J. Numer. Anal.*, 20, 3, (1984).
- [11] P. Ladevèze, M. Reynier, D. Nedjar-Parametric Correction of Finite Element Models using Modal Tests *IUTAM Symposium on Inverse problems in engineering mechanics* [11–15 May 1992, Tokyo, Japan, ed. M. Tanaka and H. D. Bui, Springer Verlag, 1993].
- [12] J. Lemaitre, J.-L. Chaboche-*Mécanique des matériaux solides*, Dunod, Paris (1985).
- [13] S. C. Cowin, M. M. Mehrabadi-The Structure of the Linear Anisotropic Elastic Symmetries *J. Mech. Phys. Solids*, 40, 7, 1459–1471 (1992).
- [14] M. Mehrabadi, St. C. Cowin-Eigentensors of Linear Elastic Materials *Q. J. Mech. appl. Math.*, 43, Pt. 1 (1990).
- [15] G. Nakamura, G. Uhlmann-Uniqueness for identifying Lamé moduli by Dirichlet to Neumann map *Inverse Problems in Engineering Sciences-ICM-90 Satellite Conference Proceedings*, ed. M. Yamaguti, Springer Verlag, Tokio (1991).
- [16] L. Ovaciuk, et al.-Progress in Impedance Imaging for Gas-Liquid Flows: Experimental Results *Inverse Problems in Engineering: Theory and Practice* ed. N. Zabarras et al., ASME-100357, P. 77 (1993).

- [17] Z. Sun, G. Uhlmann-Generic uniqueness for determined inverse problems in 2 dimensions *Inverse Problems in Engineering Sciences-ICM-90 Satellite Conference Proceedings*, ed. M. Yamaguti, Springer Verlag, Tokio (1991).

A APPENDICES

A.1 Eigentensors and eigenelastic constants

The constitutive law:

$$T = CE \quad (22)$$

is a linear application between 3-dimensional second-rank tensor spaces. We are interested in determining the strain states E , for which the tensors E and T have parallel directions. In order to determine these directions, we rewrite the constitutive law as an application between 6-dimensional vector spaces:

$$\hat{T} = \hat{C}\hat{E}$$

with:

$$\hat{T} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ 2^{1/2} T_{23} \\ 2^{1/2} T_{31} \\ 2^{1/2} T_{12} \end{bmatrix} \quad \hat{E} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2^{1/2} E_{23} \\ 2^{1/2} E_{31} \\ 2^{1/2} E_{12} \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 2^{1/2} C_{1123} & 2^{1/2} C_{1131} & 2^{1/2} C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & 2^{1/2} C_{2223} & 2^{1/2} C_{2231} & 2^{1/2} C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & 2^{1/2} C_{3323} & 2^{1/2} C_{3331} & 2^{1/2} C_{3312} \\ 2^{1/2} C_{2311} & 2^{1/2} C_{2322} & 2^{1/2} C_{2333} & 2C_{2323} & 2C_{2331} & 2C_{2312} \\ 2^{1/2} C_{3111} & 2^{1/2} C_{3122} & 2^{1/2} C_{3133} & 2C_{3123} & 2C_{3131} & 2C_{3112} \\ 2^{1/2} C_{1211} & 2^{1/2} C_{1222} & 2^{1/2} C_{1233} & 2C_{1223} & 2C_{1231} & 2C_{1212} \end{bmatrix}$$

where \hat{A} denotes the image of the 3-dimensional tensor A in the 6-dimensional vector space through a vector-space isomorphism. This image is denoted as the second-rank tensor notation [13, 14]. Parallel tensors are projected in parallel vectors and therefore our problem becomes a standard eigenvalue problem for a 6-dimensional second rank tensor:

$$(\hat{C} - c\mathbf{1})\hat{V} = 0$$

Since \hat{C} is positive definite in a 6-dimensional vector space, we have $K \leq 6$ different eigenvalues $c_{(k)}$ and we can always choose an associated orthonormal base of eigenvectors $\hat{N}^{(k)}$, $k = 1, 6$. We shall call $c_{(k)}$ eigenelastic constants and the 3-dimensional second-rank tensor image $N^{(k)}$ of $\hat{N}^{(k)}$, the eigentensors of our elastic problem. $A^{(k)}$ will denote the projections of the tensor A in the directions of eigentensors:

$$A^{(k)} = (A \cdot N^{(k)}) N^{(k)}$$

If an eigenelastic moduli has an order of multiplicity i , $A^{(k, k+1, \dots, k+i)}$ will denote the projection of A on the subspace corresponding to that eigenelastic moduli.

We present now the full expression of eigenelastic moduli and eigentensors for the case of cubic symmetry. Other cases are treated extensively in [14].

The cubic symmetry determined by 3 elastic moduli λ, μ and β related to \hat{C} by:

$$\hat{C} = \begin{bmatrix} \lambda + 2(\mu - \beta) & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2(\mu - \beta) & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2(\mu - \beta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix}$$

The first eigenelastic constant is $c_1 = 3\lambda + 2(\mu - \beta)$ related to the spherical part of the tensor:

$$A^{(1)} = \begin{bmatrix} \frac{1}{3} \text{tr}(A) & 0 & 0 \\ 0 & \frac{1}{3} \text{tr}(A) & 0 \\ 0 & 0 & \frac{1}{3} \text{tr}(A) \end{bmatrix}$$

The eigentensor corresponding to the second eigenelastic constant $c_2 = 2(\mu - \beta)$ is one representing the distortion:

$$A^{(2,3)} = \begin{bmatrix} A_{11} - \frac{1}{3} \text{tr}(A) & 0 & 0 \\ 0 & A_{22} - \frac{1}{3} \text{tr}(A) & 0 \\ 0 & 0 & A_{33} - \frac{1}{3} \text{tr}(A) \end{bmatrix}$$

The last eigentensor is one of isochoric shearing and it is related to the third eigenelastic constant $c_3 = 2\mu$:

$$A^{(4,5,6)} = \begin{bmatrix} 0 & A_{12} & A_{13} \\ A_{12} & 0 & A_{23} \\ A_{13} & A_{23} & 0 \end{bmatrix}$$

The values of the eigenelastic moduli for aluminum and copper corresponding to the values in (21) are:

$$\begin{aligned} c_1^{Al} &= 1.19 \times 10^{11} \text{ Pa} & c_2^{Al} &= c_3^{Al} = 5 \times 10^{10} \text{ Pa} \\ c_1^{Cu} &= 411 \times 10^9 \text{ Pa} & c_2^{Cu} &= 47 \times 10^9 \text{ Pa} & c_3^{Cu} &= 150 \times 10^9 \text{ Pa} \end{aligned} \quad (23)$$

A.2 Two minimization algorithms

The *alternating direction implicit* (ADI) method uses directly the fact that the minimization of J in (20) is straightforward in each of the directions $c_{(k)}$, \mathbf{E}_i , \mathbf{T}_i when all other variables are fixed. The eigenmoduli are given by (4.2) and the other two are equivalent to solving a well-defined direct elastic boundary value problem. Thus an iteration of the ADI method consist of the following steps:

1. with $c_{(k)}$ fixed (as determined by te previous step) solve the N Dirichlet problems (imposed displacements):

$$\operatorname{div}(\mathbf{C}\nabla \mathbf{u}) = 0 \quad \text{in } \Omega \quad (24)$$

$$\mathbf{u}_i|_{\partial\Omega} = \zeta_i \quad \text{on } \partial\Omega$$

2. with $c_{(k)}$ fixed (as determined by the previous step) solve the N Neumann problems (imposed forces):

$$\operatorname{div}(\mathbf{C}\nabla \mathbf{u}) = 0 \quad \text{in } \Omega \quad (25)$$

$$\mathbf{C}\nabla \mathbf{u}_i|_{\partial\Omega} = \varphi_i \quad \text{on } \partial\Omega$$

3. with $(\mathbf{E}_i)_{i=1}^N$ determined by 1 and $(\mathbf{T}_i)_{i=1}^N$ determined by 2, update $c_{(k)}$, $k = 1, 6$ by minimizing:

$$G(c_{(k)}) = I(c_{(1)}, 1, \dots, c_{(6)}, \mathbf{E}_1, \dots, \mathbf{E}_N, \mathbf{T}_1, \dots, \mathbf{T}_N) \quad (26)$$

This method has the advantage of decreasing the value of I at every iteration, even if the convergence rate proves to be small as we approach to the minimum.

The second algorithm is a *modified Newton* (MN) *method*. At every step we approximate I with a convex quadratic form, which is minimized and we proceed for the next approximation.

The detailed expressions for the approximations of $I \approx I_0 + \delta I_0 + \delta^2 I_0$ are

$$I_0 = \frac{1}{2} \sum_{k=1}^6 \int_{\Omega} |c_{(k)}^{1/2} \mathbf{T}^{(k)} - c_{(k)}^{-1/2} \mathbf{E}^{(k)}|^2 \quad (27)$$

$$\delta I_0 = \frac{1}{2} \sum_{k=1}^6 \int_{\Omega} \left[\left(|\mathbf{E}^{(k)}|^2 - \frac{1}{c_{(k)^2}} |\mathbf{T}^{(k)}|^2 \right) \delta c_{(k)} + 2c_{(k)} \mathbf{E}^{(k)} \cdot \delta \mathbf{E}^{(k)} + \frac{2}{c_{(k)^2}} \mathbf{T}^{(k)} \delta \mathbf{T}^{(k)} \right] dx$$

$$\delta^2 I_0 = \sum_{k=1}^6 \int_{\Omega} \left[\frac{1}{c_{(k)^3}} |\mathbf{T}^{(k)}|^2 \left(\delta c_{(k)} \right)^2 + 2(\mathbf{E}^{(k)} \cdot \delta \mathbf{E}^{(k)} - \frac{1}{c_{(k)^2}} \mathbf{T}^{(k)} \delta \mathbf{T}^{(k)}) \delta c_{(k)} |\delta \mathbf{E}^{(k)}|^2 + \frac{1}{c_{(k)}} |\delta \mathbf{T}^{(k)}|^2 \right] dx$$

We can develop further if we consider the relation:

$$c_{(k)} = \frac{|\mathbf{T}^{(k)}|}{|\mathbf{E}^{(k)}|} \quad (28)$$

as a constraint. This gives to the first order in $\mathbf{E}^{(k)}$ and $\mathbf{T}^{(k)}$:

$$\delta c_{(k)} = |\mathbf{E}^{(k)}|^{-1} |\mathbf{T}^{(k)}| (|\mathbf{T}^{(k)}| \delta \mathbf{T}^{(k)} - c_{(k)^2} \mathbf{E}^{(k)} \cdot \delta \mathbf{E}^{(k)} + O(|\delta \mathbf{E}^{(k)}|^2 + |\delta \mathbf{T}^{(k)}|^2)) \quad (29)$$

Substituting these relations in the first quadratic approximation we obtain a new quadratic approximation: $I \approx I_0 + \delta I'_0 + \delta^2 I'_0$ with the same I_0 , and:

$$\begin{aligned} \delta I'_0 &= \sum_{k=1}^6 \int_{\Omega} \left[c_{(k)} \mathbf{E}^{(k)} \cdot \delta \mathbf{E}^{(k)} + \frac{1}{c_{(k)}} \mathbf{T}^{(k)} \cdot \delta \mathbf{T}^{(k)} \right] dx \quad (30) \\ \delta^2 I'_0 &= \sum_{k=1}^6 \int_{\Omega} \left[c_{(k)} |\mathbf{E}^{(k)}|^2 + \frac{1}{c_{(k)}} |\delta \mathbf{T}^{(k)}|^2 - \frac{c_{(k)}}{|\mathbf{E}^{(k)}|^2} (\mathbf{E}^{(k)} \cdot \delta \mathbf{E}^{(k)} - c_{(k)^{-2}} \mathbf{T}^{(k)} \cdot \delta \mathbf{T}^{(k)})^2 \right] dx \end{aligned}$$

It is easy to see that the Hessian $\delta^2 I'_0$ is not positive definite. We can regularize this term, as it was done in the electric case [6]. With the regularization parameter $0 \leq \epsilon \ll 1$ we have:

$$\begin{aligned} \delta^2 I''_0 &= \sum_{k=1}^6 \int_{\Omega} \left[c_{(k)} |\mathbf{E}^{(k)}|^2 + \frac{1}{c_{(k)}} |\delta \mathbf{T}^{(k)}|^2 - \frac{1}{1+\epsilon} \frac{c_{(k)}}{|\mathbf{E}^{(k)}|^2} (\mathbf{E}^{(k)} \cdot [\delta \mathbf{E}^{(k)} - c_{(k)^{-2}} \delta \mathbf{T}^{(k)}])^2 \right] dx \quad (31) \end{aligned}$$